A brief introduction to quantum error correction

Andrew S. Darmawan

Yukawa Institute for theoretical physics

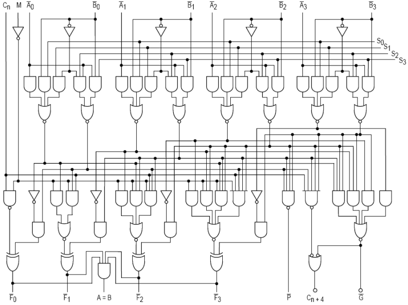
July 2, 2020

The big obstacle to building real-world quantum information technology is **noise**.

Quantum error correction is a proposed method to actively protect quantum information against noise.

Classical computers

- **Classical computers:** Basic unit of information is a bit $\{0, 1\}$. Errors on bits due to physical noise during computation are extremely rare.

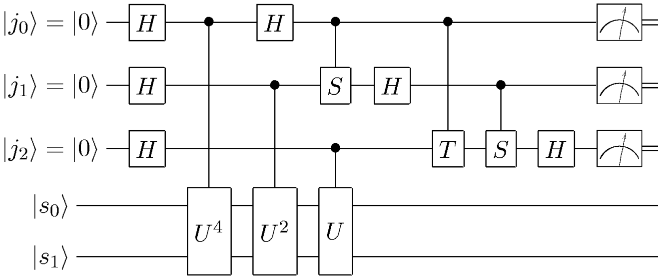


Quantum computers

- **Quantum computers:** The basic unit of information is the qubit:

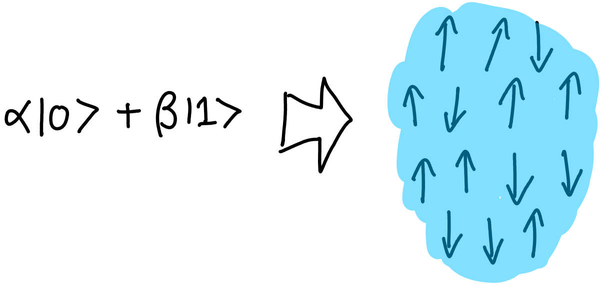
$$\alpha|0\rangle + \beta|1\rangle \tag{1}$$

  In current architectures, the probability of error per gate is at best $\sim 10^{-2} - 10^{-3}$. Errors will affect output.
- Merely *storing* quantum information is difficult.



Quantum error correction

- **Threshold theorem:** Arbitrary long quantum computations can be efficiently performed with arbitrarily high accuracy provided the error rate is below some threshold value.
- This is possible due to **Quantum error correcting codes**, where a single logical qubit is encoded into the collective state of many quantum particles.



Quantum error correction: challenges

- It is not possible to copy arbitrary quantum states (no cloning theorem).
- Superpositions must be preserved (measurements can't collapse wavefunction)
- Many types of error must be corrected.

Pauli operators

- Single qubit Pauli operators:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$$XZ = -ZX, \quad X^2 = Y^2 = Z^2 = I, \quad XZ = iY$$

- Single qubit Pauli operators with the identity form a basis for 2×2 matrices.

Pauli operators

- (n-qubit) Pauli operators are formed by tensor products $\otimes_{i=1}^{n} P_i$ where P_i are single qubit Pauli operators and the identity matrix E.g.

$$X_1 Y_3 Z_4 := X \otimes I \otimes Y \otimes Z \tag{2}$$

- Eigenvalues of Pauli operators are ± 1.
- Any two n-qubit Pauli operators either commute or anti-commute. E.g.

$$\begin{aligned}
(X \otimes I)(Z \otimes Z) &= XZ \otimes Z \\
&= -ZX \otimes Z = -(Z \otimes Z)(X \otimes I) \\
(X \otimes X)(Z \otimes Z) &= XZ \otimes XZ \\
&= (-ZX) \otimes (-ZX) = (Z \otimes Z)(X \otimes I)
\end{aligned}$$

Types of noise in quantum computers

- Noise of a single qubit interacting with an environment

$$\rho \mapsto \sum_k E_k \rho E_k^\dagger \tag{3}$$

- Common types of noise
  - Bitflip: $(1-p)\rho + pX\rho X$

$$\alpha|0\rangle + \beta|1\rangle \to \alpha|1\rangle + \beta|0\rangle \tag{4}$$

