非平衡界面成長モデルとランダム行列

T. Sasamoto

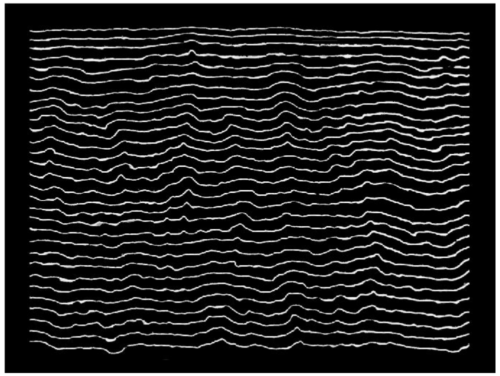
(Based on collaborations with H. Spohn, T. Imamura)

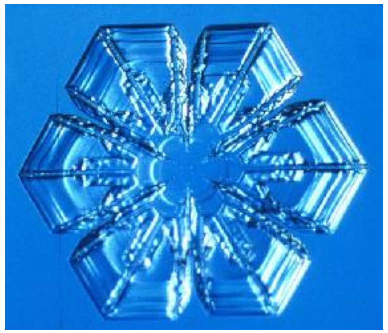
21 Feb 2012 @ 立教大学

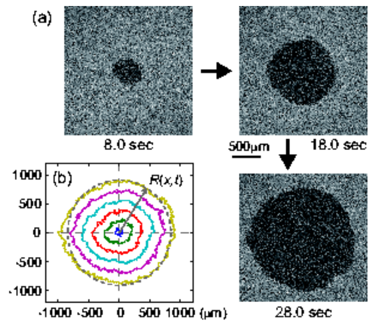
References: arxiv:1002.1883, 1108.2118, 1111.4634

1. Introduction: 1D surface growth

- Paper combustion, bacteria colony, crystal growth, liquid crystal turbulence
- Growing interface between two regions
- Non-equilibrium statistical mechanics
- Integrable systems



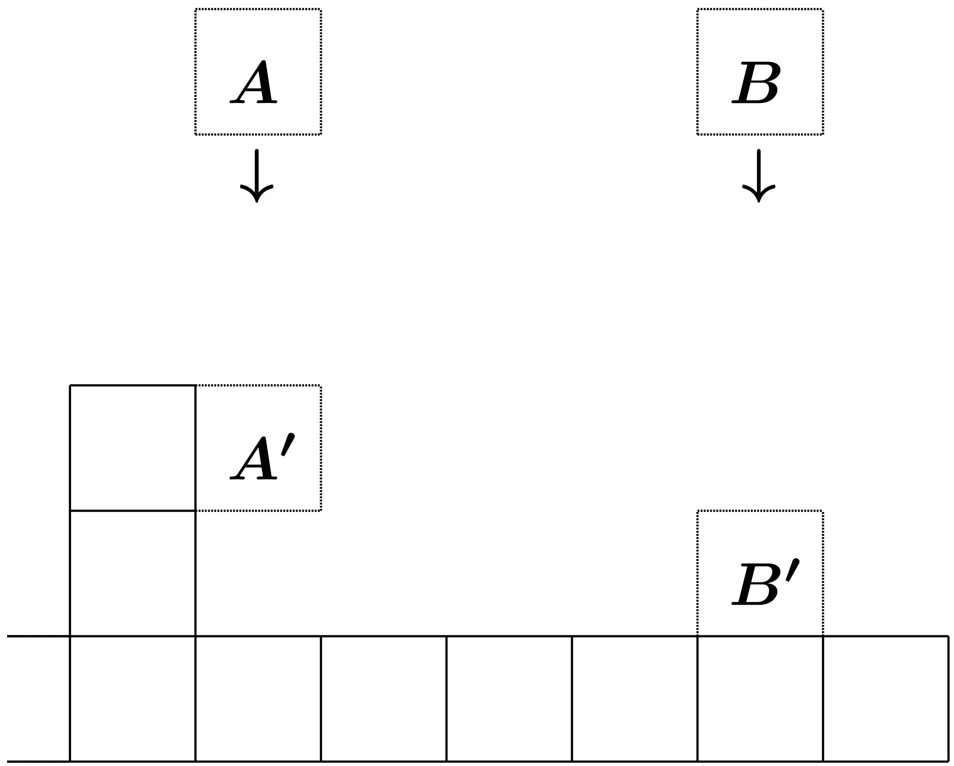


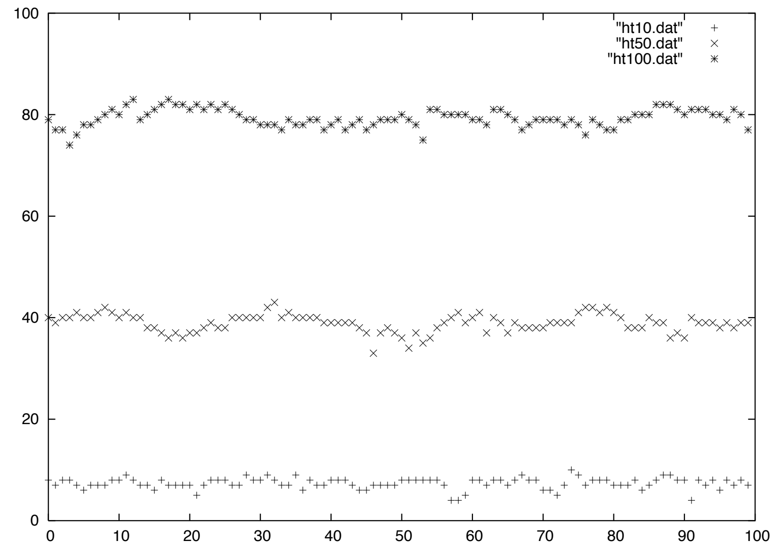


Where is matrix model?

Basics: Simulation models

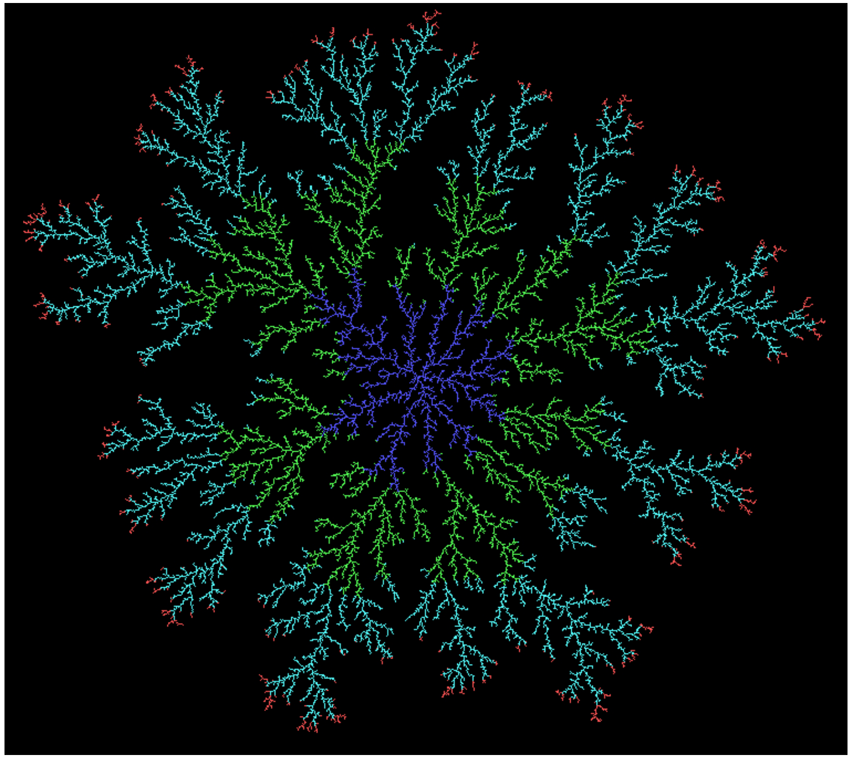
Ex: ballistic deposition model, Eden model





