Combinatorics and Matrix Models

Taro Kimura

Department of Basic Science, Univ of Tokyo
and
Mathematical Physics Laboratory, RIKEN

Feb 21, 2012

Workshop on *Matrix Models and Related Topics*
Rikkyo Univ

partially based on [arXiv:1105.6091] [arXiv:1109.0004]

Combinatorics in Physics

Combinatorics in Physics

- Gauge theory
  - $\mathcal{N} = 2, 4$ gauge theory
  - $2d$ Yang-Mills
  - $\cdots$

Combinatorics in Physics

- Gauge theory
  - $\mathcal{N} = 2, 4$ gauge theory
  - $2d$ Yang-Mills
  - $\cdots$
- Integrable models
  - Cellular Automaton (Box-Ball system, ASEP, PNG model, etc)
  - Calogero-Sutherland model (Schur, Jack, Macdonald)
  - CFT
  - $\cdots$

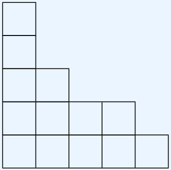
Combinatorics in Physics

- Gauge theory
  - $\mathcal{N} = 2, 4$ gauge theory
  - $2d$ Yang-Mills
  - $\cdots$
- Integrable models
  - Cellular Automaton (Box-Ball system, ASEP, PNG model, etc)
  - Calogero-Sutherland model (Schur, Jack, Macdonald)
  - CFT
  - $\cdots$

Combinatorics in Physics $\sim$ Young diagram


数理科学
1
2007
特集
ヤング図形で遊ぶ物理と数学
イメージ力を鍛える
No.523
サイエンス社

Young diagram



$$\lambda = (5, 4, 2, 1, 1)$$

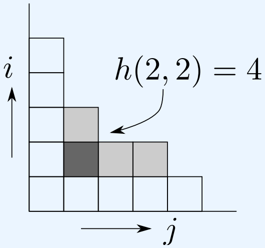
$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$$

- Representation theory
  one-to-one correspondence to irreducible representations

- Plancherel measure & hook length representation

$$\mu_n(\lambda) = \frac{(\dim \lambda)^2}{n!} = n! \prod_{(i,j)\in\lambda} \frac{1}{h(i,j)^2}$$

Hook length



$$h(i,j) = \lambda_i - j + \check{\lambda}_j - i + 1$$

Partition function

$$Z = \sum_{\lambda} \left(\frac{\Lambda}{\hbar} \right)^{2|\lambda|} \prod_{(i,j)\in\lambda} \frac{1}{h(i,j)^2}$$

[Logan-Shepp] [Vershik-Kerov] [Baik-Deift-Johansson] [Okounkov]...

- Higher Casimir model

$$Z = \sum_{\lambda} \prod_{(i,j)\in\lambda} \frac{1}{h(i,j)^2} \prod_{i=1}^{N} e^{-\sum_k g_k C_k(\lambda)}$$

- cf. representation as "norm"

$$Z = \langle \psi | \psi \rangle \,, \qquad |\psi\rangle = \exp\left(\frac{\Lambda}{\hbar} a_{-1} \right) |0\rangle$$

What to do?

What to do?

Asymptotic behavior, Large N analysis, Fluctuation...

through **Matrix Model** description

Contents

Contents

Matrix integral representation

$$
\begin{aligned}
Z_{\mathrm{matrix}} &= \int \mathcal{D}X \, e^{-\frac{1}{g}\mathrm{Tr}\, V(X)} \\
&= \int \prod_{i=1}^{N} dx_i \, \Delta(x)^2 \, e^{-\frac{1}{g}\sum V(x_i)}
\end{aligned}
$$

Vandermonde determinant

$$
\Delta(x)^2 = \prod_{i<j}^{N} (x_i - x_j)^2
$$

Combinatorics to Matrix Model

$$Z_{\mathrm{comb}} = \sum_{\lambda} \left(\frac{\Lambda}{\hbar}\right)^{2|\lambda|} \prod_{(i,j)\in\lambda} \frac{1}{h(i,j)^2}$$

$$\Downarrow$$

$$Z_{\mathrm{matrix}} = \int \prod_{i=1}^{N} dx_i \, \Delta(x)^2 \, e^{-\frac{1}{g}\sum V(x_i)}$$

- "norm" representation

$$\psi(x_1, \cdots, x_N) = \prod_{i<j}^{N} (x_i - x_j) \prod_{i=1}^{N} e^{-\frac{1}{2g} V(x_i)}$$

$$\longrightarrow \qquad Z_{\mathrm{matrix}} = \int \prod_{i=1}^{N} dx_i \, \psi(x_1, \cdots, x_N) \overline{\psi(x_1, \cdots, x_N)}$$

