2d-4d Connection through 0d Matrices

basically a review of a series of works done

with Takeshi Oota during 2009 ~ 2010.

IO5 arXiv:1003.2929 = Nucl. Phys. B

(also with Maruyoshi, T.O. and Yonezawa)

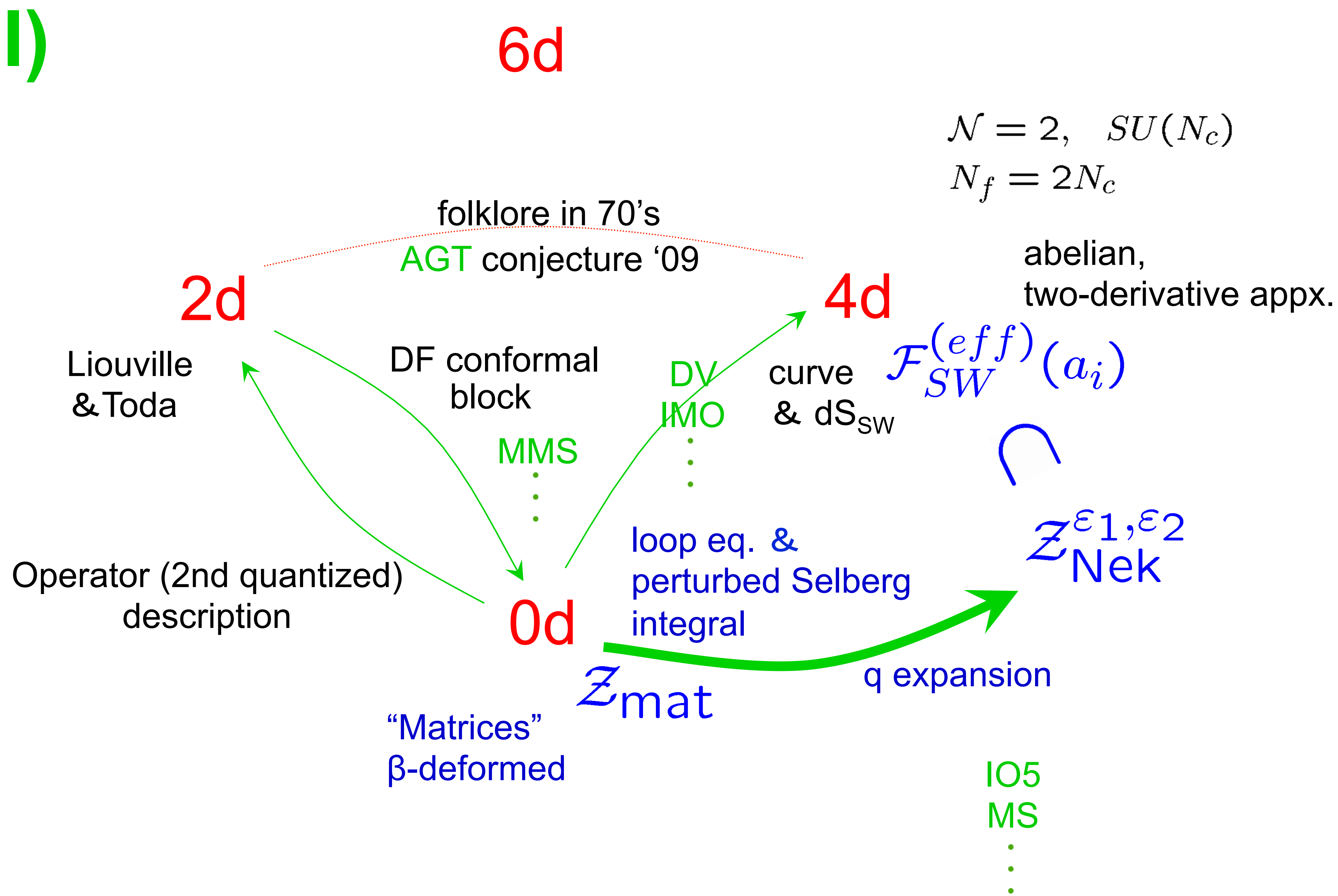
IMO 0911.4244 = PTP, IOY 1008.1861 = Phys. Rev. D

cf

recent work
with Yonezawa, arXiv:1104.2738 = IJMP on half-genus expansion with (ϵ_1, ϵ_2)
with Oota, arXiv:1106.1539 = N. P. on affine quiver matrix model

- punch lines : 2d – 4d connection,
  0d matrices acting as a bridge

The Jack polynomial and the finite N loop eq.
facilitate the computation with ϵ_i, g_s finite


I)
6d
$\mathcal{N}=2, \quad SU(N_c)$
$N_f = 2N_c$
folklore in 70's
AGT conjecture '09
2d
4d
abelian,
two-derivative appx.
Liouville
& Toda
DF conformal
block
DV
IMO
curve
& dS_{SW}
$\mathcal{F}_{SW}^{(eff)}(a_i)$
$\cap$
MMS
$\mathcal{Z}_{\text{Nek}}^{\varepsilon_1,\varepsilon_2}$
loop eq. &
perturbed Selberg
integral
Operator (2nd quantized)
description
0d
$\mathcal{Z}_{\text{mat}}$
q expansion
"Matrices"
β-deformed
IO5
MS

II) What is $\mathcal{Z}_{\text{Nek}}^{\varepsilon_1,\varepsilon_2}$?