Types of noise in quantum computers

- Noise of a single qubit interacting with an environment

$$\rho \mapsto \sum_k E_k \rho E_k^\dagger \tag{3}$$

- Common types of noise
  - Bitflip: $(1-p)\rho + pX\rho X$

$$\alpha|0\rangle + \beta|1\rangle \rightarrow \alpha|1\rangle + \beta|0\rangle \tag{4}$$

  - Phase flip: $(1-p)\rho + pZ\rho Z$

$$\alpha|0\rangle + \beta|1\rangle \rightarrow \alpha|0\rangle - \beta|1\rangle \tag{5}$$

Types of noise in quantum computers

- Noise of a single qubit interacting with an environment

$$\rho \mapsto \sum_k E_k \rho E_k^\dagger \tag{3}$$

- Common types of noise
  - Bitflip: $(1-p)\rho + pX\rho X$

$$\alpha|0\rangle + \beta|1\rangle \rightarrow \alpha|1\rangle + \beta|0\rangle \tag{4}$$

  - Phase flip: $(1-p)\rho + pZ\rho Z$

$$\alpha|0\rangle + \beta|1\rangle \rightarrow \alpha|0\rangle - \beta|1\rangle \tag{5}$$

  - Depolarising: $(1-p)\rho + p/3(X\rho X + Y\rho Y + Z\rho Z)$

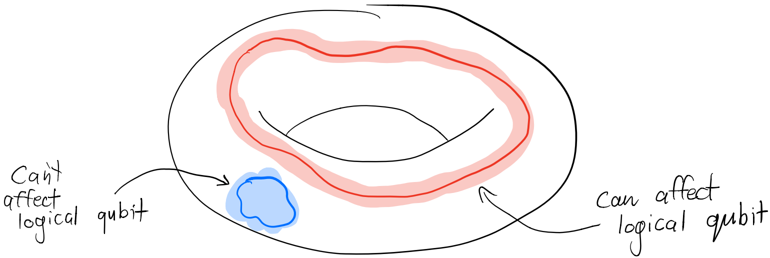
Simple example: Repetition code

$$\alpha|0\rangle + \beta|1\rangle \rightarrow \alpha|000\rangle + \beta|111\rangle \tag{6}$$

- Protects against bitflip errors.
- Does not protect against phase-flip errors.

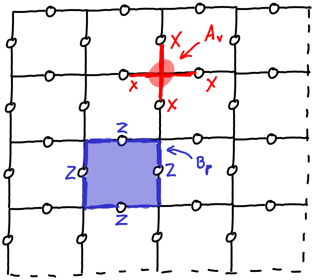
Topological error correction

- Practical: Only need nearest-neighbour interactions on a two-dimensional manifold.
- Only homologically non-trivial operators can affect encoded logical qubit. E.g. toric code/surface code
- Topological order: Homologically non-trivial observables cannot distinguish $|0_L\rangle$ and $|1_L\rangle$.



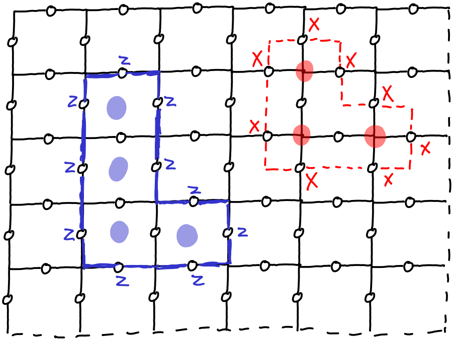
Toric code

- Physical qubits are arranged on the edges of an $L \times L$ square lattice with periodic (toric) boundary conditions.
- Set of commuting check operators $B_p = \bigotimes_{i \in p} Z_i$ and $A_v = \bigotimes_{i \in v} X_i$.
- Codespace is the simultaneous $+1$ eigenspace of all B_p and A_v operators.

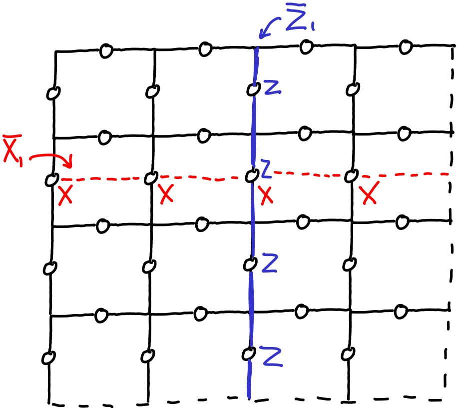


Toric code

- The check operators generate a group called the *stabilizer* $\mathcal{S}$ of the code.
- If $|\psi\rangle$ in the codespace and $g \in \mathcal{S}$ then $g|\psi\rangle = |\psi\rangle$.
- Elements of the stabilizer are homologically trivial loops.