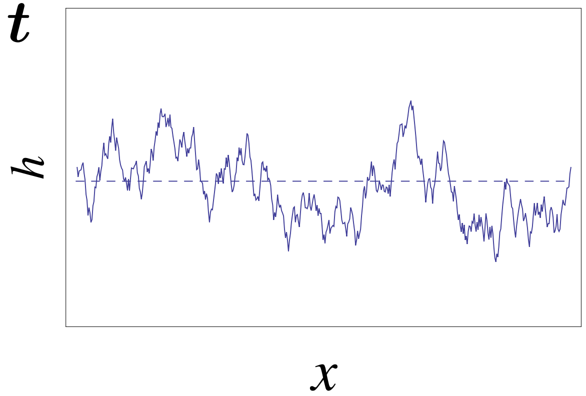
A different type of growth

Diffusion limited aggregation(DLA)...SLE?



Scaling

$h(x,t)$: surface height at position x and at time t



Scaling (L: system size)

$$W(L,t) = \langle (h(x,t) - \langle h(x,t) \rangle)^2 \rangle^{1/2}$$
$$= L^{\alpha} \Psi(t/L^z)$$

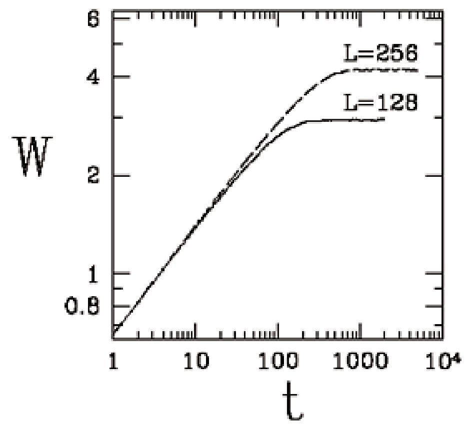


Figure 1. Interface width W versus time t for the RSOS model (Ref. [11]) in $1+1$ dimensions, in two different lattice lengths L.

<u>For $t \to \infty$</u> $\quad W(L,t) \sim L^{\alpha}$

<u>For $t \sim 0$</u> $\quad W(L,t) \sim t^{\beta}$ where $\alpha = \beta z$

In many models, $\alpha = 1/2, \beta = 1/3$

Dynamical exponent $z = 3/2$: Anisotropic scaling

KPZ equation

1986 Kardar Parisi Zhang (not Knizhnik-Polyakov-Zamolodchikov)

$$\partial_t h(x,t) = \frac{1}{2}\lambda(\partial_x h(x,t))^2 + \nu\partial_x^2 h(x,t) + \sqrt{D}\eta(x,t)$$

where η is the Gaussian noise with covariance
$\langle \eta(x,t)\eta(x',t')\rangle = \delta(x-x')\delta(t-t')$

- Dynamical RG analysis: $h(x=0,t) \simeq vt + c\xi t^{1/3}$
  KPZ universality class
- The Brownian motion is stationary.
- The issue of well-definedness.
- Now revival: New analytic and experimental developments

Another KPZ

- MBT-70 ／ KPz 70

  Tank developed in 1960s by US and West Germany. MBT(MAIN BATTLE TANK)-70 is the US name and KPz(KampfPanzer)-70 is the German name.

2. Random matrix theory

Gaussian ensembles

H: $N \times N$ matrix

$$P(H)dH = \frac{1}{Z_{N\beta}} e^{-\frac{\beta}{2}\mathrm{Tr}H^2}$$

GOE$(\beta = 1)$, **GUE**$(\beta = 2)$, **GSE**$(\beta = 4)$

Joint eigenvalue distribution

$$P_{N\beta}(x_1, x_2, \ldots, x_N) = \frac{1}{Z_{N\beta}} \prod_{1 \le i < j \le N} (x_i - x_j)^\beta \prod_{i=1}^{N} e^{-\frac{\beta}{2}x_i^2}$$

Largest eigenvalue distribution

Largest eigenvalue distribution of Gaussian ensembles

$$\mathbb{P}_{N\beta}[x_{\max} \leq s] = \frac{1}{Z_{N\beta}} \int_{(-\infty,s]^N} \prod_{i<j} (x_i - x_j)^\beta \prod_i e^{-\frac{\beta}{2}x_i^2} dx_1 \cdots dx_N$$

Scaling limit (expected to be universal)

$$\lim_{N\to\infty} \mathbb{P}_{N\beta}\left[(x_{\max} - \sqrt{2N})\sqrt{2}N^{1/6} < s\right] = F_\beta(s)$$

GUE (GOE,GSE) Tracy-Widom distribution

Tracy-Widom distributions

GUE Tracy-Widom distribution

$$F_2(s) = \det(1 - P_s K_2 P_s)$$

where P_s: projection onto $[s, \infty)$ and K_2 is the Airy kernel

$$K_2(x, y) = \int_0^\infty \mathrm{d}\lambda \mathrm{Ai}(x + \lambda)\mathrm{Ai}(y + \lambda)$$

Painlevé II representation

$$F_2(s) = \exp\left[-\int_s^\infty (x - s)u(x)^2 dx\right]$$

where $u(x)$ is the solution of the Painlevé II equation

$$\frac{\partial^2}{\partial x^2} u = 2u^3 + xu, \quad u(x) \sim \mathrm{Ai}(x) \quad x \to \infty$$