$\mathcal{F}_{SW}^{(\text{eff})}(a_i)$; LEEA of $\mathcal{N}=2$ SU(N) SUSY gauge theory

$a_i = (\phi_i)$; undetermined VEV called Coulomb moduli

$$q_{\text{bare}} = e^{\pi i \tau_{\text{bare}}}, \qquad \tau_{\text{bare}} = \frac{\theta}{\pi} + \frac{8\pi i}{g_{\text{bare}}^2}$$

"bare" omitted

$$\mathcal{F}_{SW}^{(\text{eff})} = \mathcal{F}_{\text{1-loop}}^{(SW)} + \mathcal{F}_{\text{inst}}^{(SW)}$$

$\mathcal{F}_{\text{instanton}}$; instanton contributions

- Result of Nekrasov;

$\mathcal{F}_{\text{inst}}^{(SW)}$ is microscopically calculable in the presence of Ω background

$\approx$ deformation parameter $\epsilon_1, \ \epsilon_2$

as

$$Z_{\text{Nek}}(\epsilon_1,\epsilon_2,a_i;q) = \exp\left(\frac{1}{\epsilon_1\epsilon_2}\mathcal{F}_{\text{inst}}(\epsilon_1,\epsilon_2,a_i)\right), \quad \mathcal{F}_{\text{inst}}(0,0,a_i) = \mathcal{F}_{\text{inst}}^{(SW)}$$

cf. $\exists$ higher orders in $g_s^2 = -\epsilon_1\epsilon_2$

where

$$Z_{\text{Nek}}(a_i, \epsilon_1, \epsilon_2, ; q) \equiv \sum_{k=0}^{\infty} q^k \underbrace{\int_{\widetilde{M}_k} \mathbf{1}_{\epsilon_1, \epsilon_2, a_i}}_{\begin{array}{c}|||\\ Z_k\end{array}}$$

|| ?

$$Z_{\mathcal{N}=2 \text{ SUSY}}$$

computable (in mathematics)
by the localization technique

H. Nakajima

where ϵ_1, ϵ_2 acting as Gaussian cutoffs

Its form

$$Z_k = \sum_{|Y^{(1)}|+\cdots+|Y^{(N)}|=k} Z_{Y^{(1)},\cdots,Y^{(N)}}$$

$Y^{(i)}$: partition

- The SU(2), $N_f = 2N = 4$ case, mass m_i, $i = 1 \sim 4$

Some explicit form for $N_f = 4$

$$Z_1 = Z_{(1),(0)} + Z_{(0),(1)} \qquad Z_{(1),(0)} = -\frac{1}{\epsilon_1\epsilon_2}\frac{\prod_{r=1}^4 (a+m_r)}{2a(2a+\epsilon)},$$

$$Z_{(0),(1)} = -\frac{1}{\epsilon_1\epsilon_2}\frac{\prod_{r=1}^4 (a-m_r)}{2a(2a-\epsilon)},$$

$$Z_2 = Z_{(2),(0)} + Z_{(11),(0)} + Z_{(1),(1)} + Z_{(0),(11)} + Z_{(0),(2)}$$

$$Z_{(2),(0)} = \frac{1}{2!\epsilon_1\epsilon_2^2(\epsilon_1-\epsilon_2)}\frac{\prod_{r=1}^4 (a+m_r)(a+m_r+\epsilon_2)}{2a(2a+\epsilon_2)(2a+\epsilon+\epsilon_2)},$$

$$Z_{(0),(2)} = \frac{1}{2!\epsilon_1\epsilon_2^2(\epsilon_1-\epsilon_2)}\frac{\prod_{r=1}^4 (a-m_r)(a-m_r-\epsilon_2)}{2a(2a-\epsilon_2)(2a-\epsilon)(2a-\epsilon-\epsilon_2)},$$

$$Z_{(11),(0)} = -\frac{1}{2!\epsilon_1^2\epsilon_2(\epsilon_1-\epsilon_2)}\frac{\prod_{r=1}^4 (a+m_r)(a+m_r+\epsilon_1)}{2a(2a+\epsilon_1)(2a+\epsilon)(2a+\epsilon+\epsilon_1)},$$

$$Z_{(0),(11)} = -\frac{1}{2!\epsilon_1^2\epsilon_2(\epsilon_1-\epsilon_2)}\frac{\prod_{r=1}^4 (a-m_r)(a-m_r+\epsilon_1)}{2a(2a-\epsilon_1)(2a-\epsilon)(2a-\epsilon-\epsilon_1)},$$

$$Z_{(1),(1)} = \frac{1}{\epsilon_1^2\epsilon_2^2}\frac{\prod_{r=1}^4 (a+m_r)(a-m_r)}{(4a^2-\epsilon_1^2)(4a^2-\epsilon_2^2)}$$

