QCD = a matrix model ?

Hidenori Fukaya (Osaka U.)

P.H. Damgaard (NBI,NBIA) & HF, JHEP0901:052, 2009
S. Aoki (Tsukuba) & HF, PRD 84, 014501 (2011)
JLQCD collaboration, PRL104:122002,
JLQCD collaboration, PRD 83:074501,
JLQCD collaboration, arXiv:1111.0417

1. Introduction

JLQCD (+ TWQCD) collaboration

is simulating lattice QCD

with exact chiral symmetry

using overlap fermion action.

1. Introduction

JLQCD (+ TWQCD) collaboration

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プレスリリース

大学共同利用機関法人 KEK 高エネルギー加速器研究機構

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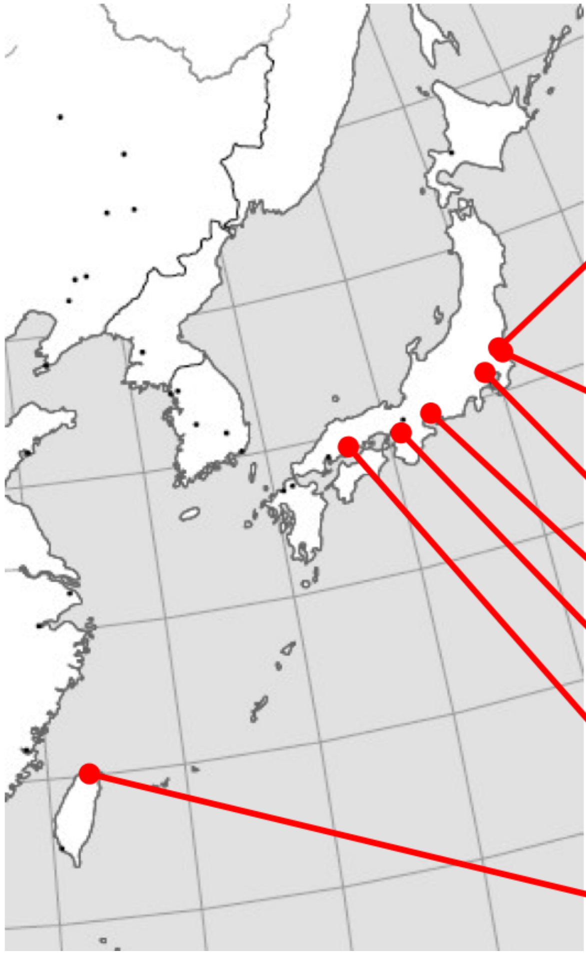
量子色力学における自発的対称性の破れを厳密に実証

大学共同利用機関法人高エネルギー加速器研究機構
国立大学法人京都大学

大学共同利用機関法人高エネルギー加速器研究機構（KEK）、国立大学法人京都大学などからなる研究チーム（研究責任者　橋本省二（高エネルギー加速器研究機構・准教授））は、物質の質量の起源となる量子色力学における自発的対称性の破れの現象を厳密な計算機シミュレーションにより世界で初めて実証しました。

1. Introduction

JLQCD (+TWQCD) collaboration



KEK: G. Cossu, X. Feng, S. Hashimoto, T. Kaneko, H. Matsufuru, S. Motoki, J. Noaki, K. Takeda, S. Ueda, N. Yamada

Univ. of Tsukuba: S. Aoki, N. Ishizuka, K. Kanaya, Y. Kuramashi, Y. Taniguchi, A. Ukawa, T. Yoshie

RIKEN(BNL): E. Shintani

Nagoya: H. Ohki

Osaka: H. Fukaya, S-W. Kim, T. Onogi,

Hiroshima: K.I. Ishikawa, M. Okawa

Taipei (TWQCD): T.W. Chiu, T.H. Hsieh, K. Ogawa

1. Introduction

Q. Is QCD a matrix model ?

A. Yes, but only in a finite volume
at very low energy.

(Lattice QCD simulation is performed in a finite V.)

Contents

2. Low energy QCD in finite V

Spontaneous chiral symmetry breaking

[Nambu, 1961] (2008 Nobel Prize)

- Chiral symmetry breaking and constituent mass

$$\langle \bar{q}q \rangle \neq 0$$

→ "effective" quark action

acquires constituent mass

$$\sim \Lambda_{QCD} \sim 300\text{MeV}$$

$$\mathcal{L} \rightarrow \bar{q}(D+m)q + C\bar{q}q\bar{q}q + \cdots$$
$$\rightarrow \bar{q}(D+m+2C\langle\bar{q}q\rangle)q + \cdots$$

→ hadron masses are ~ O(1) GeV.

- Pion effective theory

(pseudo) Nambu-Goldstone boson = pion

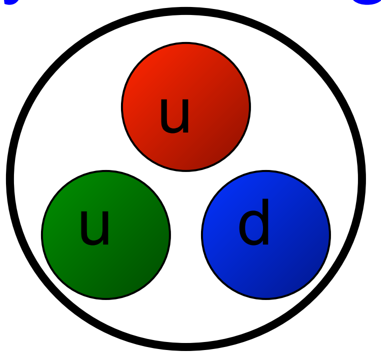
described by Chiral perturbation theory (ChPT)

[Weinberg 1979]

2. Low energy QCD in finite V

Origin of mass = chiral symmetry breaking.

- Hadrons consist of quarks.



- But

  proton mass >> quark mass ×3

  (1GeV) (3-6MeV)

- Chiral symmetry breaking generate ~90% of mass.

2. Low energy QCD in finite V

Finite volume = Pion physics.

Lattice size = 2-5fm.

Correlation length (1/M) of QCD particles

Pions(140MeV) ~ 1.4fm $\quad e^{-M_\pi L} = 0.03\text{–}0.25$

Kaons(500MeV)~ 0.4fm

Rho (800MeV)~0.26fm $\quad e^{-M_\rho L} < 0.0005$

Proton (1GeV) ~0.2fm

Finite V correction = chiral perturbation theory (ChPT)

weakly coupled = analytically calculable.

2. Low energy QCD in finite V

Our strategy

Numerical calculation

Lattice QCD simulation
in a finite volume

Analytic calculation

Finite volume correction by pion theory

QCD at $V=\infty$

A matrix Model !!

(+ corrections from field theory)

2. Low energy QCD in finite V

Pion correlator at V= ∞

(in Euclidean space-time,)

$$\int d^3x \langle P^a(x) P^b(0) \rangle = A\delta^{ab} \int \frac{dp_4}{2\pi} \frac{e^{ip_4 t}}{p_4^2 + M_\pi^2}$$

$$\propto \exp(-M_\pi t)$$

2. Low energy QCD in finite V

Pion correlator at finite V (in the p-expansion)

(periodic boundary for t-direction)

$$\int d^3x \langle P^a(x) P^b(0) \rangle = B\delta^{ab} \frac{1}{T} \sum_{p_4} \frac{e^{ip_4 t}}{p_4^2 + M_\pi^2}$$

$$p_4 = 2\pi n_t / T \quad (n_t : \text{integer})$$

$$= B\delta^{ab} \int dp \sum_n \delta(p - 2\pi n/T) \frac{1}{T} \frac{e^{ipt}}{p^2 + M_\pi^2} \qquad \left(\sum_k \delta(p - 2\pi k/T) = \sum_n \frac{T e^{ipnT}}{2\pi} \right)$$

$$\propto \frac{\cosh(M_\pi(t - T/2))}{\sinh(M_\pi T/2)}$$

2. Low energy QCD in finite V

BUT… In the limit $M_\pi \to 0$,

$$V = \infty : \quad \exp(-M_\pi t) \to 1$$

$$V \neq \infty : \quad \frac{\cosh(M_\pi(t - T/2))}{\sinh(M_\pi T/2)} \to \frac{2}{M_\pi T} \to \infty$$

Infra-red divergence due to finite V ???

despite we have IR cut-off $1/V^{1/4}$?

Something wrong ! Exp->Cosh is not enough !

2. Low energy QCD in finite V

Many vacua contribute at finite V

This fake IR divergence is due to a fixed vacuum:

$$U(x) = 1 \exp\left(i\frac{\sqrt{2}\pi(x)}{F}\right) \in SU(N_f)$$

but at finite V, the vacuum is not uniquely determined: vacuum= moduli = dynamical variable

$$U(x) = U_0 \exp\left(i\frac{\sqrt{2}\pi(x)}{F}\right),$$

U_0 should be non-perturbatively treated.

→ ε-expansion (is needed for $M_\pi V^{1/4} \ll 1.$)

2. Low energy QCD in finite V

Pion theory = Chiral perturbation theory

$$q_L \to V_L q_L, \quad q_R \to V_R q_R, \quad V_L, V_R \in SU(N)$$

but pion's transformation is non-linear. $N = \#$ of flavors

$$U(x) \equiv \exp\left(\frac{i\sqrt{2}\tau^a \pi^a(x)}{F}\right) \in SU(N) \qquad F : \text{pion decay constant}$$

$$U(x) \to V_R U(x) V_L^\dagger \qquad \det U(x) = 1, \quad U(x)^\dagger U(x) = 1.$$

Note : $\langle U \rangle \neq 0.$

[Gasser & Leutwyler 1987,1988]

2. Low energy QCD in finite V

Pion theory = Chiral perturbation theory

is constructed by collecting all possible terms invariant under $SU(N)_L \times SU_R(N)$ transformations : $U(x) \to V_R U(x) V_L^\dagger$

$$\mathrm{Tr}[U^\dagger(x)U(x)] = N. \qquad \mathrm{Tr}[\partial_\mu U^\dagger(x)\partial_\mu U(x)],\ \left(\mathrm{Tr}[\partial_\mu U^\dagger(x)\partial_\mu U(x)]\right)^2, \cdots$$

Note : all (non-trivial) terms have "derivatives".

<u>Explicit breaking term = quark mass term</u>

$$\bar{q}\mathcal{M}q = \bar{q}_L \mathcal{M}^\dagger q_R + \bar{q}_R \mathcal{M} q_L \qquad \mathcal{M} = \begin{pmatrix} m_u & & & \\ & m_d & & \\ & & m_s & \\ & & & \cdots \end{pmatrix}$$

is invariant if $\mathcal{M} \to V_R \mathcal{M} V_L^\dagger$.