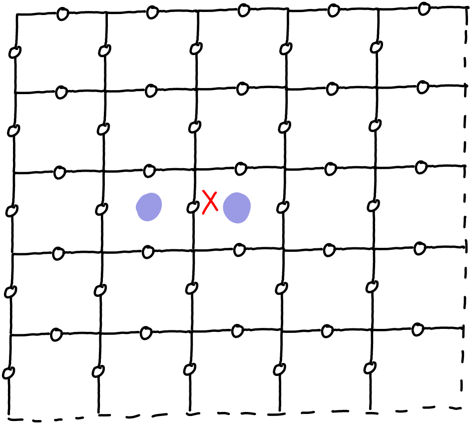
Logical operators



- The operators $\bar{Z}_1$ and $\bar{X}_1$ commute with every element in the stabilizer, but are not in the stabilizer.
- The logical qubit states $|0_L\rangle_1$, $|1_L\rangle_1$ are defined as the ± 1 eigenstates of $\bar{Z}_1$ in the code space.
- $\bar{Z}_2$ and $\bar{X}_2$ wrap around the torus in the other way.

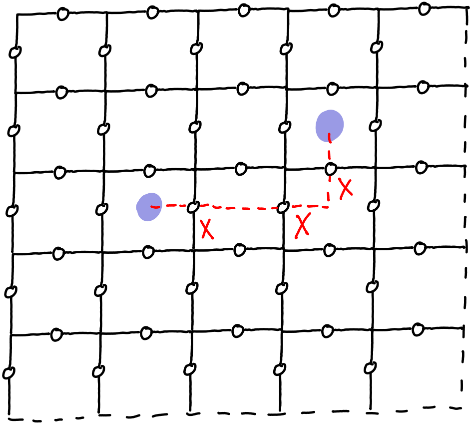
Error-correction with the toric code

- A single X error flips adjacent plaquettes.



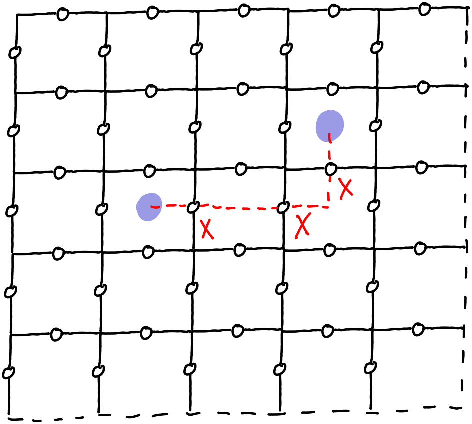
Error-correction with the toric code

- A chain of X or Z errors will only flip the checks at the ends of the chain.
- The set of flipped checks is called they syndrome.



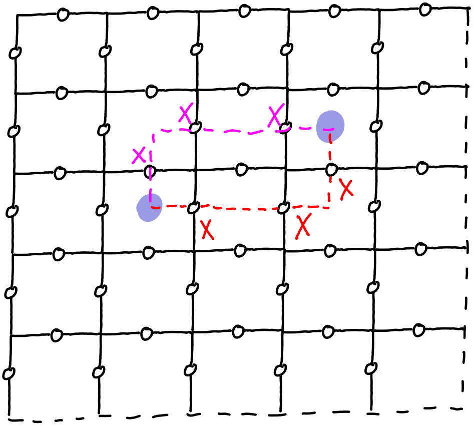
Error-correction with the toric code

- A chain of X or Z errors will only flip the checks at the ends of the chain.
- The set of flipped checks is called they syndrome.



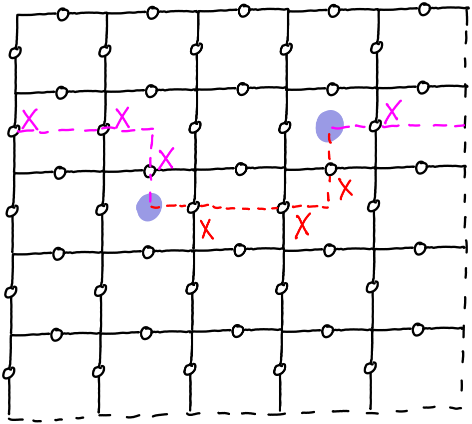
Error-correction with the toric code

- We can correct the error by matching the flipped checks (Z-checks with strings of Pauli X and vice versa).