- **AGT conjecture**

$$\mathcal{F}_{\text{conf}} = \mathcal{Z}_{\text{Nek}}^{\varepsilon_1,\varepsilon_2}$$

$\mathcal{F}_{\text{conf}}$: generic Virasoro conformal block

• Generic Virasoro conformal block

BPZ

- 4-point conformal block $\mathcal{F}$

  Four-point function for primary operators $\Phi_{\Delta_i}(z_i, \bar{z}_i)$ in 2d CFT with the central charge c:

$$\langle \Phi_{\Delta_1}(\infty,\infty)\Phi_{\Delta_2}(1,1)\Phi_{\Delta_3}(q,\bar{q})\Phi_{\Delta_4}(0,0)\rangle = \sum_I C^{\Delta_I}_{\Delta_1\Delta_2} K_{\Delta_I} C^{\Delta_I}_{\Delta_3\Delta_4} \left| \mathcal{F}(q|c;\Delta_1,\Delta_2,\Delta_3,\Delta_4,\Delta_I) \right|^2$$

In more detail, the OPE: $V_{\hat{\Delta}_1}(z)V_{\hat{\Delta}_2}(z') = \sum_{\hat{\Delta}} \frac{\mathcal{C}^{\Delta}_{\hat{\Delta}_1\hat{\Delta}_2} V_{\hat{\Delta}}(z')}{(z-z')^{\hat{\Delta}_1+\hat{\Delta}_2-\Delta}}$

the scalar product; $\langle V_{\hat{\Delta}}(0)V_{\hat{\Delta}'}(\infty)\rangle = K_\Delta \ \delta_{\Delta,\Delta'}\delta_{|Y|,|Y'|} \ Q_\Delta(Y,Y'),$

$\mathcal{C}^{\hat{\Delta}}_{\hat{\Delta}_1\hat{\Delta}_2} = C^{\Delta}_{\Delta_1\Delta_2}(\gamma Q^{-1})^{\Delta}_{\Delta_1\Delta_2}$; the factorization of structure constant "Wigner-Eckert"

$$\mathcal{F}(q|c;\Delta_1,\Delta_2,\Delta_3,\Delta_4,\Delta_I) = \sum_{|Y|=|Y'|} q^{|Y|}\gamma_{\Delta_I,\Delta_1,\Delta_2}(Y) Q^{-1}_{\Delta_I}(Y,Y')\gamma_{\Delta_I,\Delta_3,\Delta_4}(Y')$$

The Shapovalov form $Q_\Delta(Y,Y') = \langle \Delta | L_Y L_{-Y'} | \Delta \rangle$

$$L_Y = L_{k_1}L_{k_2}\cdots L_{k_\ell}, \quad \text{for} \quad Y = (k_1,k_2,\ldots,k_\ell),$$

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}$$

The three point function $\gamma_{\Delta,\Delta_1,\Delta_2}(Y) = \prod_{i=1}^{\ell}\left(\Delta + k_i\Delta_1 - \Delta_2 + \sum_{j<i} k_j\right)$

Note that $\mathcal{F}$ is model independent (representation theoretic) quantity.

• Integral representation for $\mathcal{F}_{\text{conf}}$ which exploits free fields

$$\langle \phi(z)\phi(w)\rangle = 2\log(z-w), \qquad T = \frac{1}{4} : (\partial\phi)^2 : + \frac{Q_E}{2}\partial^2\phi$$

$$c = 1 - 6Q_E^2, \qquad Q_E = b_E - \frac{1}{b_E}$$

$$\text{V.O.} \ : \quad e^{\frac{1}{2}\alpha(q)} \ \ \text{has} \ \ \Delta_\alpha = \frac{1}{4}\alpha(\alpha - 2Q_E)$$

Begin with $\left\langle \prod_{I=1}^{4} : e^{\frac{1}{2}\alpha_I(q_I)} : \right\rangle$

- should get rid of momentum conservation which follows from the free field OPE.
- The presence of integrated dim 1 V.O. is harmless for building $\mathcal{F}$: screening op.