$\to$

$$\mathrm{Tr}[\mathcal{M}^\dagger U(x) + U^\dagger(x)\mathcal{M}],\quad \left(\mathrm{Tr}[\mathcal{M}^\dagger U(x) + U^\dagger(x)\mathcal{M}]\right)^2, \cdots$$

The theory is "perturbative" when $E \sim p \sim m \sim 0.$

[Gasser & Leutwyler 1987,1988]

2. Low energy QCD in finite V

ε expansion

Near the massless limit : $M_\pi \ll 1/L,$

we have to change the parametrization

$$U(x) = \exp\left(\frac{i\sqrt{2}\xi(x)}{F}\right) \to U_0 \exp\left(\frac{i\sqrt{2}\xi(x)}{F}\right)$$

and non-perturbatively treat U_0 $\left(\text{with} \int d^4x\, \xi(x) = 0.\right)$

The ε counting rule :

$$\partial_\mu \sim \xi(x) \sim \mathcal{M}^{1/4} \sim 1/L \sim 1/T \sim \epsilon \qquad U_0 \sim \mathcal{O}(1)$$

The mass term is an NLO perturbation in the ε expansion.

[Gasser & Leutwyler 1987,1988]

2. Low energy QCD in finite V

Pion Lagrangian in the ε expansion

$$\mathcal{S}_{\rm ChPT} = \int d^4x \left[\frac{F^2}{4}{\rm Tr}[\partial_\mu U^\dagger(x)\partial_\mu U(x)] - \frac{\Sigma}{2}{\rm Tr}[\mathcal{M}^\dagger U(x) + U^\dagger(x)\mathcal{M}] + \cdots\right]$$

$$\to -\frac{\Sigma V}{2}{\rm Tr}[\mathcal{M}^\dagger U_0 + U_0^\dagger \mathcal{M}] + \int d^4x \left[\frac{1}{2}(\partial_\mu \pi^a(x))^2 + \frac{m^2}{2}(\pi^a(x))^2 + \cdots\right]$$

LO=SU(N) matrix model ! NLO correction = field theory

cf. in the p expansion (at large V),

$$\left(\to -\frac{\Sigma V}{2}{\rm Tr}[\mathcal{M}^\dagger + \mathcal{M}] + \int d^4x \left[\frac{1}{2}(\partial_\mu \pi^a(x))^2 + \frac{m^2}{2}(\pi^a(x))^2 + \cdots\right]\right)$$

constant (we can ignore) + field theory

[Gasser & Leutwyler 1987,1988]

2. Low energy QCD in finite V

Fixing topology = SU(N) -> U(N)

$$\mathcal{Z}_Q^{U_0} = \int \frac{d\theta}{2\pi} e^{i\theta Q} \mathcal{Z}(\theta)$$

$$= \int \frac{d\theta}{2\pi} e^{i\theta Q} \int_{SU(N)} dU_0 \exp\left(\frac{\Sigma V}{2} \mathrm{Tr}\left[\mathcal{M}^\dagger e^{i\theta/N_f} U_0 + U_0^\dagger \mathcal{M} e^{-i\theta/N_f}\right]\right)$$

$$= \int_{U(N)} dU_0 (\det U_0)^Q \exp\left(\frac{\Sigma V}{2} \mathrm{Tr}\left[\mathcal{M}^\dagger U_0 + U_0^\dagger \mathcal{M}\right]\right)$$

Using Toda-lattice equations, $\mathcal{Z}_Q^{U_0}$ is analytically integrable:

$$\mathcal{Z}_{Q(n,m)}^{U_0}(\{\mu_i\}) = \frac{\det[\mu_i^{j-1} \mathcal{J}_{Q+j-1}(\mu_i)]_{i,j=1,\cdots n+m}}{\prod_{j>i=1}^{n}(\mu_j^2 - \mu_i^2) \prod_{j>i=n+1}^{n+m}(\mu_j^2 - \mu_i^2)},$$

$$\mu_i = m_i \Sigma V, \qquad \mathcal{J}_{Q+j-1}(\mu_i) = \begin{cases} (-1)^{j-1} K_{Q+j-1}(\mu_i) & \text{for bosons} \\ I_{Q+j-1}(\mu_i) & \text{for fermions} \end{cases}$$

[Splittorff et al. 2003,2004, Fyodorov et al. 2003]

2. Low energy QCD in finite V

Another matrix model ?

[Shuryak & Verbaarschot, 1993, Akemann et al. 1997, and many…]

Let us consider a different approach from QCD itself:

$$\mathcal{Z}_Q^{QCD}(m) = \int dA_\mu \det(D+m)^{N_f} \exp(-S_G(A_\mu))$$

$$= \int dA_\mu [m^{|Q|} \prod_i (\lambda_i(A_\mu)^2 + m^2)]^{N_f} \exp(-S_G(A_\mu))$$

High modes $(\lambda_i \gg \Lambda_{\rm QCD})$ -> weak coupling

$$\lambda_i \sim \pm p + O(g^2)$$

Low modes $(\lambda_i \ll \Lambda_{\rm QCD})$ -> strong coupling

$$\lambda_i \sim \text{random and non-degenerate}$$

2. Low energy QCD in finite V

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$$= \int dA_\mu [m^{|Q|} \prod_i (\lambda_i(A_\mu)^2 + m^2)]^{N_f} \exp(-S_G(A_\mu))$$

For low energy part, can we replace ???

$$= \int \prod_i d\lambda_i [m^{|Q|} \prod_i (\lambda_i^2 + m^2)]^{N_f} \exp(-S_{\mathrm{matrix}}(\lambda_i))???$$

2. Low energy QCD in finite V

Chiral random matrix theory (ChRMT)

$$\mathcal{Z}_Q^{\mathrm{ChRMT}} = \int d\Phi \prod_f^{N_f} \det \begin{pmatrix} \hat{m} & i\Phi \\ i\Phi^\dagger & \hat{m} \end{pmatrix} \exp(-S_{\mathrm{matrix}}(\Phi))$$

$$\Phi : (n+Q) \times n \quad \mathrm{matrix} \qquad \hat{m} = m \times const.$$

Universality : the spectrum in the limit $n \to \infty$ is the same for different $S_{\mathrm{matrix}}(\Phi)$ (up to an overall factor).

[Nishigaki 1996, Akemann et al. 1997]

Simple choice : $S_{\mathrm{matrix}}(\Phi) = \frac{2n+Q}{2} \mathrm{Tr}(\Phi^\dagger \Phi)$

2. Low energy QCD in finite V

ChRMT = ChPT

$$S_{\mathrm{matrix}}(\Phi) = \frac{N}{2}\mathrm{Tr}(\Phi^\dagger\Phi), \quad (N = 2n + Q)$$

$$\mathcal{Z}_Q^{\mathrm{ChRMT}} = \int d\Phi \prod_f^{N_f} \det \begin{pmatrix} \hat{m} & i\Phi \\ i\Phi^\dagger & \hat{m} \end{pmatrix} \exp(-S_{\mathrm{matrix}}(\Phi))$$

$$= \int d\Phi \int d\psi^* d\psi \exp\left[-\frac{N}{2}\Phi_{ij}^*\Phi_{ij} - i\sum_f^{N_f} \begin{pmatrix} \psi_L^{f*} & \psi_R^{f*} \end{pmatrix} \begin{pmatrix} i\hat{m} & -\Phi \\ -\Phi^\dagger & i\hat{m} \end{pmatrix} \begin{pmatrix} \psi_L^f \\ \psi_R^f \end{pmatrix} \right]$$

$$= \int d\psi^* d\psi \exp\left[\hat{m}\sum_f (\psi_{Li}^{f*}\psi_{Li}^f + \psi_{Ri}^{f*}\psi_{Ri}^f) - \sum_{f,g} \frac{2}{N}(\psi_{Rj}^{f*}\psi_{Li}^{f*})(\psi_{Li}^{g*}\psi_{Rj}^{g*}) \right]$$

$$\text{second term} = -\frac{2}{N}\sum_{f,g}(B_{fg}^2 - C_{fg}^2) \qquad B_{fg} = \frac{1}{2}\sum_i(\psi_{Li}^{*g}\psi_{Li}^f - \psi_{Ri}^{*f}\psi_{Ri}^g), \quad C_{fg} = \frac{1}{2}\sum_i(\psi_{Li}^{*g}\psi_{Li}^f + \psi_{Ri}^{*f}\psi_{Ri}^g)$$

We can use Hubbard–Stratonovich (HS) transformation :

$$\exp(-A_{fg}B_{fg}^2) \int d\sigma_{fg} \exp\left[-\frac{1}{A_{fg}}(\sigma_{fg})^2 + 2i\sigma_{fg}B_{fg} \right]$$

2. Low energy QCD in finite V

ChRMT = ChPT

$$\mathcal{Z}_Q^{\mathrm{ChRMT}} = \int d\sigma e^{-\frac{N}{2}\mathrm{Tr}[\sigma\sigma^\dagger]} \det(\sigma + \hat{m})^n \det(\sigma^\dagger + \hat{m})^{n+Q},$$

$$\sigma : N_f \times N_f \quad \text{complex matrix}$$

In the limit $n \to \infty$, integration around the suddle point, which satisfies $\sigma(\sigma^\dagger + m) = 1,$ leaves unirary part :

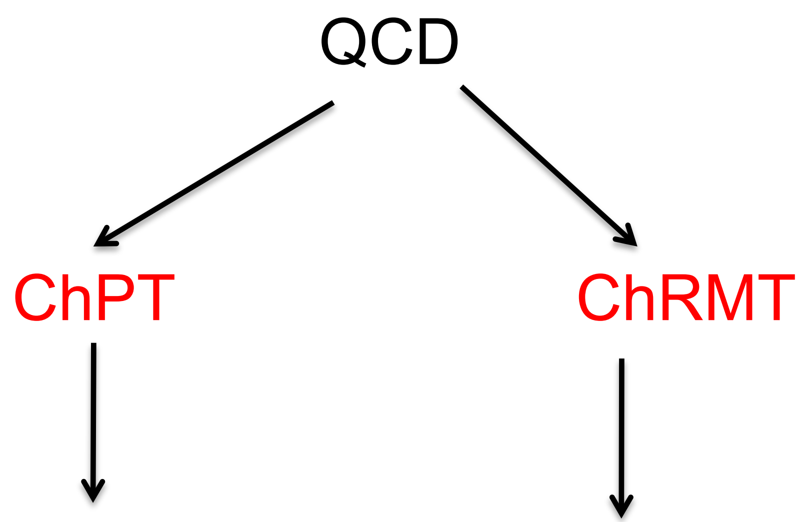
$$\mathcal{Z}_Q^{\mathrm{ChRMT}} \to \int_{U(N_f)} dU_0 \exp(n\hat{m}\mathrm{Tr}(U_0 + U_0^\dagger)) \det(U_0^\dagger)^Q$$

$$= \mathcal{Z}_Q^{\mathrm{ChPT}} \;\; !$$

if we identify $n\hat{m} = m\Sigma V.$

2. Low energy QCD in finite V

Banks-Casher relation [1980]



$$\langle \bar{\psi}\psi \rangle = \pi \rho(0)$$

$$\left(\rho(\lambda) \equiv \lim_{V\to\infty} \sum_{\lambda_i \geq 0} \left\langle \frac{\delta(\lambda_i - \lambda)}{V} \right\rangle . \right)$$