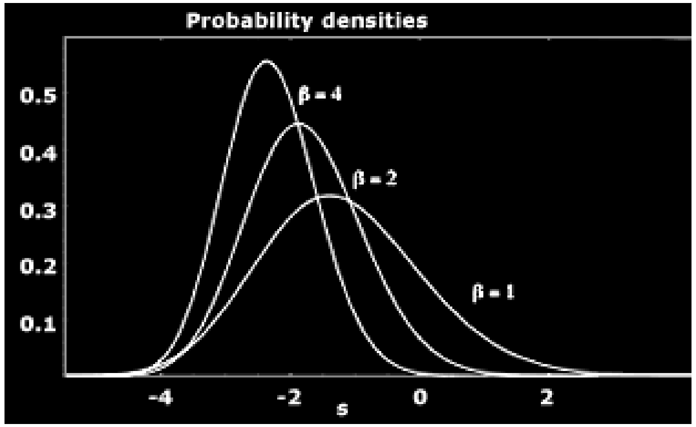
GOE Tracy-Widom distribution

$$F_1(s) = \exp\left[-\frac{1}{2}\int_s^\infty u(x)dx\right](F_2(s))^{1/2}$$

GSE Tracy-Widom distribution

$$F_4(s) = \cosh\left[-\frac{1}{2}\int_s^\infty u(x)dx\right](F_2(s))^{1/2}$$

Figures for Tracy-Widom distributions

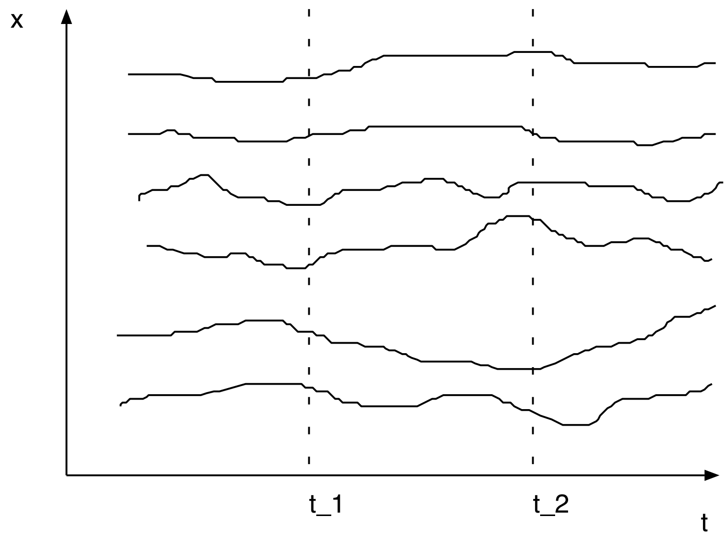


Time dependent version: Dyson's BM model

$$S = \int dt \frac{1}{2} \mathrm{Tr}(\dot{M}^2 + M^2)$$

Or, each matrix element performs the Ornstein-Uhlenbeck process(Brownian motion in a harmonic potential).

Dynamics of eigenvalues



A kind of vicious walk

Talk by Katori

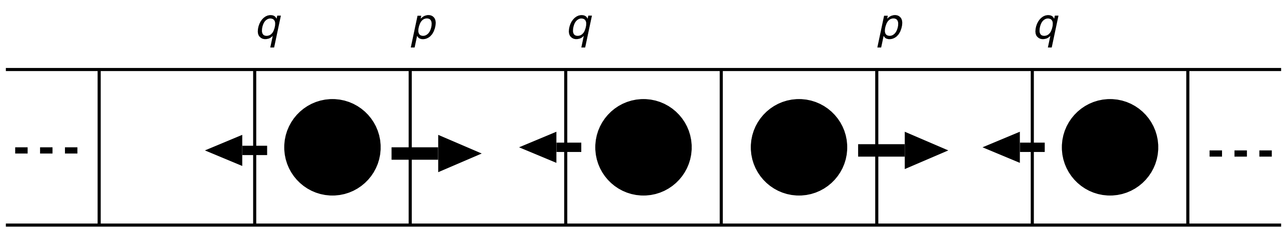
Dynamics of largest eigenvalue

= Airy process

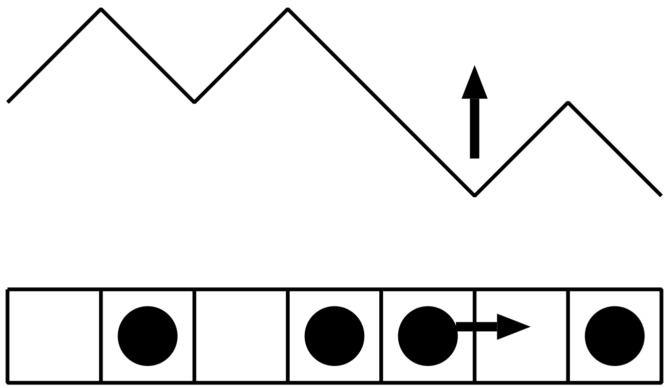
Prähofer-Spohn,Brezin-Hikami

3. Surface growth and random matrix

A discrete model: ASEP(asymmetric simple exclusion process)



Mapping to surface growth

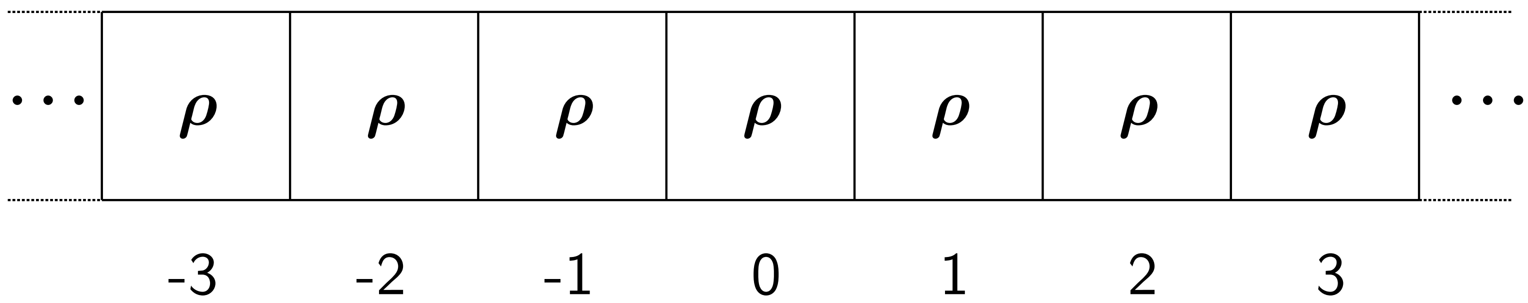


A few comments on ASEP

- Some applications
  - Traffic flow
  - Ionic conductor
  - Ribosome on mRNA, Molecular motor
- A standard model in nonequilibrium statistical physics
  - A system far from equilibrium
  - Shock wave
  - Boundary induced phase transitions
  - Exactly solvable