- Weight function

$$
\begin{aligned}
\prod_{(i,j)\in\lambda} \frac{1}{h(i,j)^2} &= \prod_{i<j}^{N} (\lambda_i - \lambda_j - i + j)^2 \prod_{i=1}^{N} \frac{1}{\Gamma(\lambda_i + N - i + 1)^2} \\
&= \prod_{i<j}^{N} (x_i - x_j)^2 \prod_{i=1}^{N} \frac{1}{\Gamma(x_i)^2} \\
&\equiv \Delta^2(x) \exp\left(- \sum_{i=1}^{N} V(x_i) \right), \\
V(x) &= 2 \log \Gamma(x) \sim 2\,(x \log x - x)
\end{aligned}
$$

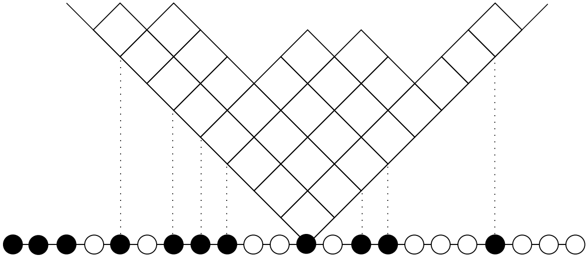
- Weight function

$$\begin{aligned}
\prod_{(i,j)\in\lambda} \frac{1}{h(i,j)^2} &= \prod_{i<j}^{N} (\lambda_i - \lambda_j - i + j)^2 \prod_{i=1}^{N} \frac{1}{\Gamma(\lambda_i + N - i + 1)^2} \\
&= \prod_{i<j}^{N} (x_i - x_j)^2 \prod_{i=1}^{N} \frac{1}{\Gamma(x_i)^2} \\
&\equiv \Delta^2(x) \exp\left(- \sum_{i=1}^{N} V(x_i) \right), \\
V(x) &= 2 \log \Gamma(x) \sim 2 \left(x \log x - x \right)
\end{aligned}$$

- Matrix integral from summation over the partition

$$\sum_{\lambda} = \sum_{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N} = \sum_{x_1 > x_2 > \cdots > x_N} = \frac{1}{N!} \sum_{x_1, \cdots, x_N} \longrightarrow \int \prod_{i=1}^{N} dx_i$$

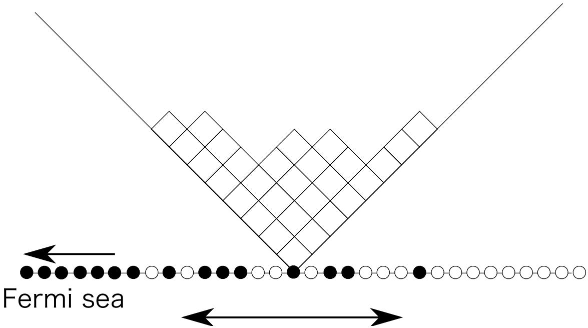
1-dim particle description



- Fermionization: $x_i = \lambda_i + N - i + 1$

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \quad \longrightarrow \quad x_1 > x_2 > \cdots > x_N$$

Young diag $\Longleftrightarrow$ 1-dim particle



Fluctuation around Fermi surface

- Chiral decomposition at large N

Summary

Combinatorics to Matrix Model

$$Z_{\mathrm{comb}} = \sum_{\lambda} \left(\frac{\Lambda}{\hbar} \right)^{2|\lambda|} \prod_{(i,j)\in\lambda} \frac{1}{h(i,j)^2}$$

$$\Downarrow$$

$$Z_{\mathrm{matrix}} = \int \prod_{i=1}^{N} dx_i \, \Delta(x)^2 \, e^{-\frac{1}{g}\sum V(x_i)}$$

- $\mathbb{C}\mathbf{P}^1$ matrix model [Eguchi-Yang]

$$V(x) = 2\,(x \log x - x)$$

Contents

Example 1

β-deformation

$$\mu_n^{(\beta)}(\lambda) = n! \prod_{(i,j)\in\lambda} \frac{1}{h^\beta(i,j)h_\beta(i,j)}$$

- Inhomogeneous hook-length

$$h_\beta(i,j) = \lambda_i - j + \beta(\check{\lambda}_j - i) + 1, \quad h^\beta(i,j) = \lambda_i - j + \beta(\check{\lambda}_j - i) + \beta$$