λ : Dirac eigenvalue

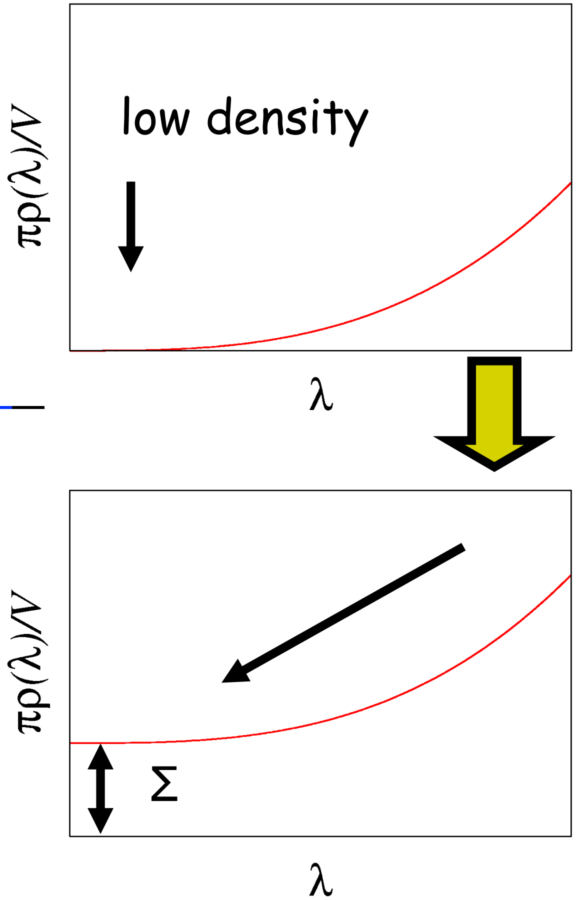
2. Low energy QCD in finite V

B-C relation for non-zero $\lambda, m, V,$ and Q

$$
\begin{aligned}
\rho(\lambda) &= \int_0^\infty d\lambda' \delta(\lambda-\lambda')\rho(\lambda') \\
&= \lim_{\epsilon\to 0}\frac{1}{\pi}\int_0^\infty d\lambda' \frac{2\epsilon}{(\lambda-\lambda')^2+\epsilon^2}\rho(\lambda') \\
&= \lim_{\epsilon\to 0}\frac{1}{\pi}\int_0^\infty d\lambda' \left[\frac{1}{i(\lambda-\lambda')+\epsilon} - \frac{1}{i(\lambda-\lambda')-\epsilon}\right]\rho(\lambda') \\
&= \lim_{\epsilon\to 0}\frac{1}{\pi}\int_0^\infty d\lambda' \left[\frac{1}{i(\lambda-\lambda')+\epsilon} - \frac{1}{i(\lambda-\lambda')-\epsilon}\right]\left\langle \sum_{\lambda_i}\frac{\delta(\lambda'-\lambda_i)}{V}\right\rangle \\
&= \lim_{\epsilon\to 0}\frac{1}{\pi V}\left\langle \mathrm{Tr}\frac{1}{D+i\lambda+\epsilon} - \mathrm{Tr}\frac{1}{D+i\lambda-\epsilon}\right\rangle \\
&= \frac{1}{\pi}\lim_{\epsilon\to 0}\left(\langle\bar{q}q\rangle_{m_v=i\lambda+\epsilon} - \langle\bar{q}q\rangle_{m_v=i\lambda-\epsilon}\right) \\
&= \frac{1}{\pi}\mathrm{Re}\langle\bar{q}q\rangle_{m_v=i\lambda},
\end{aligned}
$$

2. Low energy QCD in finite V

Why Chiral Symmetry broken ?



$$i\not{D} = \not{p} + g\not{A}$$

Free fermon When $g = 0, \quad \lambda = \pm p$

$$Area(S^3) = 2\pi^2 R^3$$

$$\rho(\lambda) = \frac{2\pi^2\lambda^3}{V} \times \left(\frac{L}{2\pi}\right)^4 \times 3 \times 4 \times \frac{1}{2}$$

$$= \frac{3}{4\pi^2}\lambda^3$$

Strong coupling

$$\Sigma = \langle \bar{q} q \rangle$$

2. Low energy QCD in finite V

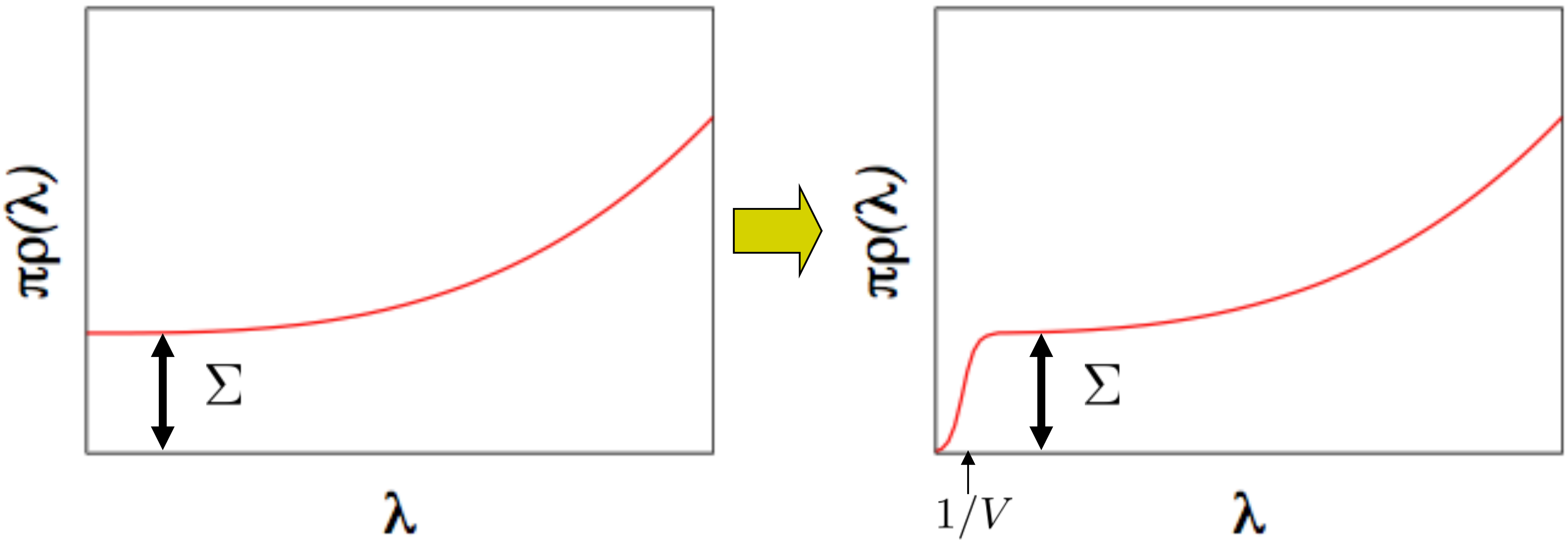
Finite V effects in BC relation

Finite V never allows SSB $\rightarrow \rho(0) = 0.$

but SSB occurs at infinite V

$\longrightarrow$ only $\lambda < 1/V$ modes know finite V.

infinite V finite V

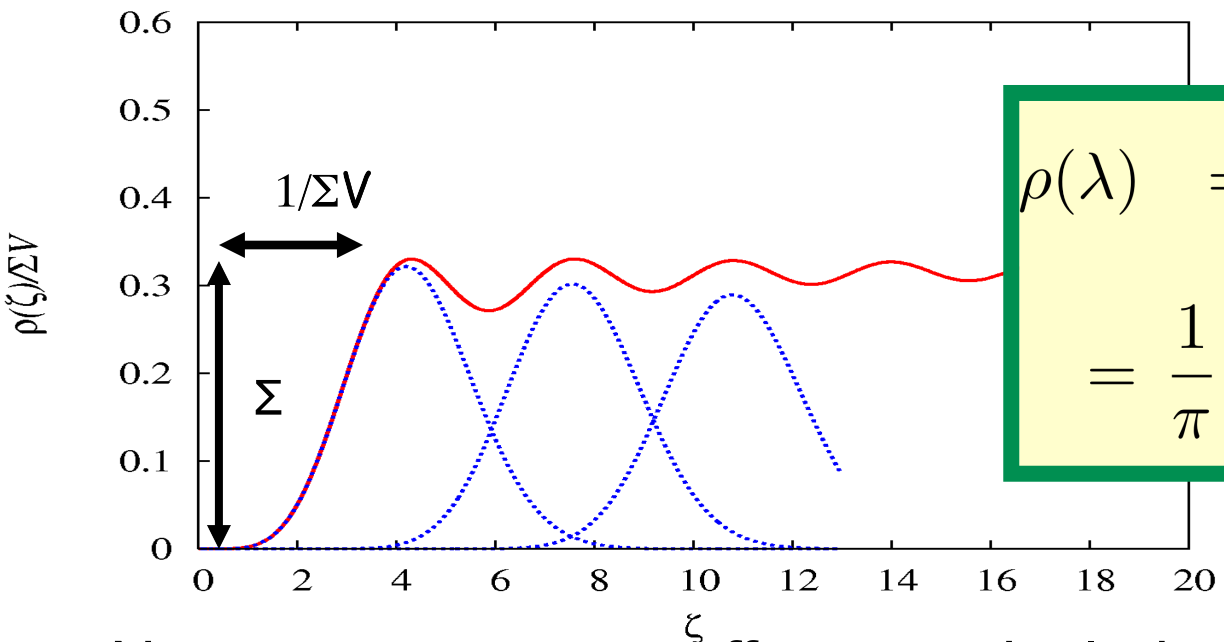


2. Low energy QCD in finite V

Result of ChPT (= ChRMT) at LO

[Akemann & Damgaard (1998), Damgaard & Nishigaki (2001)]

Drop around zero = non-perturbative zero-mode fluctuation



$$\rho(\lambda) = \left.\frac{\langle \bar{q}q \rangle}{\pi}\right|_{m_v=i\lambda}$$
$$= \left.\frac{1}{\pi}\frac{\partial}{\partial m_v} \ln Z^{\mathrm{ChPT}}\right|_{m_v=i\lambda}$$

Note: m, Q, V effects are included.