$\Longrightarrow$ insert these btn $[q_1, q_2]$ and btn $[q_3, q_4]$

Contents:

III) ·β-deformed one-matrix model (β-ensemble):

$$Z = \int \mathrm{d}^N\lambda \left(\Delta(\lambda)\right)^{+2b_E^2} \exp\left(\frac{b_E}{g_s}\sum_{I=1}^{N} W(\lambda_I)\right) \qquad \Delta(\lambda) = \prod_{1\le I<J\le N}(\lambda_I - \lambda_J)$$

generic for a while

· Vir. constraints; insert $\sum_{I=1}^{N}\frac{\partial}{\partial\lambda_I}\frac{1}{z-\lambda_I}$ into ↑

David
H.I. & Y. Matsuo
⋮

identify as $T(z)\Big|_{+}$, $i.e.$ $\langle\!\langle T(z)\Big|_{+}\rangle\!\rangle = 0$

· Introduce $J(z) = \mathrm{i}\partial\phi(z) = \frac{1}{\sqrt{2}g_s}W'(z) + \sqrt{2}b_E \,\mathrm{Tr}\,\frac{1}{z-M}$

$$T(z) = -\frac{1}{2} : \partial\phi(z)^2 : + \frac{iQ_E}{\sqrt{2}}\partial^2\phi(z), \qquad Q_E = b_E - \frac{1}{b_E}$$

and write as $\langle\!\langle g_s^2 T(z)\rangle\!\rangle = \frac{1}{4}W'(z)^2 - \frac{Q_E}{2}g_s W''(z) - f(z)$

where $f(z) \equiv \left\langle\!\!\left\langle b_E g_s \sum_{I=1}^{N}\frac{W'(z)-W'(\lambda_I)}{z-\lambda_I}\right\rangle\!\!\right\rangle$

· Separately define the curve $(x,z) = (y(z), z)$ by

$$\left\langle\!\!\left\langle \left(x + \frac{\mathrm{i}g_s}{\sqrt{2}}\partial\phi(z)\right)\left(x - \frac{\mathrm{i}g_s}{\sqrt{2}}\partial\phi(z)\right)\right\rangle\!\!\right\rangle = x^2 - g_s^2\langle\!\langle T(z)\rangle\!\rangle = 0$$

$$[x, z] = Q_E g_s$$

• A_{n-1} quiver matrix model:

$r = n-1$

ITEP 1991
IMO 0911.4244
=PTP

constructed s.t. obeying W_n constraints at finite N_a

$$Z \equiv \int \prod_{a=1}^{r} \left\{ \prod_{I=1}^{N_a} \mathrm{d}\lambda_I^{(a)} \right\} \left(\Delta_{A_{n-1}}(\lambda) \right)^{b_E^2} \exp \left(\frac{b_E}{g_s} \sum_{a=1}^{r} \sum_{I=1}^{N_a} W_a(\lambda_I^{(a)}) \right)$$

$$\Delta_{A_{n-1}}(\lambda) = \prod_{a=1}^{r} \prod_{1 \le I < J \le N_a} (\lambda_I^{(a)} - \lambda_J^{(a)})^2 \prod_{1 \le a < b \le r} \prod_{I=1}^{N_a} \prod_{J=1}^{N_b} (\lambda_I^{(a)} - \lambda_J^{(b)})^{(\alpha_a, \alpha_b)}$$

- $\exists n$ spin 1 currents s.t. $\sum_{i=1}^{n} J_i(z) = 0$

$$J_i(z) = \mathrm{i}\partial\varphi_i(z) = \frac{1}{g_s} t_i(z) + b_E \sum_{a=1}^{n-1} (\delta_{i,a} - \delta_{i,a+1}) \mathrm{Tr} \frac{1}{z - M_a}$$

$$t_i(z) = \sum_{a=i}^{n-1} W_a'(z) - \frac{1}{n} \sum_{a=1}^{n-1} a\, W_a'(z)$$

- $: \det(x - \mathrm{i} g_s \partial\phi(z)) :=: \overleftarrow{\prod_{1 \le i < n}} (x - g_s J_i(z)) :$ contains W_n generators

- W_n constraints $\left\langle\!\!\left\langle \det(x - \mathrm{i} g_s \partial\phi(z)) \Big|_+ \right\rangle\!\!\right\rangle = 0$

- the curve Σ $\left\langle\!\!\left\langle \det(x - \mathrm{i} g_s \partial\phi(z)) \right\rangle\!\!\right\rangle = 0$

Isomorphism with the Witten-Gaiotto curve established in the planar limit this way.

· **The planar limit:** the singlet factorization and the curve factorizes as

$$0 = \underset{\substack{\wr \\ \lim g_s J_i}}{\prod_{i=1}^{n}} (x - y_i(z)) \qquad (x, z) = (y_i(z), z)$$

· **3-Penner:** choose $W_a(z) = \sum_{p=1}^{3} (\mu_p, \alpha_a) \log(q_p - z)$

$q_0 = \infty,\ q_1 = 0,\ q_2 = 1,\ q_3 = q$

IMO

The curve is found to agree with the $(n, N_f = 2n)$ curve in Witten-Gaiotto form.