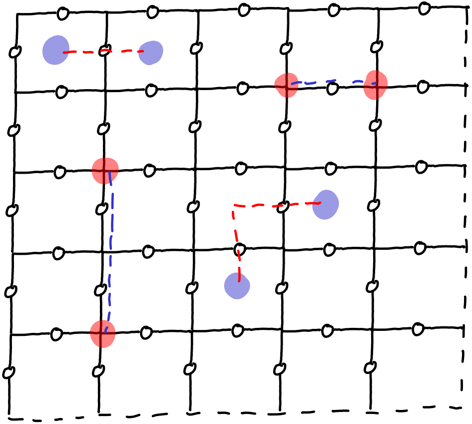
Error-correction with the toric code

- However in doing so, it is possible to apply a non-trivial operation to the encoded qubits (a logical error).
- A classical *decoding algorithm* is used to choose which correction to apply. It's goal: return to the code space while minimising the probablity of a logical error.



Minimum-weight matching decoder

- Minimum-weight matching: Consider B_p and A_v syndromes separately. For each apply a correction with smallest possible weight.



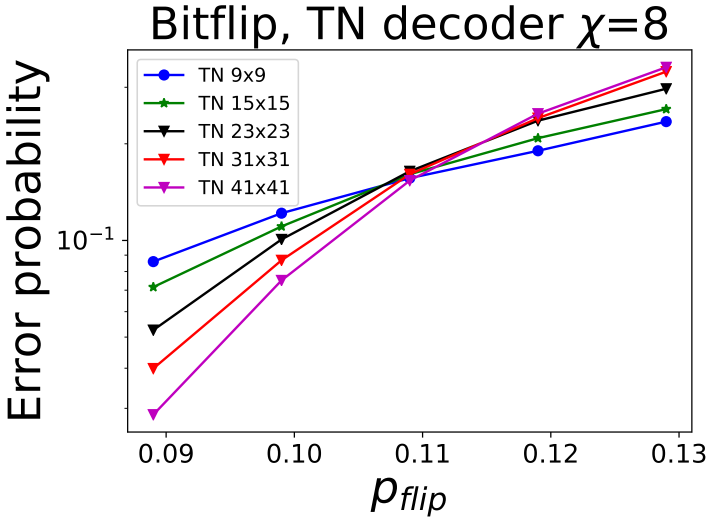
- Works well provided the number of errors is not too large.

Threshold

- If the error rate is below certain value, called the *threshold* we can exponentially supress errors on the logical qubits by increasing the lattice size.
- The threshold depends on a number of factors:
  - The code being used
  - The decoder
  - The type of noise

Results: Bitflip

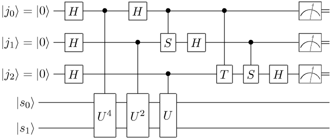
- Each qubit has independent probability p_{flip} of being flipped.



Full fault tolerance

- In the real world, gates, measurements, state-preparation are all imperfect and prone to errors.
- Remarkably, an error threshold exists even when all the operations in error correction are faulty.

Universal quantum computation



- For universal quantum computation, it must be possible to perform a universal set of gates in a fault-tolerant way.

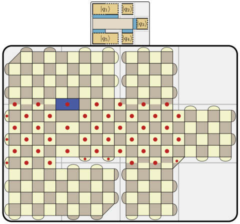


Figure: Performing gates by lattice surgery (From "A Game of Surface Codes: Large-scale Quantum Computing with Lattice Surgery" Litinski D. 2019)

Open problems

- Error correction is extremely expensive: What can we do to lower the cost?
  - Reducing noise in hardware
  - More efficient codes
  - More efficient decoding

Summary

- Quantum error correction is a way to actively protect quantum information against noise.
- Quantum error correction involves encoding a single logical qubit into many physical qubits and performing operations that detect and correct errors.
- Quantum error correction is challenging: it requires a huge number of extra qubits and operations.

References:

- Daniel Gottesman's course on QEC at Perimeter Institute 2007 and arXiv:0904.2557
- Lectures on Topological Codes and Quantum Computation by Dan Browne at UCL