2. Low energy QCD in finite V (Review)

Summary of Sec.2

Low energy QCD in finite V = Martix model(s)

QCD

Chiral symmetry breaking

Random Dirac eigenvalues

U(Nf) matrix model (pion's moduli) = n=∞ random matrix model (Gaussian unitary ensemble)

Banks-Casher relation

3. Corrections from *field theory*

ε expansion of Chiral Lagrangian

$$\mathcal{L} = -\frac{\Sigma}{2}\mathrm{Tr}\left[\mathcal{M}^\dagger U_0 + U_0^\dagger \mathcal{M}\right]$$

LO = matrix model

$$+\frac{1}{2}\mathrm{Tr}(\partial_\mu \xi)^2$$

NLO = massless pions (10-20%)

$$+\frac{\Sigma}{2F^2}\mathrm{Tr}[\mathcal{M}^\dagger U_0 \xi^2 + \xi^2 U_0^\dagger \mathcal{M}] + \cdots ,$$

NNLO = (perturbative) interactions (10-20%)

The mass term contribution is underestimated.

3. Corrections from *field theory*

ε expansion is really useful ?

ε expansion

$$M_\pi L \ll 1.$$

p expansion

$$M_\pi L \gg 1.$$

Group	Nf	Action	a(fm)	L	Mπ (MeV)
ETMC	2	Twisted mass	0.05-0.100	~3fm	280
MILC	2+1	Staggered	0.045-0.12	3~6 fm	250
RBC/UKQCD	2+1	Domain wall	0.085-0.11	3~4fm	290
JLQCD	2+1	Overlap	0.11	1.8fm	310
PACS-CS	2+1	Wilson	0.09	~3fm	140
BMW	2+1	Wilson	0.065-0.125	3~5fm	190
ALV	2+1	DW on MILC	0.06-0.12	3~4fm	250
HPQCD	2+1	HISQ	0.045-0.15	3~4fm	360

But on the lattice,

$$M_\pi L = 2 \sim 5.$$

No requirement in original theory (if $E, p \ll m_\rho$):

both expansions are bad. → Better way of expansion ?

3. Corrections from *field theory*

New expansion

p-expansion

$$U(x) = 1 \exp\left(i\frac{\sqrt{2}\pi(x)}{F}\right), \qquad M_\pi \sim \text{LO}$$

ε-expansion

$$U(x) = U_0 \exp\left(i\frac{\sqrt{2}\pi(x)}{F}\right), \qquad M_\pi \sim \text{NLO}$$

3. Corrections from *field theory*

New expansion

p-expansion

$$U(x) = 1 \exp\left(i\frac{\sqrt{2}\pi(x)}{F}\right), \qquad M_\pi \sim \mathrm{LO}$$

ε-expansion

$$U(x) = U_0 \exp\left(i\frac{\sqrt{2}\pi(x)}{F}\right), \qquad M_\pi \sim \mathrm{NLO}$$

New *i* (interpolating)- expansion

$$U(x) = U_0 \exp\left(i\frac{\sqrt{2}\pi(x)}{F}\right), \qquad M_\pi \sim \mathrm{LO}$$

[Damgaard & HF, 2008]

3. Corrections from *field theory*

The pion (chiral) Lagrangian at finite V

$$\mathcal{L} = -\frac{\Sigma}{2}\mathrm{Tr}\left[\mathcal{M}^\dagger U_0 + U_0^\dagger \mathcal{M}\right]$$

LO = matrix model

$$+\frac{1}{2}\mathrm{Tr}(\partial_\mu \xi)^2 + \frac{1}{2}\sum_a M_a^2 (\xi^a)^2$$

NLO = massive pions

$$+\frac{\Sigma}{2F^2}\mathrm{Tr}[\mathcal{M}^\dagger (U_0 - 1)\xi^2 + \xi^2 (U_0 - 1)^\dagger \mathcal{M}]$$

$$+\cdots$$

NNLO = (perturbative) interactions

[Damgaard & HF, 2008]

= a hybrid system of

matrix model and massive bosonic fields

3. Corrections from *field theory*

Calculation at NLO

$$\langle f(U_0,\xi)\rangle = \frac{\left\langle \left\langle f(U_0,\xi) e^{-\mathcal{S}_I^{(1)}(U_0,\,\xi)} \right\rangle_\xi e^{-\mathcal{S}_I^{(2)}(U_0)} \right\rangle_{U_0}}{\left\langle \left\langle e^{-\mathcal{S}_I^{(1)}(U_0,\,\xi)} \right\rangle_\xi e^{-\mathcal{S}_I^{(2)}(U_0)} \right\rangle_{U_0}},$$

$S_I^{(i)}$'s : (NLO) interaction terms

$\langle \cdots \rangle_{U_0}$: non-perturbative group integrals

$\langle \cdots \rangle_{\xi}$: perturbative integrals

3. Corrections from *field theory*

Pion correlator at 1-loop

[Aoki & HF, 2011]

Because of the mixing of zero and non-zero modes, the calculation is very tedious :

$$\begin{aligned}
\langle P(x)P(0)\rangle = & -\frac{\Sigma^2}{4}(Z_M^{12}Z_F^{12})^4\mathcal{C}^{0a} + \frac{\Sigma^2}{\mu_1+\mu_2}\left(\frac{\Sigma_{\rm eff}}{\Sigma} - (Z_M^{12}Z_F^{12})^2\right)\mathcal{C}^{0b} \\
& + \frac{\Sigma^2}{2}(\Delta Z_{11}^{\Sigma} - \Delta Z_{22}^{\Sigma})\mathcal{C}^{0c} + \frac{\Sigma^2}{2F^2}\Big[(Z_F^{12}(Z_M^{12})^2)^2\mathcal{C}^1\bar{\Delta}(x, M_{12}'^2) \\
& + \mathcal{C}^2\left(\frac{\Sigma}{F^2}\partial_{M^2}\right)\bar{\Delta}(x, M^2)\Big|_{M^2=M_{12}^2} \\
& + \mathcal{C}_{12}^4\left(\bar{\Delta}(x, M_{11}^2) - \bar{\Delta}(x, M_{12}^2)\right) + \mathcal{C}_{21}^4\left(\bar{\Delta}(x, M_{22}^2) - \bar{\Delta}(x, M_{12}^2)\right) \\
& + \sum_{j\neq 1}\mathcal{C}_{1j}^5\left(\bar{\Delta}(x, M_{2j}^2) - \bar{\Delta}(x, M_{12}^2)\right) + \sum_{i\neq 2}\mathcal{C}_{2i}^5\left(\bar{\Delta}(x, M_{1i}^2) - \bar{\Delta}(x, M_{12}^2)\right) \\
& + \mathcal{C}^6\bar{G}(x, M_{11}^2, M_{22}^2) \\
& + \mathcal{C}_{12}^7\left(\bar{G}(x, M_{11}^2, M_{22}^2) - \bar{G}(x, M_{11}^2, M_{11}^2)\right) \\
& + \mathcal{C}_{21}^7\left(\bar{G}(x, M_{11}^2, M_{22}^2) - \bar{G}(x, M_{22}^2, M_{22}^2)\right)\Big],
\end{aligned}$$

3. Corrections from *field theory*

[Aoki & HF, 2011]

where

$$\mathcal{C}^{0a} \equiv \left\langle ([U_0]_{12} - [U_0^\dagger]_{21})([U_0]_{21} - [U_0^\dagger]_{12}) + \frac{1}{2}([U_0]_{12} - [U_0^\dagger]_{21})^2 + \frac{1}{2}([U_0]_{21} - [U_0^\dagger]_{12})^2 \right\rangle_{U_0},$$

$$\mathcal{C}^{0b} \equiv \left\langle \frac{[U_0 + U_0^\dagger]_{11}}{2} + \frac{[U_0 + U_0^\dagger]_{22}}{2} \right\rangle_{U_0},$$

$$U_0 \in SU(N) \text{ in } \theta = 0 \text{ vacuum},$$
$$U_0 \in U(N) \text{ in a fixed } Q \text{ sector}$$

$$\mathcal{C}^{0c} \equiv \frac{1}{4}\langle ([U_0]_{12} - [U_0^\dagger]_{21})^2 - ([U_0]_{21} - [U_0^\dagger]_{12})^2 \rangle_{U_0},$$

$$\mathcal{C}^{1} \equiv \left\langle ([U_0]_{11} + [U_0^\dagger]_{22})([U_0]_{22} + [U_0^\dagger]_{11}) + \sum_{j\neq 1}^{N_f} [U_0]_{1j}[U_0^\dagger]_{j1} + \sum_{i\neq 2}^{N_f} [U_0]_{2i}[U_0^\dagger]_{i2} \right.$$

$$\mathcal{C}^{2} \equiv \left\langle 2([\mathcal{R}]_{11} + [\mathcal{R}]_{22}) - \sum_{j\neq 1} \frac{[\mathcal{R}]_{1j}[\mathcal{R}]_{j1}}{m_j - m_1} - \sum_{i\neq 2} \frac{[\mathcal{R}]_{2i}[\mathcal{R}]_{i2}}{m_i - m_2} \right\rangle_{U_0},$$