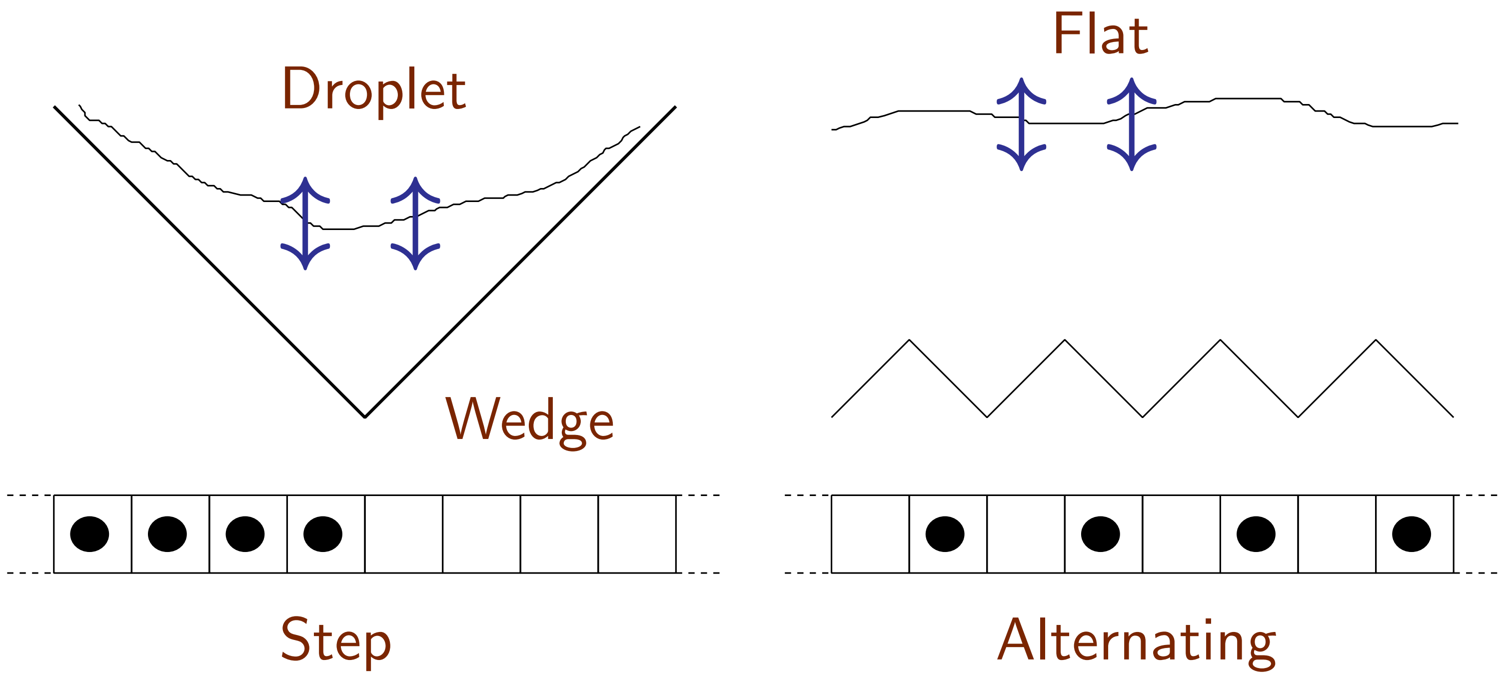
Stationary measure

ASEP $\cdots$ Bernoulli measure: each site is independent and occupied with prob. ρ $(0 < \rho < 1)$. Current is $\rho(1-\rho)$.



Surface growth $\cdots$ Random walk height profile

Surface growth and 2 initial conditions besides stationary

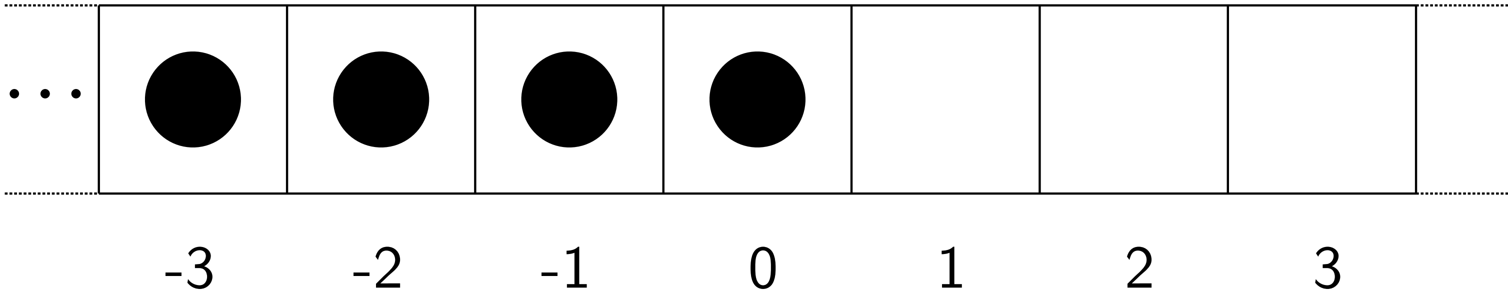


Integrated current $N(x,t)$ in ASEP $\Leftrightarrow$ Height $h(x,t)$ in surface growth

Connection to random matrix: Johansson

TASEP(Totally ASEP, hop only in one direction)

Step initial condition $(t = 0)$



$N(t)$: Number of particles which crossed (0,1) up to time t

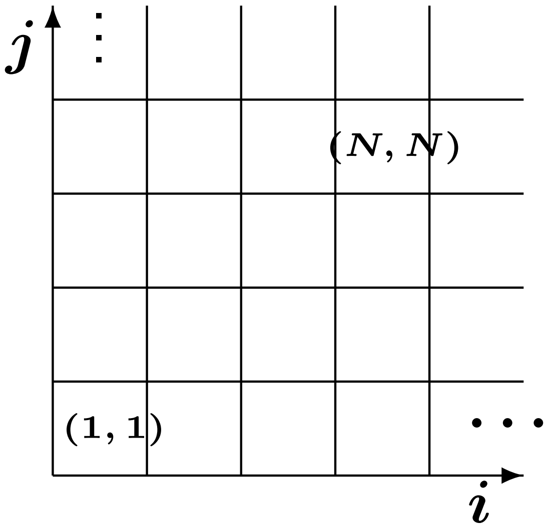
$$\mathbb{P}[N(t) \geq N] = \frac{1}{Z_N} \int_{[0,t]^N} \prod_{i<j} (x_i - x_j)^2 \prod_i e^{-x_i} dx_1 \cdots dx_N$$

(cf. chiral GUE)

Zero temperature directed polymer

- Time at which Nth particle arrives at the origin

$$= \max_{\substack{\text{up-right paths from} \\ (1,1)\text{to}(N,N)}} \left\{ \sum_{(i,j) \text{ on a path}} w_{i,j} \right\}$$



w_{ij} on (i,j): exponentially distributed waiting time of ith hop of jth particle

- RSK algorithm $\Rightarrow$ Combinatorics of Young tableaux

  (cf. Schur measure $\rightarrow$ Plancherel measure $\rightarrow$ Gross-Witten)

Dynamics on Gelfand-Zetlin Cone

Blocking and pushing dynamics

$$
\begin{array}{ccccccccccc}
 & & & & & x_1^1 & & & & & \\
 & & & & x_1^2 & & x_2^2 & & & & \\
 & & & x_1^3 & & x_2^3 & & x_3^3 & & & \\
 & & \iddots & & & \vdots & & & \ddots & & \\
x_1^n & x_2^n & x_3^n & & & \cdots & & & & x_{n-1}^n & x_n^n
\end{array}
$$

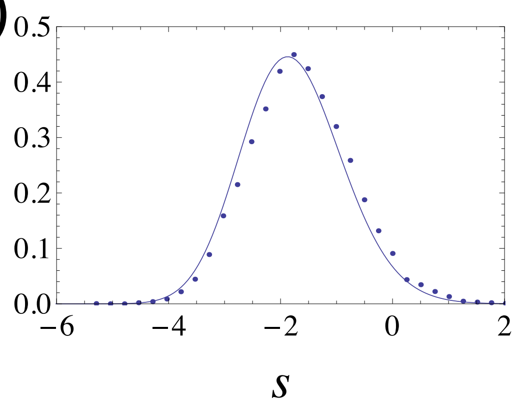
Current distributions for ASEP with wedge initial conditions

2000 Johansson (TASEP) 2008 Tracy-Widom (ASEP)

$$N(0, t/(q-p)) \simeq \frac{1}{4}t - 2^{-4/3}t^{1/3}\xi_{\mathrm{TW}}$$

Here $N(x=0,t)$ is the integrated current of ASEP at the origin and ξ_{TW} obeys the GUE Tracy-Widom distributions;

$$F_{\mathrm{TW}}(s) = \mathbb{P}[\xi_{\mathrm{TW}} \le s] = \det(1 - P_s K_{\mathrm{Ai}} P_s)$$



where K_{Ai} is the Airy kernel

$$K_{\mathrm{Ai}}(x,y) = \int_0^\infty \mathrm{d}\lambda \mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda)$$

Current Fluctuations of TASEP with flat initial conditions: GOE TW distribution

More generalizations: stationary case: $\boldsymbol{F_0}$ distribution, multi-point fluctuations: Airy process, etc

Experimental relevance?

What about the KPZ equation itself?