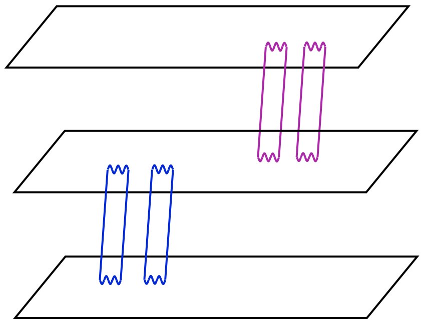
$\rightarrow$ identification of μ_p with m_i in $n = 3$ case given

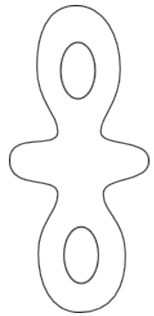
· Original reasoning leading to the computability

$n = 3,\ N_f = 2n = 6,\ n-1 = 2$ kinds of e.v. distributions IO5

$q \neq 0,\ \ g = 2$

3 Penner

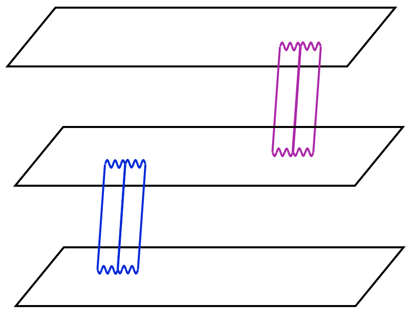


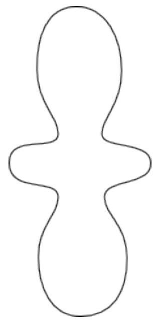


$q \to 0 \ \Downarrow$ $\Downarrow\ q \to 0$

$q = 0,\ \ g = 0$

2 Penner





- This has provided us an important insight
- Need only to take derivatives at $q = 0$

III)

The Detsenko-Fateev multiple integral is an integral representation of the arbitrary 4-point conformal block. $\mathcal{F}(q|c;\Delta_1,\Delta_2,\Delta_3,\Delta_4,\Delta_I)$

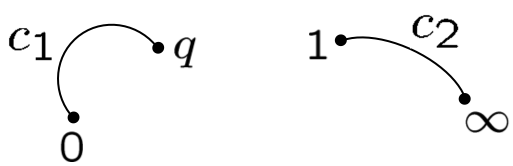
$$c=1-6Q_E^2,\ \ \Delta_i=\frac{1}{4}\alpha_i(\alpha_i-2Q_E),\ \ \Delta_I=\frac{1}{4}\alpha_I(\alpha_I-Q_E)$$

We regard this as a version of β-deformed one-matrix model with special attention to the integration domain. Actually, it is a "perturbed double-Selberg matrix model":

$$\begin{aligned} Z_{\text{pert-(Selberg)}^2}(q\,|\,b_E;N_L,\alpha_1,\alpha_2;N_R,\alpha_4,\alpha_3) &= q^{\sigma}(1-q)^{(1/2)\alpha_2\alpha_3} \\ &\times\left(\prod_{I=1}^{N_L}\int_0^1 \mathrm{d}x_I\right)\prod_{I=1}^{N_L} x_I^{b_E\alpha_1}(1-x_I)^{b_E\alpha_2}(1-qx_I)^{b_E\alpha_3}\prod_{1\le I<J\le N_L}|x_I-x_J|^{2b_E^2} \\ &\times\left(\prod_{J=1}^{N_R}\int_0^1 \mathrm{d}y_J\right)\prod_{J=1}^{N_R} y_J^{b_E\alpha_4}(1-y_J)^{b_E\alpha_3}(1-q\,y_J)^{b_E\alpha_2}\prod_{1\le I<J\le N_R}|y_I-y_J|^{2b_E^2} \\ &\times\prod_{I=1}^{N_L}\prod_{J=1}^{N_R}(1-q\,x_I y_J)^{2b_E^2} \end{aligned}$$

under $\alpha_1+\alpha_2+\alpha_3+\alpha_4+2(N_L+N_R)b_E=2Q_E$

- originally



· Z to continue

$$Z_{\text{pert}-(\text{Selberg})^2}(q \,|\, b_E; N_L, \alpha_1, \alpha_2; N_R, \alpha_4, \alpha_3)$$
$$= q^{\Delta_I - \Delta_1 - \Delta_2}\, \mathcal{B}_0(b_E; N_L, \alpha_1, \alpha_2; N_R, \alpha_4, \alpha_3)\, \mathcal{B}(q \,|\, b_E; N_L, \alpha_1, \alpha_2; N_R, \alpha_4, \alpha_3)$$