$$\mathcal{C}^{3}_{ij} \equiv \frac{1}{2}\langle ([U_0]_{ji})^2 + ([U_0^\dagger]_{ij})^2 \rangle_{U_0} + \frac{\langle [\mathcal{R}]_{ij}[U_0^\dagger]_{ij} + [U_0]_{ji}[\mathcal{R}]_{ji} \rangle_{U_0}}{m_i - m_j} + \frac{\langle ([\mathcal{R}]_{ij})^2 + ([\mathcal{R}]_{ji})^2 \rangle_{U_0}}{2(m_i - m_j)^2},$$

$$\mathcal{C}^{4}_{ij} \equiv \langle [U_0]_{ij}[U_0^\dagger]_{ji} \rangle_{U_0} + \frac{\langle [\mathcal{R}]_{ji}[U_0]_{ij} + [\mathcal{R}]_{ij}[U_0^\dagger]_{ji} \rangle_{U_0}}{m_j - m_i} + \frac{\langle [\mathcal{R}]_{ij}[\mathcal{R}]_{ji} \rangle_{U_0}}{(m_j - m_i)^2},$$

$$\mathcal{C}^{5} \equiv -\left\langle ([U_0]_{12} + [U_0^\dagger]_{21})([U_0]_{21} + [U_0^\dagger]_{12}) + \frac{1}{2}([U_0]_{12} + [U_0^\dagger]_{21})^2 + \frac{1}{2}([U_0]_{21} + [U_0^\dagger]_{12})^2 \right\rangle_{U_0},$$

$$\mathcal{C}^{6}_{ij} \equiv \frac{1}{2}\langle ([U_0]_{ji} + [U_0^\dagger]_{ij})^2 \rangle_{U_0} + \frac{\langle ([\mathcal{R}]_{ij} + [\mathcal{R}]_{ji})([U_0]_{ji} + [U_0^\dagger]_{ij}) \rangle_{U_0}}{m_i - m_j}$$

3. Corrections from *field theory*

The zero-mode integrals

"simplest" example (in 2+1 flavor theory)

$$\mathcal{S}_v \equiv \left\langle \frac{[U_0 + U_0^\dagger]_{vv}}{2} \right\rangle_{U_0} = -\frac{1}{(\mu^2-\mu_v^2)^2(\mu_s^2-\mu_v^2)} \times \frac{\det\begin{pmatrix} \partial_{\mu_v} K_\nu(\mu_v) & I_\nu(\mu_v) & I_\nu(\mu) & \mu^{-1}I_{\nu-1}(\mu) & I_\nu(\mu_s) \\ -\partial_{\mu_v}(\mu_v K_{\nu+1}(\mu_v)) & \mu_v I_{\nu+1}(\mu_v) & \mu I_{\nu+1}(\mu) & I_\nu(\mu) & \mu_s I_{\nu+1}(\mu_s) \\ \partial_{\mu_v}(\mu_v^2 K_{\nu+2}(\mu_v)) & \mu_v^2 I_{\nu+2}(\mu_v) & \mu^2 I_{\nu+2}(\mu) & \mu I_{\nu+1}(\mu) & \mu_s^2 I_{\nu+2}(\mu_s) \\ -\partial_{\mu_v}(\mu_v^3 K_{\nu+3}(\mu_v)) & \mu_v^3 I_{\nu+3}(\mu_v) & \mu^3 I_{\nu+3}(\mu) & \mu^2 I_{\nu+2}(\mu) & \mu_s^3 I_{\nu+3}(\mu_s) \\ \partial_{\mu_v}(\mu_v^4 K_{\nu+4}(\mu_v)) & \mu_v^4 I_{\nu+4}(\mu_v) & \mu^4 I_{\nu+4}(\mu) & \mu^3 I_{\nu+3}(\mu) & \mu_s^4 I_{\nu+4}(\mu_s) \end{pmatrix}}{\det\begin{pmatrix} I_\nu(\mu) & \mu^{-1}I_{\nu-1}(\mu) & I_\nu(\mu_s) \\ \mu I_{\nu+1}(\mu) & I_\nu(\mu) & \mu_s I_{\nu+1}(\mu_s) \\ \mu^2 I_{\nu+2}(\mu) & \mu I_{\nu+1}(\mu) & \mu_s^2 I_{\nu+2}(\mu_s) \end{pmatrix}},$$

where I_ν's and K_ν's are modified Bessel functions,

$$\mu_v = m_v \Sigma V,\ \mu = m_{ud}\Sigma V,\ \mu_s = m_s \Sigma V.$$

3. Corrections from *field theory*

Pion correlator at 1-loop [Aoki & HF, 2011]

$$\int d^3x \langle P(x)P(0)\rangle = C_{PP}\frac{\cosh(M_\pi^{NLO}(t-T/2))}{\sinh(M_\pi^{NLO}T/2)} + D_{PP}$$

D_{PP} cancels IR divergence: $D_{PP} \underset{M\to 0}{\sim} -2\frac{C_{PP}}{M_\pi^{NLO}T} + E + \cdots ,$

disappears in the p-regime: $\lim_{M_\pi \to \text{large}} D_{PP} \sim \exp(-m\Sigma V) \to 0.$

($C_{PP}, D_{PP}, M_\pi^{NLO}$: functions of Σ and F_π)

$$\to_{V\to\infty} \exp(-M_\pi t)$$

3. Corrections from *field theory*

Dirac eigenvalue density at one-loop

$$\rho_Q(\lambda) = \Sigma_{\text{eff}}\ \hat{\rho}_Q^{\epsilon}(\lambda\Sigma_{\text{eff}}V, \{m_{sea}\Sigma V\}) + \rho^p(\lambda),$$

$\hat{\rho}_Q^{\epsilon}$: same function as at LO of ε expansion

(in terms of modified Bessel functions)

but Σ_{eff} includes m_{sea} dependence at NLO.

ρ^p : NLO logarithmic curve (chiral-logs)

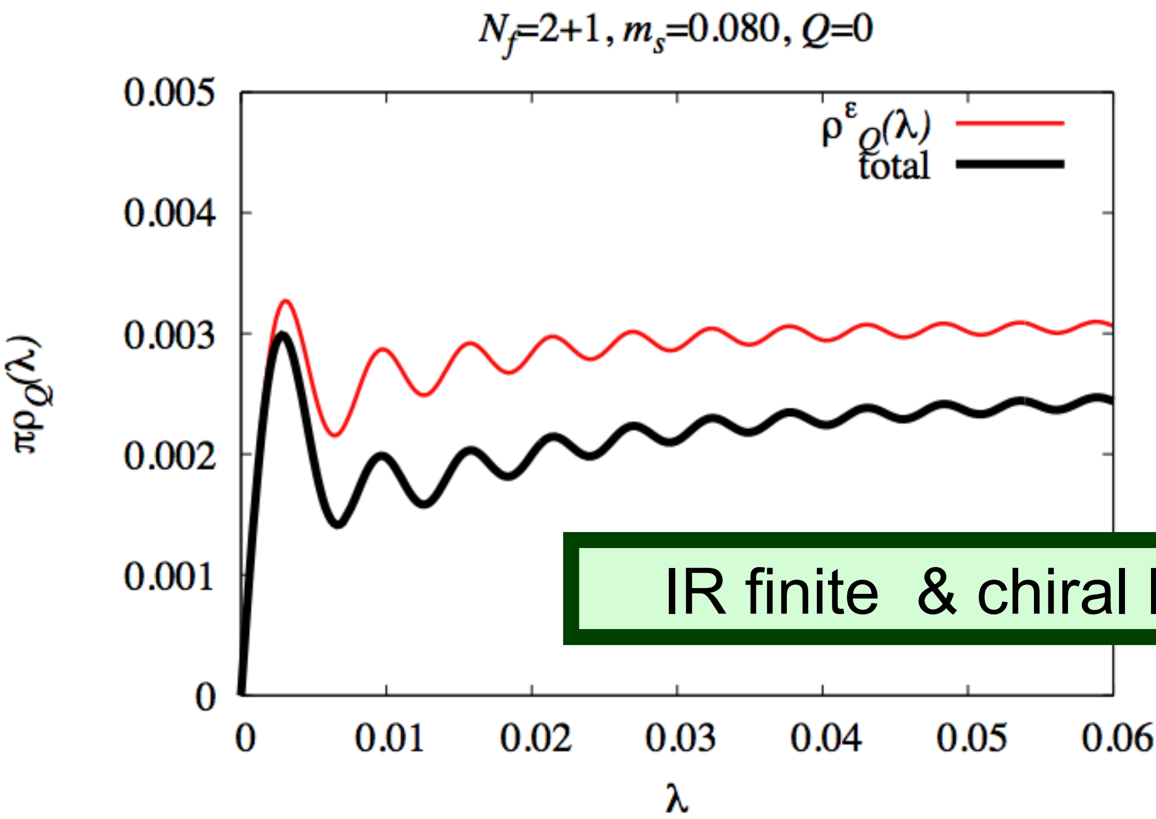
from the non-zero mode in the bulk.

(IR finite. Q-independent.)

[P.H.Damgaard & HF JHEP0901:052,2009]

3. Corrections from *field theory*

Dirac eigenvalue density at one-loop



$$
\begin{aligned}
L &= T/3 = 2\text{fm}, \\
m_{ud} &= 20\text{MeV}, \\
m_s &= 100\text{MeV}, \\
\Sigma &= [250\text{MeV}]^3, \\
F &= 90\text{MeV},
\end{aligned}
$$

$$L_6 = 0.$$

$$a^{-1} = 1.8\text{GeV}$$

IR finite & chiral log

4. Application to lattice QCD analysis

QCD simulation with exact chiral symmetry

[JLQCD & TWQCD collaborations, 2006-2011]

2+1-flavor overlap Dirac fermions [Neuberger 98]

Iwasaki gauge action, β=2.3, 1/a ~ 1.759 GeV.

Lattice size : L=16 [1.8 fm], T=48.