- cf. Ω-background

$$\beta = -\frac{\epsilon_1}{\epsilon_2} \longrightarrow 1 \quad \text{when} \quad \epsilon_1 = -\epsilon_2 = \hbar$$

- Weight function

$$
\begin{aligned}
&\frac{1}{h_\beta(i,j)h^\beta(i,j)} \\
= \quad &\prod_{i<j}^{N} \frac{\Gamma(\lambda_i - \lambda_j + \beta(j-i) + \beta)\Gamma(\lambda_i - \lambda_j + \beta(j-i) + 1)}{\Gamma(\lambda_i - \lambda_j + \beta(j-i))\Gamma(\lambda_i - \lambda_j + \beta(j-i) + 1 - \beta)} \\
&\times \prod_{i=1}^{N} \frac{\Gamma(\lambda_i + \beta(N-i))\Gamma(\lambda_i + \beta(N-i) + 1 - \beta)}{\Gamma(\lambda_i + \beta(N-i) + \beta)\Gamma(\lambda_i + \beta(N-i) + 1)}
\end{aligned}
$$

Matrix measure

$$
\Delta^2(x) \longrightarrow \prod_{i<j}^{N} (x_i - x_j)^{2\beta}
$$

β-ensemble matrix model

Example 2

q-deformation

$$\mu_{q,t}(\lambda) = [n]_q! \prod_{(i,j)\in\lambda} \frac{(1-q)(1-q^{-1})}{(1-q^{\lambda_i-j+1}t^{\check{\lambda}_j-i})(1-q^{-\lambda_i+j}t^{-\check{\lambda}_j+i-1})}$$

- q-integer

$$[n]_q = \frac{1-q^n}{1-q} \longrightarrow n \quad (q\to 1)$$

- Reduction: $t = q^\beta$, $q \longrightarrow 1 \quad \Longrightarrow \quad \beta$-deformation
  - cf. $5d$ partition function on $\mathbb{R}^4 \times S^1$

Matrix measure

$$\Delta^2(x) \longrightarrow \prod_{i<j}^{N} \left(\frac{2}{R} \sinh \frac{R}{2} (x_i - x_j) \right)^{2\beta}$$

Matrix potential

$$V(x) = -\frac{1}{R} \left[\mathrm{Li}_2 \left(e^{Rx} \right) - \mathrm{Li}_2 \left(e^{-Rx} \right) \right]$$

Example 3

Nekrasov partition function

$$Z = \sum_{\vec{\lambda}} \Lambda^{2n|\vec{\lambda}|} \prod_{(l,i)\neq(m,j)} \frac{a_l - a_m + \lambda_i^{(l)} - \lambda_j^{(m)} + j - i}{a_l - a_m + j - i}$$

[Klemm-Sułkowski] [Sułkowski]

- "external fields": a_l $\quad$ ($\longleftarrow$ Coulomb moduli)
- Matrix measure

$$\Delta^2(x) = \prod_{i<j}^{N} (x_i - x_j)^2$$

- Matrix potential

$$V(x) = 2\sum_{l=1}^{n} \left[(x - a_l) \log\left(\frac{x - a_l}{\Lambda}\right) - (x - a_l) \right]$$

Example 4

Orbifold partition function

$$Z = \sum_{\lambda} \prod_{\Gamma\text{-inv}\subset\lambda} \frac{(\Lambda/\hbar)^2}{h(i,j)^2}$$

Γ-invariant sector: $\quad h(i,j) \equiv 0 \pmod{r}$

- Matrix measure

$$\Delta^2(x) = \prod_{u=1}^{r} \prod_{i<j}^{N} (x_i^{(u)} - x_j^{(u)})^{2(\beta-1)/r+2} \prod_{u<v}^{r} \prod_{i,j}^{N} (x_i^{(u)} - x_j^{(v)})^{2(\beta-1)/r}$$

Multi-matrix model

- Large N analysis [Kimura]
  - cf. gauge theory on $\mathbb{C}^2/\mathbb{Z}_r$

Summary

$$
\begin{array}{lccc}
 & \text{Schur: } (x_i - x_j)^2 & \longrightarrow & \text{Schur w/ ext.} \\
\beta\text{-def} & \downarrow & & \downarrow \\
 & \text{Jack: } (x_i - x_j)^{2\beta} & \longrightarrow & \text{Jack w/ ext.} \\
q\text{-def} & \downarrow & & \downarrow \\
 & \text{Macdonald: } \sinh(x_i - x_j)^{2\beta} & \longrightarrow & \text{Macdonald w/ ext.}
\end{array}
$$

- Orbifolding $\mathbb{C}^2/\mathbb{Z}_r$
- Other possibilities
  - Hall-Littlewood: $q \to 0$
  - Type BC root system
  - Elliptic function: $6d$, $\mathbb{R}^4 \times T^2$

Contents

- What to study in this model?