$$\mathcal{B}_0(b_E; N_L, \alpha_1, \alpha_2; N_R, \alpha_4, \alpha_3)$$
$$= S_{N_L}(1 + b_E\alpha_1, 1 + b_E\alpha_2, b_E^2)\, S_{N_R}(1 + b_E\alpha_4, 1 + b_E\alpha_3, b_E^2)$$

$$\mathcal{B}(q \,|\, b_E; N_L, \alpha_1, \alpha_2; N_R, \alpha_4, \alpha_3)$$
$$= (1-q)^{(1/2)\alpha_2\alpha_3} \left\langle\!\!\left\langle \prod_{I=1}^{N_L} (1 - qx_I)^{b_E\alpha_3} \prod_{J=1}^{N_R} (1 - qy_J)^{b_E\alpha_2} \prod_{I=1}^{N_L} \prod_{J=1}^{N_R} (1 - qx_I y_J)^{2b_E^2} \right\rangle\!\!\right\rangle_{N_L, N_R}$$

Here S_{N_L}, S_{N_R} are the celebrated Selberg integral (an extension of Beta fn!!)

$$S_N(\beta_1, \beta_2, \gamma) = \left(\prod_{I=1}^{N} \int_0^1 \mathrm{d}x_I \right) \prod_{I=1}^{N} x_I^{\beta_1 - 1}(1 - x_I)^{\beta_2 - 1} \prod_{1 \le I < J \le N} |x_I - x_J|^{2\gamma}$$
$$= \prod_{j=1}^{N} \frac{\Gamma(1 + j\gamma)\Gamma(\beta_1 + (j-1)\gamma)\Gamma(\beta_2 + (j-1)\gamma)}{\Gamma(1+\gamma)\Gamma(\beta_1 + \beta_2 + (N + j - 2)\gamma)}$$

and averaging is w.r.t. these.

· Two kinds of generating functions :

IO5

$$\mathcal{B}(q) = 1 + \sum_{\ell=1}^{\infty} q^{\ell}\,\mathcal{B}_{\ell}$$
$$= \left\langle\!\!\left\langle \exp\left[-2\sum_{k=1}^{\infty}\frac{q^k}{k}\left(b_E\sum_{I=1}^{N_L} x_I^k + \frac{1}{2}\alpha_2\right)\left(b_E\sum_{J=1}^{N_L} y_J^k + \frac{1}{2}\alpha_3\right)\right]\right\rangle\!\!\right\rangle_{N_L,N_R}$$
$$= (1-q)^{(1/2)\alpha_2\alpha_3}\mathcal{A}(q)$$

$$\mathcal{A}(q) = 1 + \sum_{\ell=1}^{\infty} q^{\ell}\,\mathcal{A}_{\ell}$$
$$= \left\langle\!\!\left\langle \exp\left[-\sum_{k=1}^{\infty}\frac{q^k}{k}\left(\alpha_2 + b_E\sum_{I=1}^{N_L} x_I^k\right)\left(b_E\sum_{J=1}^{N_R} y_J^k\right) - \sum_{k=1}^{\infty}\frac{q^k}{k}\left(b_E\sum_{I=1}^{N_L} x_I^k\right)\left(\alpha_3 + b_E\sum_{J=1}^{N_R} y_J^k\right)\right]\right\rangle\!\!\right\rangle_{N_L,N_R}$$

takes the form

$$= \sum_{k=0}^{\infty} q^k \sum_{|Y_1|+|Y_2|=k} \mathcal{A}_{Y_1,Y_2}$$

a pair of partitions (Y_1, Y_2) naturally appears.

· The rest of the plan :

i) some exact cal from special fn

ii) some by solving finite N loop eq.

i)

Jack polynomial $P_\lambda^{(1/\gamma)}(x) \qquad x=(x_1,\cdots,x_N)$

$\lambda=(\lambda_1,\lambda_2,\cdots)$ is a partition $\lambda_1\geq\lambda_2\geq\cdots\geq 0$

$$\left\langle\!\!\left\langle P_\lambda^{(1/b_E^2)}(x)\right\rangle\!\!\right\rangle_{N_L}=\prod_{i\geq 1}\frac{\left(1+b_E\alpha_1+b_E^2(N-i)\right)_{\lambda_i}\left(b_E^2(N_L+1-i)\right)_{\lambda_i}}{\left(2+b_E(\alpha_1+\alpha_2)+b_E^2(2N_L-1-i)\right)_{\lambda_i}}$$

$$\times\prod_{(i,j)\in\lambda}\frac{1}{(\lambda_i-j+b_E^2(\lambda'_j-i+1))}$$

conj by Mcdonald '87

proven by Kadell '97

where λ' ; the conjugate partition of λ

$(a)_n=a(a+1)\cdots(a+n-1)$; Pochhammer symbol

- From an explicit form of Jack poly. $|\lambda|\leq 2$
  we obtain

$$\left\langle\!\!\left\langle b_E\sum_{I=1}^{N_L}x_I\right\rangle\!\!\right\rangle_{N_L}=\frac{b_EN_L(b_EN_L-Q_E+\alpha_1)}{(\alpha_I-2Q_E)}$$