Topology fixed: Q=0 (or 1)

Quark masses : m_s=0.08, 0.100,

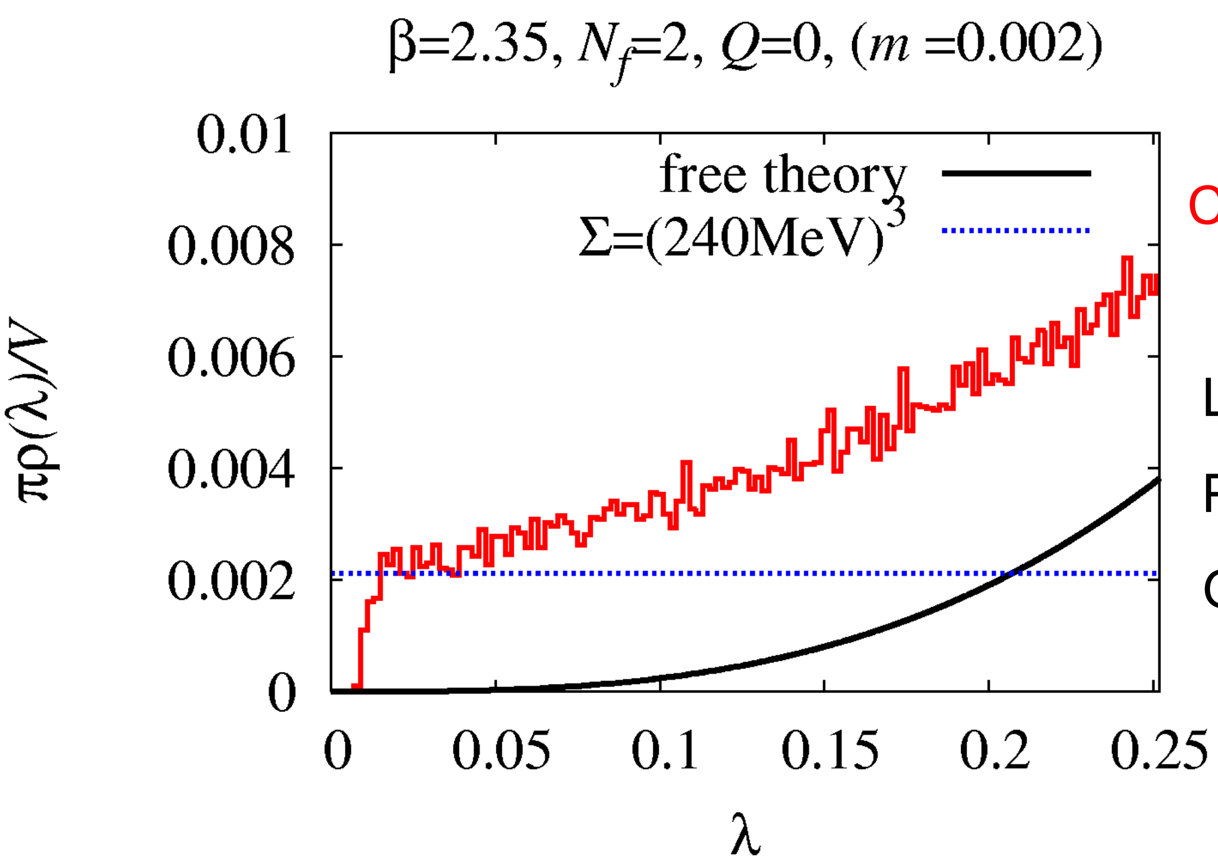
m_{ud}= 0.002, 0.015,0.025,0.035,0.050,0.080,0.100

(~3 MeV) (30MeV <)

ε-regime, p-regime

4. Application to lattice QCD analysis

lattice QCD results JLQCD [2007 - 2010]



Consistent with

Banks- Casher's scenario !

Low-mode's accumulation.

Plateau suggests $\Sigma \sim (240\text{MeV})^3$.

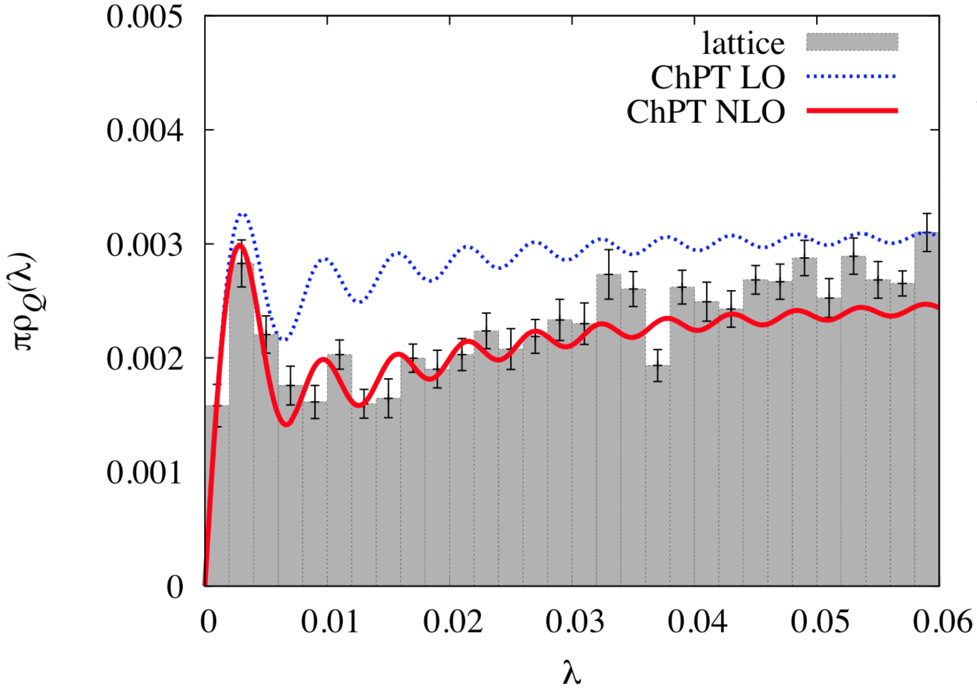
Gap from 0 : finite V effects

4. Application to lattice QCD analysis

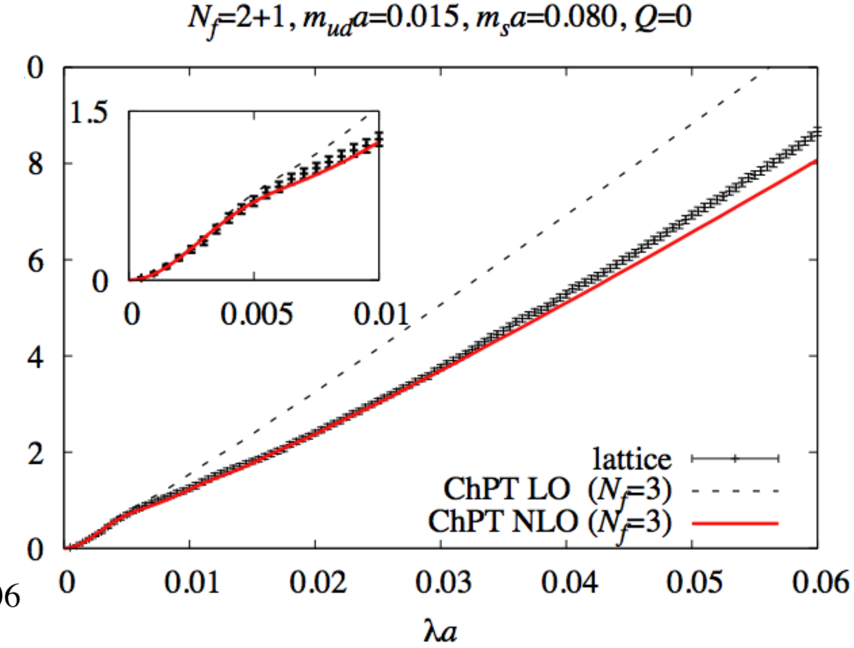
[JLQCD, PRL 104,122002 (2010)]

Comparison with NLO ChPT

consistent with our NLO ChPT analysis !



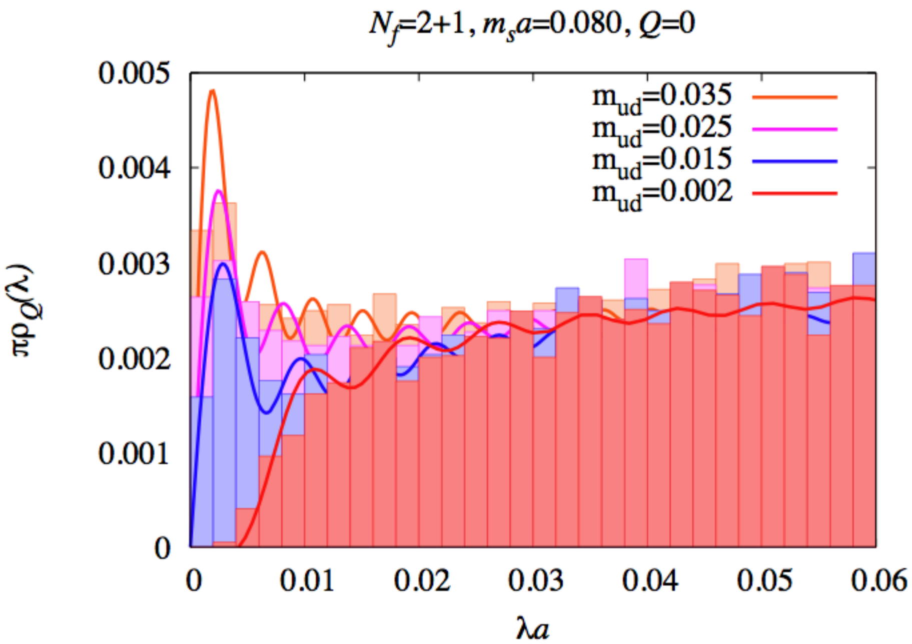
$$N_Q(\lambda) = V \int_0^{\lambda} d\lambda' \rho_Q(\lambda')$$