- What to study in this model?

$$\boxed{\textbf{Asymptotic behavior in large } N}$$

- What to study in this model?

$\boxed{\textbf{Asymptotic behavior in large } N}$

$\boxed{\textbf{Universality}}$

- 't Hooft limit and eigenvalue distribution

$$g \longrightarrow 0, \qquad N \longrightarrow \infty, \qquad gN : \text{fixed}$$

  - Eigenvalue distribution

$$\rho(x) = \frac{1}{N} \left\langle \sum_{i=1}^{N} \delta(x - x_i) \right\rangle = -\frac{\omega(x + i\epsilon) - \omega(x - i\epsilon)}{2gN\pi i}$$

Resolvent

$$\omega(x) = gN \int dz \, \frac{\rho(z)}{x - z}$$

Equation of motion

$$V'(z) = \omega(z + i\epsilon) + \omega(z - i\epsilon)$$

- Auxiliary analytic function cf. [Halmagyi-Yasnov]

$$e^{y(z+i\epsilon)/2} + e^{-y(z+i\epsilon)/2} = e^{y(z-i\epsilon)/2} + e^{-y(z-i\epsilon)/2}$$

  - $y(z) = V'(z) - 2\omega(z)$

- Asymptotics $(z \to \infty)$

$$\omega(z) \longrightarrow \frac{1}{z}, \qquad V'(z) = 2\sum_{l=1}^{n} \log\left(\frac{z - a_l}{\Lambda}\right) \longrightarrow 2\log\left(\frac{z}{\Lambda}\right)^n$$

- Analytic function (monic polynomial: $P_n(z) = z^n + \cdots$)

$$P_n(z) = \Lambda^n \left(e^{y/2} + e^{-y/2} \right)$$

- Analytic function (monic polynomial: $P_n(z) = z^n + \cdots$)

$$P_n(z) = \Lambda^n \left(e^{y/2} + e^{-y/2} \right) \equiv \Lambda^n \left(w + \frac{1}{w} \right)$$

- Analytic function (monic polynomial: $P_n(z) = z^n + \cdots$)

$$P_n(z) = \Lambda^n \left(e^{y/2} + e^{-y/2} \right) \equiv \Lambda^n \left(w + \frac{1}{w} \right)$$

$$\boxed{\textbf{Seiberg-Witten curve}}$$

- Analytic function (monic polynomial: $P_n(z) = z^n + \cdots$)

$$P_n(z) = \Lambda^n \left(e^{y/2} + e^{-y/2} \right) \equiv \Lambda^n \left(w + \frac{1}{w} \right)$$

Seiberg-Witten curve

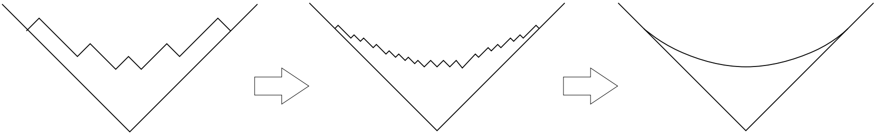
- Seiberg-Witten differential

$$dS = \frac{1}{2\pi i} z \frac{dw}{w} = \frac{1}{4\pi i} y(z) dz$$

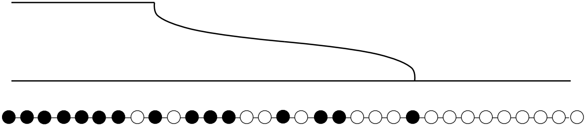
SW curve $\Longleftrightarrow$ Spectral curve

- Profile function [Logan-Shepp] [Vershik-Kerov] cf. [Okounkov]

$$\Omega(x) = \begin{cases} \frac{2}{\pi}\left(x \arcsin \frac{x}{2} + \sqrt{4-x^2}\right) & \text{for} \quad |x| \le 2 \\ |x| & \text{for} \quad |x| > 2 \end{cases}$$



- Density function $\rho(x)$



Contents

Summary

- Combinatorics to Matrix model
  $\longrightarrow$ boson-fermion correspondence, norm representation
- Some extensions
  $\longrightarrow$ Jack, Macdonald, Nekrasov, Orbifolding
- Large N analysis
  $\longrightarrow$ Seiberg-Witten curve, Limit shape
- Correlation functions, fluctuations