IO5

$$2\left\langle\!\!\left\langle b_E^2\sum_{1\leq I<J\leq N_L}x_Ix_J\right\rangle\!\!\right\rangle_{N_L}=\frac{b_EN_L(b_EN_L-b_E)(\alpha_1+b_EN_L-Q_E)(\alpha_1+b_EN_L-Q_E-b_E)}{(\alpha_I-2Q_E)(\alpha_I-2Q_E-b_E)}$$

$$\left\langle\!\!\left\langle b_E\sum_{I=1}^{N_L}x_I(1-x_I)\right\rangle\!\!\right\rangle_{N_L}=\frac{b_EN_L(\alpha_1+b_EN_L-Q_E)(\alpha_2+b_EN_L-Q_E)(\alpha_1+\alpha_2+b_EN_L-2Q_E)}{(\alpha_I-2Q_E)(\alpha_I-3Q_E+b_E)(\alpha_I-2Q_E-b_E)}$$

• More on Jack polynomial

• Jack polynomials are the eigenstates of

$$\sum_{I=1}^{N}\left(x_I\frac{\partial}{\partial x_I}\right)^2+\gamma\sum_{1\le I<J\le N}\left(\frac{x_I+x_J}{x_I-x_J}\right)\left(x_I\frac{\partial}{\partial x_I}-x_J\frac{\partial}{\partial x_J}\right)$$

with homogeneous degree $|\lambda|=\lambda_1+\lambda_2+\cdots$

• normalization

$$P_\lambda^{(1/\gamma)}(x)=m_\lambda(x)+\sum_{\mu<\lambda}a_{\lambda\mu}m_\mu(x)$$

Here $m_\lambda(x)$ is the monomial symmetric polynomial.

$``<"$ is dominance ordering.

Explicit forms of the Jack polynomials for $|\lambda| \leq 2$

$$P_{(1)}^{(1/\gamma)}(x) = m_{(1)}(x) = \sum_{I=1}^{N} x_I,$$

$$P_{(2)}^{(1/\gamma)}(x) = m_{(2)}(x) + \frac{2\gamma}{1+\gamma} m_{(1^2)}(x) = \sum_{I=1}^{N} x_I^2 + \frac{2\gamma}{1+\gamma} \sum_{1 \leq I < J \leq N} x_I x_J,$$

$$P_{(1^2)}^{(1/\gamma)}(x) = m_{(1^2)}(x) = \sum_{1 \leq I < J \leq N} x_I x_J.$$

ii)

Back to the model (perturbed double-Selberg $\approx$ 3 Penner).

Recall, at $q = 0$, a pair of decoupled Selbergs $\approx$ 2 Penner's .

Build the original model $(q \neq 0)$ through resolvent.

$$Z_{\text{Selberg}}(b_E; N_L, \alpha_1, \alpha_2) = \left(\prod_{I=1}^{N_L} \int_0^1 \mathrm{d}x_I\right) \prod_{1 \le I < J \le N_L} |x_I - x_J|^{2b_E^2} \exp\left(b_E \sum_{I=1}^{N_L} \widetilde{W}(x_I)\right)$$

$$\tilde{W}(x) = \alpha_1 \log x + \alpha_2 \log(1-x)$$

- The loop eq. at finite N

$$\left\langle\!\left\langle \left(\widehat{w}_{N_L}(z)\right)^2 \right\rangle\!\right\rangle_{N_L} + \left(\widetilde{W}'(z) + Q_E \frac{\mathrm{d}}{\mathrm{d}z}\right) \left\langle\!\left\langle \widehat{w}_{N_L}(z) \right\rangle\!\right\rangle_{N_L} - \tilde{f}_{N_L}(z) = 0$$

$$\widehat{w}_{N_L}(z) := b_E \sum_{I=1}^{N_L} \frac{1}{z - x_I}, \quad \tilde{f}_{N_L}(z) := \left\langle\!\left\langle b_E \sum_{I=1}^{N_L} \frac{\widetilde{W}'(z) - \widetilde{W}'(x_I)}{z - x_I} \right\rangle\!\right\rangle_{N_L}$$

$$\widetilde{w}_{N_L}(z) := \left\langle\!\left\langle \widehat{w}_{N_L}(z) \right\rangle\!\right\rangle_{N_L} = \left\langle\!\left\langle b_E \sum_{I=1}^{N_L} \frac{1}{z - x_I} \right\rangle\!\right\rangle_{N_L}$$