4. Experiments by liquid crystal turbulence

2010-2011 Takeuchi Sano

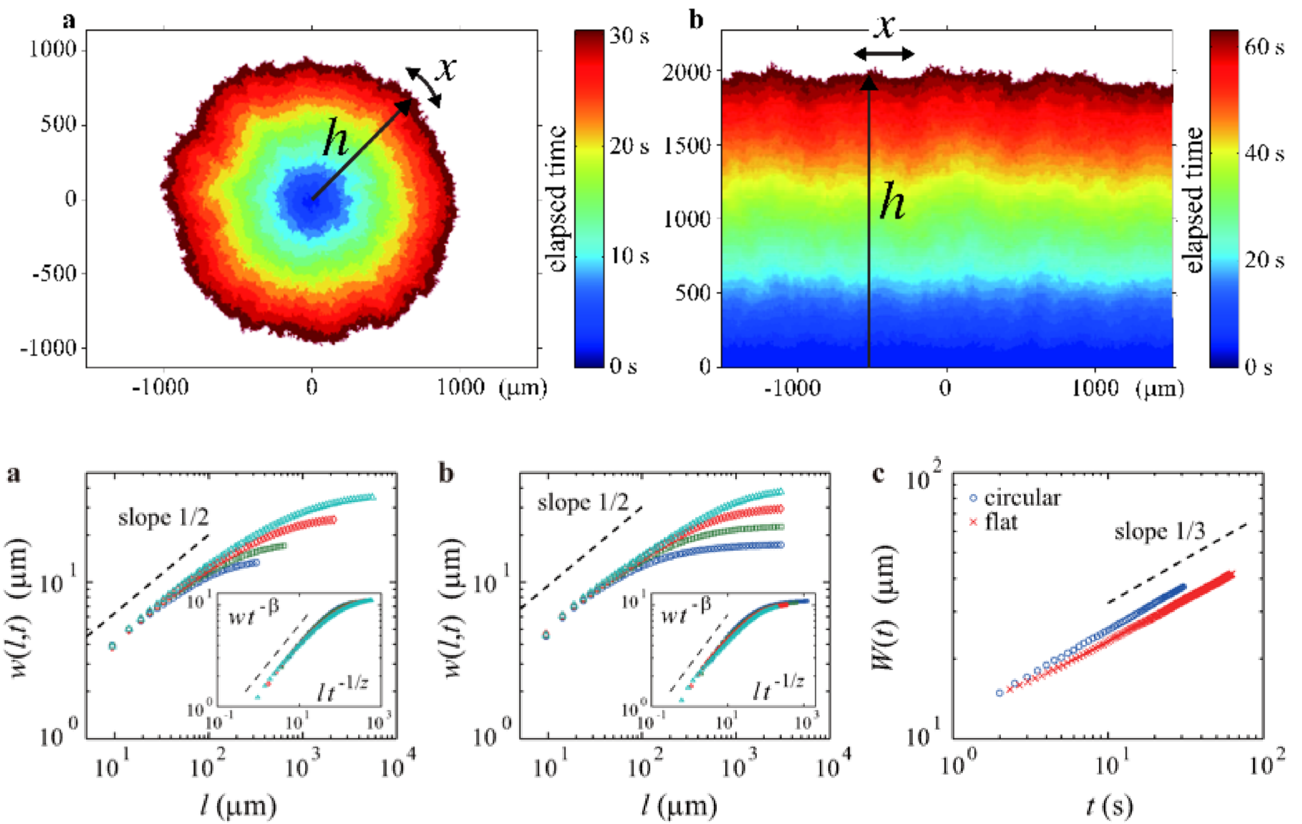


Figure 2 | Family-Vicsek scaling. a,b, Interface width $w(l, t)$ against the length scale l at different times t for the circular (a) and flat (b) interfaces. The four data correspond, from bottom to top, to $t = 2.0$ s, 4.0 s, 12.0 s and 30.0 s for the panel a and to $t = 4.0$ s, 10.0 s, 25.0 s and 60.0 s for the panel b. The insets show the same data with the rescaled axes. c, Growth of the overall width $W(t) \equiv \sqrt{\langle [h(x,t) - \langle h \rangle]^2 \rangle}$. The dashed lines are guides for the eyes showing the exponent values of the KPZ class.

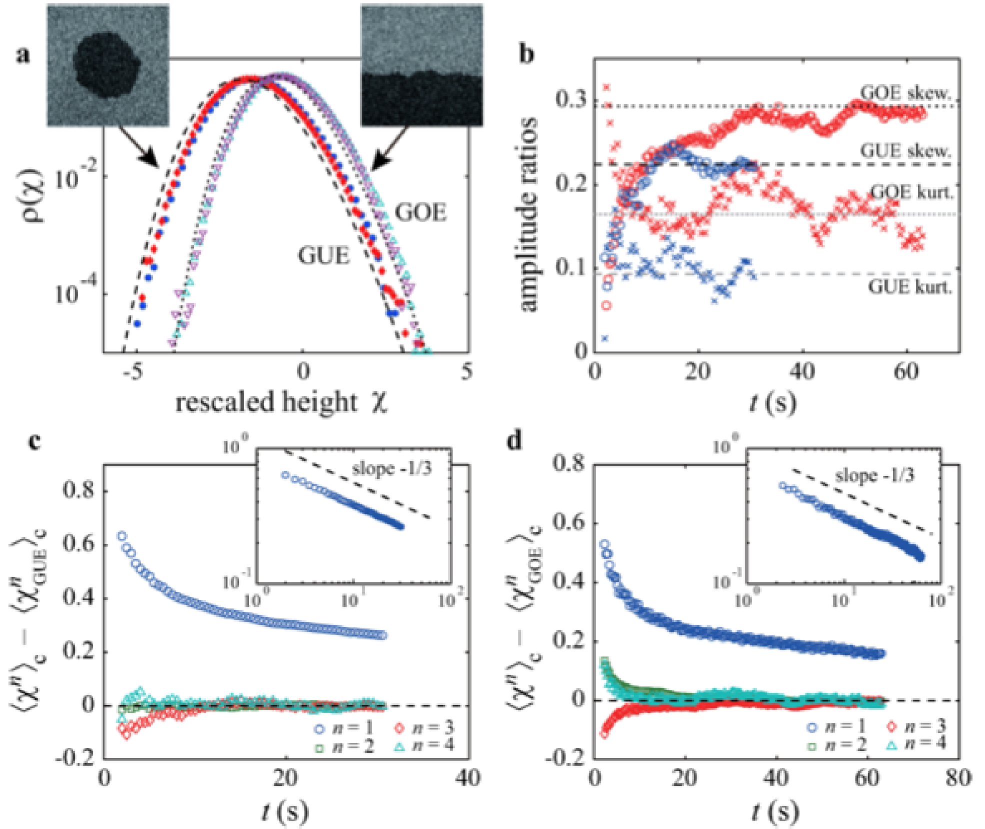


Figure 3 | Universal fluctuations. a, Histogram of the rescaled local height $\chi = (h - v_\infty t)/(\Gamma t)^{1/3}$. The blue and red solid symbols show the histograms for the circular interfaces at $t = 10$ s and 30 s; the light blue and purple open symbols are for the flat interfaces at $t = 20$ s and 60 s, respectively. The dashed and dotted curves show the GUE and GOE TW distributions, respectively. Note that for the GOE TW distribution χ is multiplied by $2^{-2/3}$ in view of the theoretical prediction[31]. b, The skewness (circle) and the kurtosis (cross) of the distribution of the interface fluctuations for the circular (blue) and flat (red) interfaces. The dashed and dotted lines indicate the values of the skewness and the kurtosis of the GUE and GOE TW distributions[31]. c, d, Differences in the cumulants between the experimental data $\langle \chi^n \rangle_c$ and the corresponding TW distributions $\langle \chi^n_{\mathrm{GUE}} \rangle_c$ for the ciruclar interfaces (c) and $\langle \chi^n_{\mathrm{GOE}} \rangle_c$ for the flat interfaces (d). The insets show the same data for $n = 1$ in logarithmic scales. The dashed lines are guides for the eyes with the slope $-1/3$.