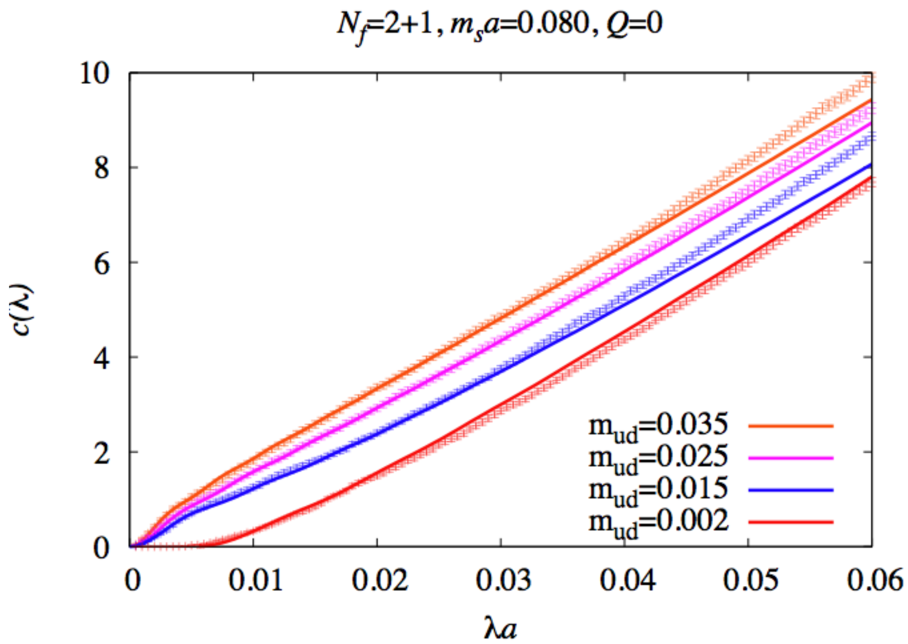


4. Application to lattice QCD analysis

[JLQCD, PRL 104,122002 (2010)]

m_{ud} dependence

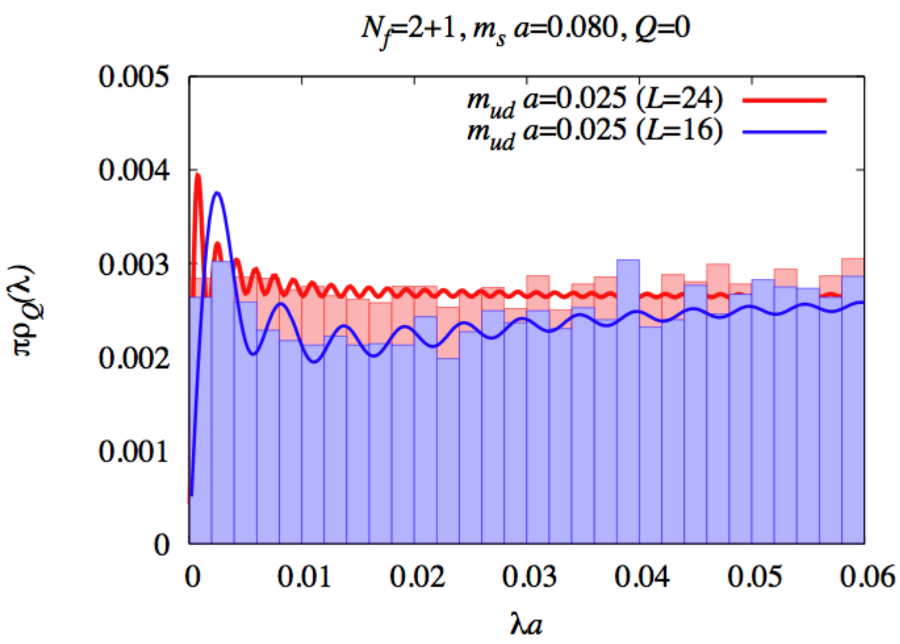


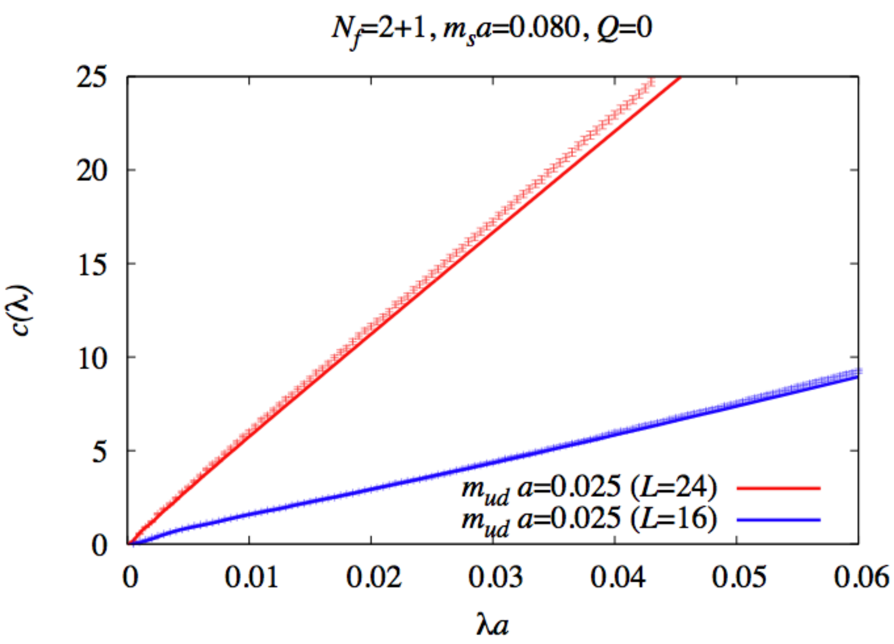


4. Application to lattice QCD analysis

[JLQCD, PRL 104,122002 (2010)]

volume dependence

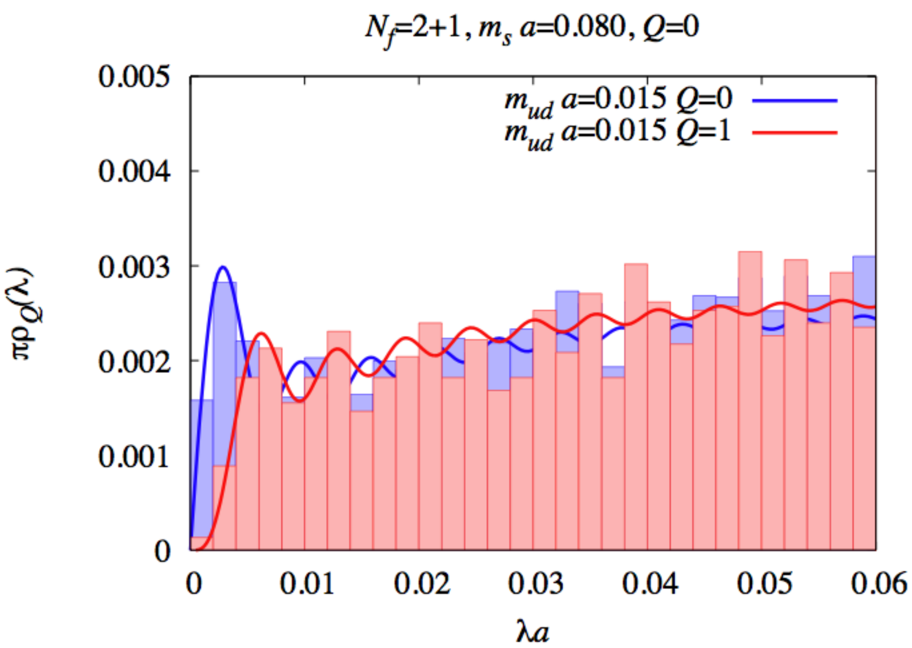


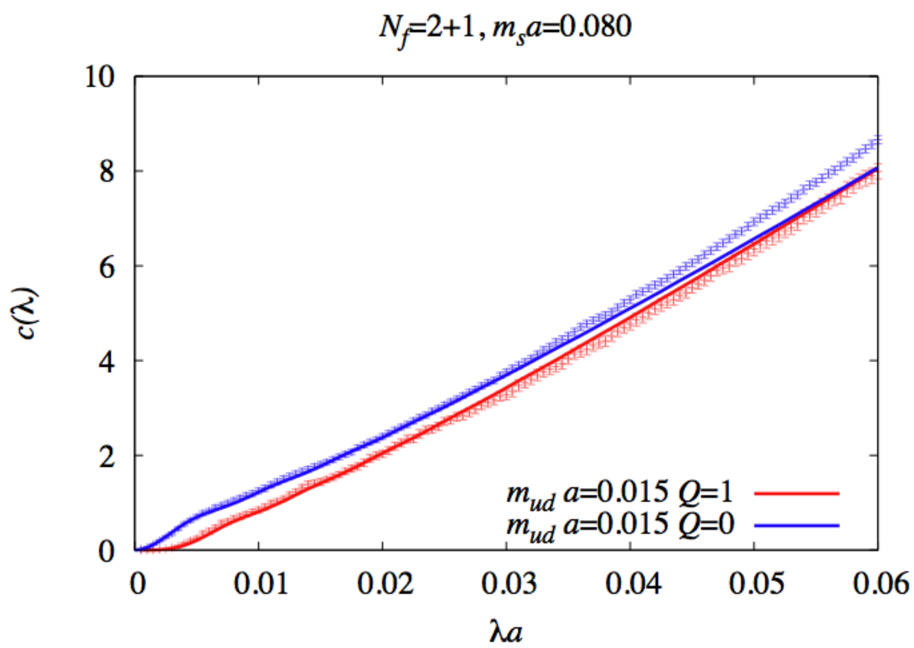


4. Application to lattice QCD analysis

[JLQCD, PRL 104,122002 (2010)]

topology dependence





4. Application to lattice QCD analysis

[JLQCD, PRL 104,122002 (2010)]

Extraction of low-energy constants

Using the NLO formula, we determine 3 free parameters:

$$\Sigma = [234(04)(17)\ \mathrm{MeV}\ ]^3, \quad \text{(MS-bar scheme at 2GeV)}$$