• By looking at $O\left(\frac{1}{z}\right), O\left(\frac{1}{z^2}\right), O\left(\frac{1}{z^3}\right)$, IO5

we obtain exact results

$$\left\langle\!\left\langle b_E\, p_{(1)}(\mu) \right\rangle\!\right\rangle_{N_L} = \left\langle\!\!\left\langle b_E \sum_{I=1}^{N_L} x_I \right\rangle\!\!\right\rangle_{N_L} = \frac{b_E N_L (b_E N_L - Q_E + \alpha_1)}{(\alpha_1 + \alpha_2 + 2b_E N_L - 2Q_E)},$$

$$\tilde{f}_{N_L}(z) = -\frac{b_E N_L(\alpha_1 + \alpha_2 + b_E N_L - Q_E)}{z(z-1)},$$

$$-\tilde{w}_{N_L}(0) = \left\langle\!\!\left\langle b_E \sum_{I=1}^{N_L} \frac{1}{x_I} \right\rangle\!\!\right\rangle_{N_L} = \frac{b_E N_L(\alpha_1 + \alpha_2 + b_E N_L - Q_E)}{\alpha_1},$$

$$\tilde{w}_{N_L}(1) = \left\langle\!\!\left\langle b_E \sum_{I=1}^{N_L} \frac{1}{1 - x_I} \right\rangle\!\!\right\rangle_{N_L} = \frac{b_E N_L(\alpha_1 + \alpha_2 + b_E N_L - Q_E)}{\alpha_2}$$

The first one agrees with that from i).

• 0d – 4d relation

IO5

- matrix side parameters:

  seven parameters with one constraint

  $b_E,\ N_L,\ \alpha_1,\ \alpha_2,\ N_R,\ \alpha_4,\ \alpha_3$

- $\mathcal{N}=2,\ SU(2),\ N_f=4$, six parameters

  $\frac{\epsilon_1}{g_s},\ \frac{a}{g_s},\ \frac{m_1}{g_s},\ \frac{m_2}{g_s},\ \frac{m_3}{g_s},\ \frac{m_4}{g_s}$

- By looking at $\mathcal{B}_1=\mathcal{A}_1-\frac{1}{2}\alpha_2\alpha_3$, and explicit form of $\mathcal{A}_1^{\text{Nek}}=\mathcal{A}_{[1],[0]}^{\text{Nek}}+\mathcal{A}_{[0],[1]}^{\text{Nek}}$,

$$\mathcal{A}_{[1],[0]}^{\text{Nek}}=\frac{(a+m_1)(a+m_2)(a+m_3)(a+m_4)}{2a(2a+\epsilon)g_s^2}$$
$$\mathcal{A}_{[0],[1]}^{\text{Nek}}=\frac{(a-m_1)(a-m_2)(a-m_3)(a-m_4)}{2a(2a-\epsilon)g_s^2}$$

we get

$$b_EN_L=\frac{a-m_2}{g_s},\qquad b_EN_R=-\frac{a+m_3}{g_s},$$
$$\alpha_1=\frac{1}{g_s}(m_2-m_1+\epsilon),\qquad \alpha_2=\frac{1}{g_s}(m_2+m_1),$$
$$\alpha_3=\frac{1}{g_s}(m_3+m_4),\qquad \alpha_4=\frac{1}{g_s}(m_3-m_4+\epsilon).$$

- Splitting of our $\mathcal{A}_1$ into $\mathcal{A}_{[1],[0]}+\mathcal{A}_{[0],[1]}$ done rather nontrivially, $\mathcal{A}_2$ computed and checked to agree

· More on $\mathcal{A}_2$ (and higher)

IO5

Rearrangements

$$\mathcal{A}_2 = \sum_{|Y_1|+|Y_2|=2} \mathcal{A}_{Y_1,Y_2} = \mathcal{A}_{(2),(0)} + \mathcal{A}_{(1^2),(0)} + \mathcal{A}_{(1),(1)} + \mathcal{A}_{(0),(1^2)} + \mathcal{A}_{(0),(2)}$$

$$\mathcal{A}_{Y_1,Y_2} = \left\langle\!\!\left\langle\; M_{Y_1,Y_2}(x) \;\right\rangle\!\!\right\rangle_{N_L} \left\langle\!\!\left\langle\; \widetilde{M}_{Y_1,Y_2}(y) \;\right\rangle\!\!\right\rangle_{N_R}$$

Unfortunately, contrary to our original expectation, finding M and $\widetilde{M}$ are not straightforward.

$$\widetilde{M}_{(2),(0)}(y) = b_E^2 P_{(2)}^{(1/b_E^2)}(y)$$

$$\widetilde{M}_{(1^2),(0)}(y) = \frac{2b_E^2}{1+b_E^2} P_{(1^2)}^{(1/b_E^2)}(y)$$

The rest is more complicated.

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G. E. Arutyunov (Utrecht)
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T. Eguchi (YITP)
S. Gukov (Caltech)
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