See Takeuchi Sano Sasamoto Spohn, Sci. Rep. 1,34(2011)

5. The narrow wedge KPZ equation

Remember the KPZ equation

$$\partial_t h(x,t) = \frac{1}{2}\lambda(\partial_x h(x,t))^2 + \nu \partial_x^2 h(x,t) + \sqrt{D}\eta(x,t)$$

2010 Sasamoto Spohn, Amir Corwin Quastel

- Narrow wedge initial condition
- Based on (i) the fact that the weakly ASEP is KPZ equation (1997 Bertini Giacomin) and (ii) a formula for step ASEP by 2009 Tracy Widom
- The explicit distribution function for finite t
- The KPZ equation is in the KPZ universality class

Narrow wedge initial condition

Scalings

$$x \to \alpha^2 x, \quad t \to 2\nu\alpha^4 t, \quad h \to \frac{\lambda}{2\nu} h$$

where $\alpha = (2\nu)^{-3/2}\lambda D^{1/2}$.

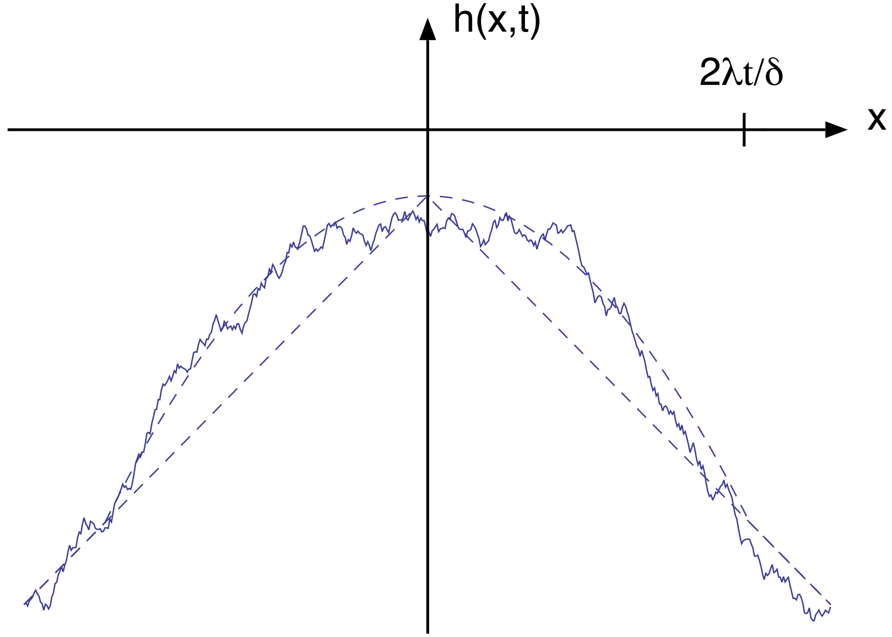
We can and will do set $\nu = \frac{1}{2}, \lambda = D = 1$.

We consider the droplet growth with macroscopic shape

$$h(x,t) = \begin{cases} -x^2/2t & \text{for } |x| \le t/\delta\,, \\ (1/2\delta^2)t - |x|/\delta & \text{for } |x| > t/\delta \end{cases}$$

which corresponds to taking the following narrow wedge initial conditions:

$$h(x,0) = -|x|/\delta\,, \quad \delta \ll 1$$


h(x,t)
2λt/δ
x

Distribution

$$h(x,t) = -x^2/2t - \tfrac{1}{12}\gamma_t^3 + \gamma_t \xi_t$$

where $\gamma_t = (2t)^{-1/3}$.

The distribution function of ξ_t

$$F_t(s) = \mathbb{P}[\xi_t \leq s] = 1 - \int_{-\infty}^{\infty} \exp\big[-\mathrm{e}^{\gamma_t(s-u)}\big]$$
$$\times\Big(\det(1 - P_u(B_t - P_{\mathrm{Ai}})P_u) - \det(1 - P_u B_t P_u)\Big)\mathrm{d}u$$

where $P_{\mathrm{Ai}}(x,y) = \mathrm{Ai}(x)\mathrm{Ai}(y)$.

$\boldsymbol{P_u}$ is the projection onto $[\boldsymbol{u}, \infty)$ and the kernel $\boldsymbol{B_t}$ is

$$\boldsymbol{B_t(x,y) = K_{\mathbf{Ai}}(x,y) + \int_0^\infty \mathbf{d}\lambda(\mathbf{e}^{\gamma_t \lambda} - 1)^{-1}}$$
$$\times\big(\mathbf{Ai}(\boldsymbol{x}+\boldsymbol{\lambda})\mathbf{Ai}(\boldsymbol{y}+\boldsymbol{\lambda}) - \mathbf{Ai}(\boldsymbol{x}-\boldsymbol{\lambda})\mathbf{Ai}(\boldsymbol{y}-\boldsymbol{\lambda})\big)\,.$$

A question: Random matrix interpretation?

Developments

- 2010 Calabrese Le Doussal Rosso, Dotsenko Replica
- 2010 Corwin Quastel Half-BM by step Bernoulli ASEP
- 2010 O'Connell A directed polymer model related to quantum Toda lattice
- 2010 Prolhac Spohn Multi-point distributions by replica
- 2011 Calabrese Le Dossal Flat case by replica
- 2011 Corwin et al Tropical RSK for inverse gamma polymer
- 2011 Borodin Corwin Macdonald process
- 2011 Imamura Sasamoto Half-BM and stationary case by replica

6. Stationary case

- Narrow wedge is technically the simplest.
- Flat case is a well-studied case in surface growth
- Stationary case is important for stochastic process and nonequilibrium statistical mechanics
  - Two-point correlation function
  - Experiments: Scattering, direct observation
  - A lot of approximate methods (renormalization, mode-coupling, etc.) have been applied to this case.
  - Nonequilibrium steady state(NESS): No principle. Dynamics is even harder.

Modification of initial condition

Two sided BM

$$h(x,0) = \begin{cases} B_-(-x), & x < 0, \\ B_+(x), & x > 0, \end{cases}$$

where $B_\pm(x)$ are two independent standard BMs

We consider a generalized initial condition

$$h(x,0) = \begin{cases} \tilde{B}(-x) + v_- x, & x < 0, \\ B(x) - v_+ x, & x > 0, \end{cases}$$

where $B(x), \tilde{B}(x)$ are independent standard BMs and $v_\pm$ are the strength of the drifts.