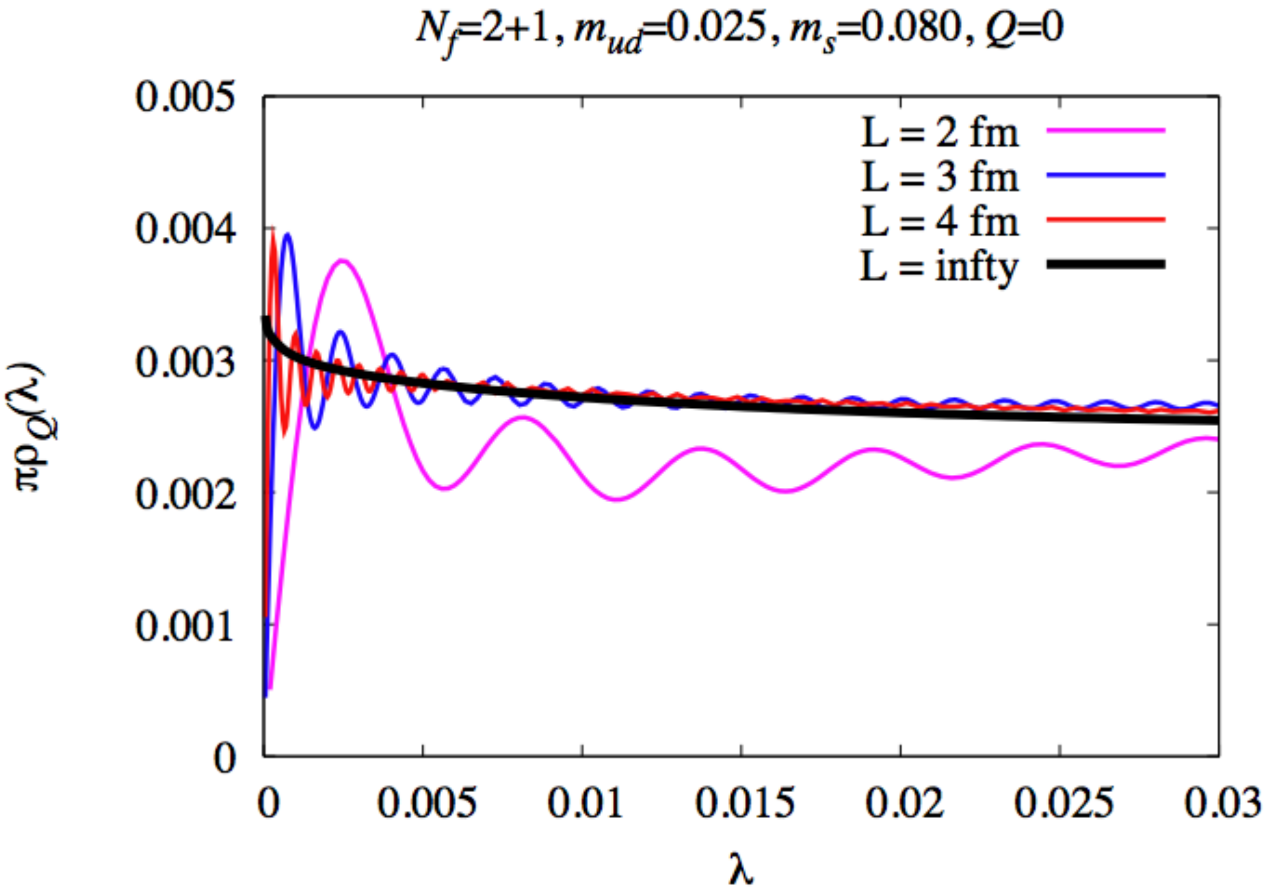
$$F = 71(3)(8)\ \mathrm{MeV}\ ,$$

$$L_6^r = 0.00003(07)(17) \quad \text{(at 770 MeV)}$$

4. Application to lattice QCD analysis

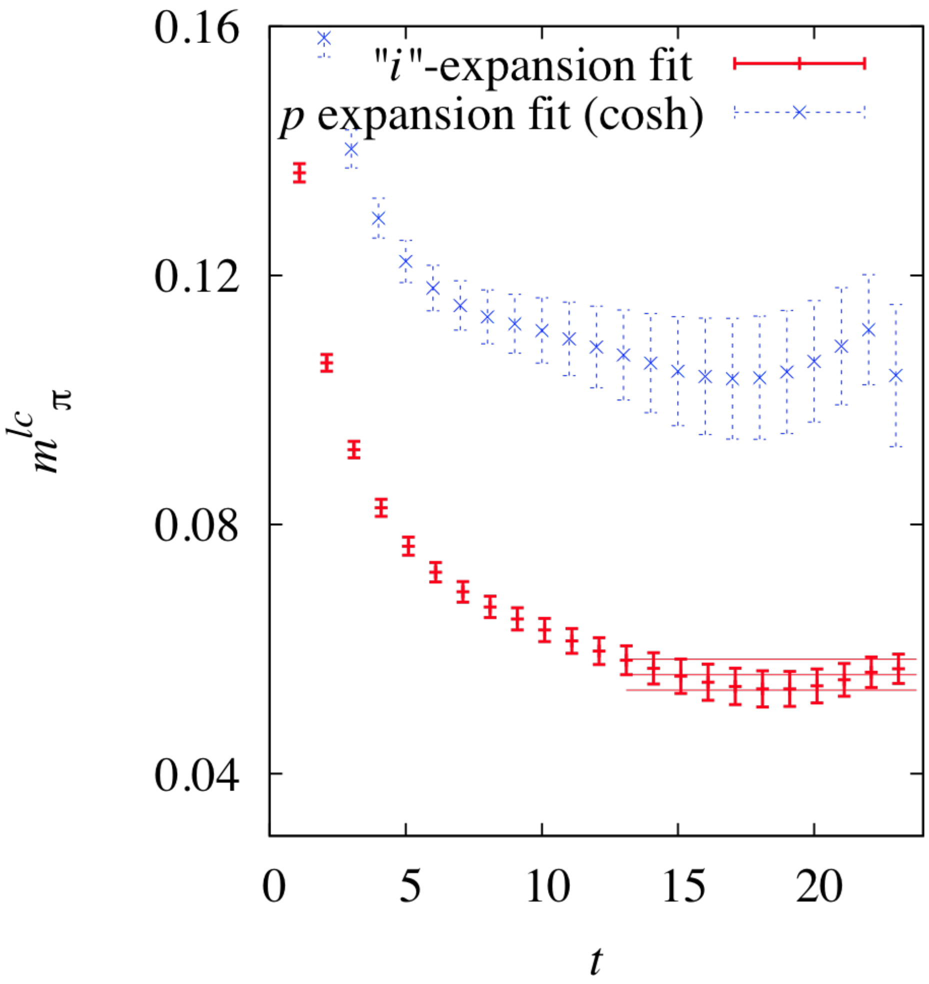
[JLQCD, PRL 104,122002 (2010)]

From finite V to infinite V



4. Application to lattice QCD analysis

Pion 2pt function



Fitting with

$$C_{PP}\frac{\cosh(M_\pi^{NLO}(t-T/2))}{\sinh(M_\pi^{NLO}T/2)}+D_{PP}$$

$$[\Sigma_{\rm eff}=0.002041(70)\ \ \text{input}]$$

after 1-loop (of non-zero modes) volume correction, we obtain

$$M_\pi^{V=\infty}=97.6(4.2)\ \ \text{MeV},$$

$$F_\pi^{V=\infty}=128.6(5.6)\ \ \text{MeV}.$$

$$(1/a=1.759\ \text{GeV})$$

4. Application to lattice QCD analysis

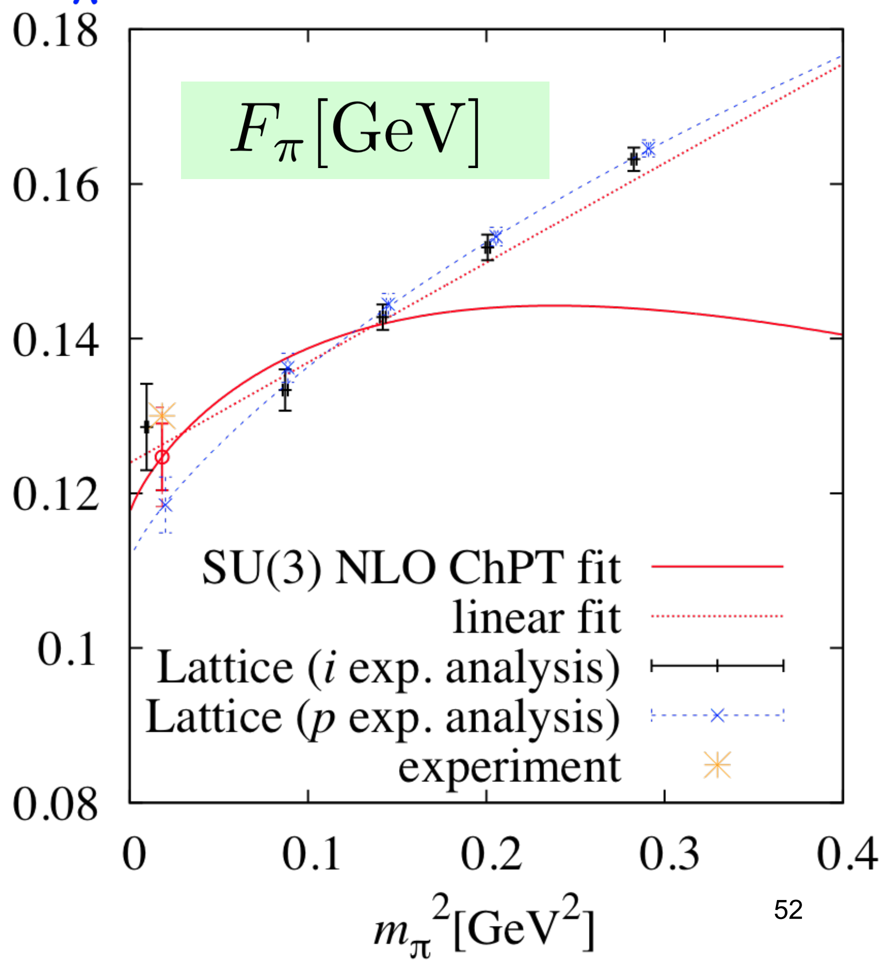
Chiral “interpolation” for F_π

$$F_\pi = 125(4)(^{+5}_{-0})\ \ \mathrm{MeV}$$

bigger than our previous analysis using p-regime data only :

$$F_\pi = 119(4)\ \ \mathrm{MeV}$$

Linear fit looks better than NLO ChPT fit, though…



5. Summary

Low energy QCD in finite V

= Martix model(s)

+ ~10-20% correction from field theory,

which is useful for lattice QCD analysis.

QCD

Chiral symmetry breaking

Random Dirac eigenvalues

U(Nf) matrix model
(pion's moduli)

=

n=∞ random matrix model
(Gaussian unitary ensemble)

Banks-Casher relation

5. Summary

$$\int d^3x \langle P(x)P(0)\rangle = C_{PP}\frac{\cosh(M_\pi^{NLO}(t-T/2))}{\sinh(M_\pi^{NLO}T/2)} + D_{PP}$$