Result

For the generalized initial condition with $v_\pm$

$$F_{v_\pm,t}(s) := \mathbf{Prob}\left[h(x,t) + \gamma_t^3/12 \le \gamma_t s\right]$$
$$= \frac{\Gamma(v_+ + v_-)}{\Gamma(v_+ + v_- + \gamma_t^{-1} d/ds)}\left[1 - \int_{-\infty}^{\infty} du e^{-e^{\gamma_t(s-u)}} \nu_{v_\pm,t}(u)\right]$$

Here $\nu_{v_\pm,t}(u)$ is expressed as a difference of two Fredholm determinants,

$$\nu_{v_\pm,t}(u) = \det\left(1 - P_u(B_t^\Gamma - P_{\mathrm{Ai}}^\Gamma)P_u\right) - \det\left(1 - P_u B_t^\Gamma P_u\right),$$

where P_s represents the projection onto (s,∞),

$$P_{\mathrm{Ai}}^\Gamma(\xi_1,\xi_2) = \mathrm{Ai}_\Gamma^\Gamma\left(\xi_1, \frac{1}{\gamma_t}, v_-, v_+\right)\mathrm{Ai}_\Gamma^\Gamma\left(\xi_2, \frac{1}{\gamma_t}, v_+, v_-\right)$$

$$
\begin{aligned}
B_t^{\Gamma}(\xi_1, \xi_2) = \int_{-\infty}^{\infty} dy \frac{1}{1 - e^{-\gamma_t y}} \mathrm{Ai}_{\Gamma}^{\Gamma}\left(\xi_1 + y, \frac{1}{\gamma_t}, v_-, v_+\right) \\
\times \mathrm{Ai}_{\Gamma}^{\Gamma}\left(\xi_2 + y, \frac{1}{\gamma_t}, v_+, v_-\right),
\end{aligned}
$$

and

$$
\mathrm{Ai}_{\Gamma}^{\Gamma}(a, b, c, d) = \frac{1}{2\pi} \int_{\Gamma_{i\frac{d}{b}}} dz e^{iza + i\frac{z^3}{3}} \frac{\Gamma(ibz + d)}{\Gamma(-ibz + c)},
$$

where Γ_{z_p} represents the contour from $-\infty$ to ∞ and, along the way, passing below the pole at $z = id/b$.

Height distribution for the stationary KPZ equation

$$F_{0,t}(s) = \frac{1}{\Gamma(1+\gamma_t^{-1}d/ds)} \int_{-\infty}^{\infty} du \gamma_t e^{\gamma_t(s-u)+e^{-\gamma_t(s-u)}} \nu_{0,t}(u)$$

where $\nu_{0,t}(u)$ is obtained from $\nu_{v_\pm,t}(u)$ by taking $v_\pm \to 0$ limit.

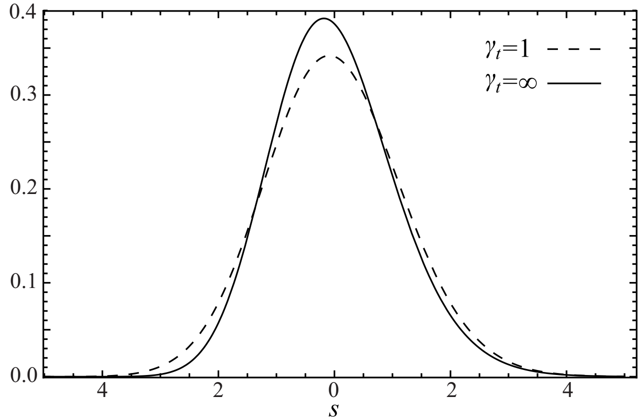


Figure 1: Stationary height distributions for the KPZ equation for $\gamma_t = 1$ case. The solid curve is F_0.

Stationary 2pt correlation function

$$C(x,t) = \langle (h(x,t) - \langle h(x,t)\rangle)^2\rangle$$

$$g_t(y) = (2t)^{-2/3} C\left((2t)^{2/3}y, t\right)$$

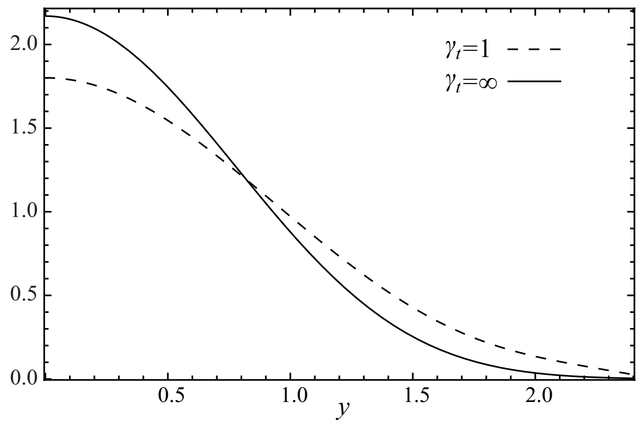


Figure 2: Stationary 2pt correlation function $g_t''(y)$ for $\gamma_t = 1$. The solid curve is the corresponding quantity in the scaling limit $g''(y)$.

Derivation

Cole-Hopf transformation

1997 Bertini and Giacomin

$$h(x,t) = \log\left(Z(x,t)\right)$$

$Z(x,t)$ is the solution of the stochastic heat equation,

$$\frac{\partial Z(x,t)}{\partial t} = \frac{1}{2}\frac{\partial^2 Z(x,t)}{\partial x^2} + \eta(x,t)Z(x,t).$$

and can be considered as directed polymer in random potential η.

cf. Hairer Well-posedness of KPZ equation without Cole-Hopf

Feynmann-Kac and Generating function

Feynmann-Kac expression for the partition function,

$$Z(x,t) = \mathbb{E}_x \left(\exp\left[\int_0^t \eta\left(b(s), t-s\right) ds \right] Z(b(t), 0) \right)$$

We consider the Nth replica partition function $\langle Z^N(x,t) \rangle$ and compute their generating function $G_t(s)$ defined as

$$G_t(s) = \sum_{N=0}^{\infty} \frac{\left(-e^{-\gamma_t s}\right)^N}{N!} \left\langle Z^N(0,t) \right\rangle e^{N \frac{\gamma_t^3}{12}}$$

with $\gamma_t = (t/2)^{1/3}$.

δ-Bose gas

Taking the Gaussian average over the noise η, one finds that the replica partition function can be written as

$$
\begin{aligned}
&\langle Z^N(x,t)\rangle \\
&= \prod_{j=1}^{N} \int_{-\infty}^{\infty} dy_j \int_{x_j(0)=y_j}^{x_j(t)=x} D[x_j(\tau)] \exp\left[-\int_0^t d\tau \left(\sum_{j=1}^{N} \frac{1}{2}\left(\frac{dx}{d\tau}\right)^2\right.\right. \\
&\quad \left.\left. - \sum_{j\neq k=1}^{N} \delta\left(x_j(\tau)-x_k(\tau)\right)\right)\right] \times \left\langle \exp\left(\sum_{k=1}^{N} h(y_k,0)\right)\right\rangle \\
&= \langle x|e^{-H_N t}|\Phi\rangle.
\end{aligned}
$$

H_N is the Hamiltonian of the δ-Bose gas,

$$H_N = -\frac{1}{2}\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2} - \frac{1}{2}\sum_{j\neq k}^{N}\delta(x_j - x_k),$$

$|\Phi\rangle$ represents the state corresponding to the initial condition. We compute $\langle Z^N(x,t)\rangle$ by expanding in terms of the eigenstates of H_N,

$$\langle Z(x,t)^N\rangle = \sum_z \langle x|\Psi_z\rangle\langle\Psi_z|\Phi\rangle e^{-E_z t}$$

where E_z and $|\Psi_z\rangle$ are the eigenvalue and the eigenfunction of H_N: $H_N|\Psi_z\rangle = E_z|\Psi_z\rangle$.

The state $|\Phi\rangle$ can be calculated because the initial condition is Gaussian. For the region where $x_1 < \ldots < x_l < 0 < x_{l+1} < \ldots < x_N, 1 \leq l \leq N$ it is given by

$$\langle x_1, \cdots, x_N | \Phi \rangle = e^{v_- \sum_{j=1}^{l} x_j - v_+ \sum_{j=l+1}^{N} x_j}$$

$$\times \prod_{j=1}^{l} e^{\frac{1}{2}(2l-2j+1)x_j} \prod_{j=1}^{N-l} e^{\frac{1}{2}(N-l-2j+1)x_{l+j}}$$

We symmetrize wrt $x_1, \ldots, x_N$.

Bethe states

By the Bethe ansatz, the eigenfunction is given as

$$\langle x_1, \cdots, x_N | \Psi_z \rangle = C_z \sum_{P \in S_N} \mathrm{sgn} P$$

$$\times \prod_{1 \le j < k \le N} \left(z_{P(j)} - z_{P(k)} + i\,\mathrm{sgn}(x_j - x_k) \right) \exp\left(i \sum_{l=1}^{N} z_{P(l)} x_l \right)$$

N momenta z_j $(1 \le j \le N)$ are parametrized as

$$z_j = q_\alpha - \frac{i}{2}(n_\alpha + 1 - 2r_\alpha), \quad \text{for } j = \sum_{\beta=1}^{\alpha-1} n_\beta + r_\alpha.$$

$(1 \le \alpha \le M$ and $1 \le r_\alpha \le n_\alpha)$. They are divided into M groups where $1 \le M \le N$ and the αth group consists of n_α quasimomenta $z_j's$ which shares the common real part q_α.

$$C_z = \left(\frac{\prod_{\alpha=1}^{M} n_\alpha}{N!} \prod_{1 \le j < k \le N} \frac{1}{|z_j - z_k - i|^2} \right)^{1/2}$$

$$E_z = \frac{1}{2} \sum_{j=1}^{N} z_j^2 = \frac{1}{2} \sum_{\alpha=1}^{M} n_\alpha q_\alpha^2 - \frac{1}{24} \sum_{\alpha=1}^{M} \left(n_\alpha^3 - n_\alpha \right).$$

Expanding the moment in terms of the Bethe states, we have

$$\begin{aligned} &\langle Z^N(x,t) \rangle \\ &= \sum_{M=1}^{N} \frac{N!}{M!} \prod_{j=1}^{N} \int_{-\infty}^{\infty} dy_j \left(\int_{-\infty}^{\infty} \prod_{\alpha=1}^{M} \frac{dq_\alpha}{2\pi} \sum_{n_\alpha=1}^{\infty} \right) \delta_{\sum_{\beta=1}^{M} n_\beta, N} \\ &\quad \times e^{-E_z t} \langle x | \Psi_z \rangle \langle \Psi_z | y_1, \cdots, y_N \rangle \langle y_1, \cdots, y_N | \Phi \rangle. \end{aligned}$$

The completeness of Bethe states was proved by Prolhac Spohn

We see

$$\langle \Psi_z | \Phi \rangle = N! C_z \sum_{P \in S_N} \mathrm{sgn} P \prod_{1 \le j < k \le N} \left(z^*_{P(j)} - z^*_{P(k)} + i \right)$$
$$\times \sum_{l=0}^{N} (-1)^l \prod_{m=1}^{l} \frac{1}{\sum_{j=1}^{m} (-i z^*_{P_j} + v_-) - m^2/2}$$
$$\times \prod_{m=1}^{N-l} \frac{1}{\sum_{j=N-m+1}^{N} (-i z^*_{P_j} - v_+) + m^2/2}.$$

Combinatorial identities

(1)

$$\sum_{P\in S_N} \mathrm{sgn} P \prod_{1\le j<k\le N} \big(w_{P(j)} - w_{P(k)} + i f(j,k)\big)$$

$$= N! \prod_{1\le j<k\le N} (w_j - w_k)$$

(2)For any complex numbers w_j $(1 \leq j \leq N)$ and a,

$$\sum_{P \in S_N} \mathrm{sgn} P \prod_{1 \leq j < k \leq N} \left(w_{P(j)} - w_{P(k)} + a \right)$$
$$\times \sum_{l=0}^{N} (-1)^l \prod_{m=1}^{l} \frac{1}{\sum_{j=1}^{m} (w_{P(j)} + v_-) - m^2 a/2}$$
$$\times \prod_{m=1}^{N-l} \frac{1}{\sum_{j=N-m+1}^{N} (w_{Pj} - v_+) + m^2 a/2}$$
$$= \frac{\prod_{m=1}^{N} (v_+ + v_- - am) \prod_{1 \leq j < k \leq N} (w_j - w_k)}{\prod_{m=1}^{N} (w_m + v_- - a/2)(w_m - v_+ + a/2)}.$$

A similar identity in the context of ASEP has not been found.

Generating function

$$G_t(s) = \sum_{N=0}^{\infty} \prod_{l=1}^{N} (v_+ + v_- - l) \sum_{M=1}^{N} \frac{(-e^{-\gamma_t s})^N}{M!}$$

$$\prod_{\alpha=1}^{M} \left(\int_0^{\infty} d\omega_\alpha \sum_{n_\alpha=1}^{\infty} \right) \delta_{\sum_{\beta=1}^{M} n_\beta, N}$$

$$\det \left(\int_C \frac{dq}{\pi} \frac{e^{-\gamma_t^3 n_j q^2 + \frac{\gamma_t^3}{12} n_j^3 - n_j(\omega_j + \omega_k) - 2iq(\omega_j - \omega_k)}}{\prod_{r=1}^{n_j} (-iq + v_- + \frac{1}{2}(n_j - 2r))(iq + v_+ + \frac{1}{2}(n_j - 2r))} \right)$$

where the contour is $C = \mathbb{R} - ic$ with c taken large enough.

This generating function itself is not a Fredholm determinant due to the novel factor $\prod_{l=1}^{N}(v_+ + v_- - l)$.

We consider a further generalized initial condition in which the initial overall height χ obeys a certain probability distribution.

$$\tilde{h} = h + \chi$$

where h is the original height for which $h(0,0) = 0$. The random variable χ is taken to be independent of h.

Moments
$$\langle e^{N\tilde{h}} \rangle = \langle e^{Nh} \rangle \langle e^{N\chi} \rangle.$$

We postulate that χ is distributed as the inverse gamma distribution with parameter $v_+ + v_-$, i.e., if $1/\chi$ obeys the gamma distribution with the same parameter. Its Nth moment is $1/\prod_{l=1}^{N}(v_+ + v_- - l)$ which compensates the extra factor.

Distributions

$$\boldsymbol{F(s) = \frac{1}{\kappa(\gamma_t^{-1}\frac{d}{ds})}\tilde{F}(s),}$$

where $\boldsymbol{\tilde{F}(s)} = \mathrm{Prob}[\boldsymbol{\tilde{h}(0,t) \le \gamma_t s}]$,
$\boldsymbol{F(s)} = \mathrm{Prob}[\boldsymbol{h(0,t) \le \gamma_t s}]$ and $\boldsymbol{\kappa}$ is the Laplace transform of the pdf of $\boldsymbol{\chi}$. For the inverse gamma distribution, $\boldsymbol{\kappa(\xi) = \Gamma(v+\xi)/\Gamma(v)}$, by which we get the formula for the generating function.

Summary

- Some surface growth models are related to random matrix.
- The developments in the last 12 years allow us to study also the KPZ equation.
- We have presented some formulas for the height distributions and stationary two point correlation functions.