Continuity of commutative limit of non-commutative (NC) $\mathcal{N}=4$ Super Yang-Mills (SYM)

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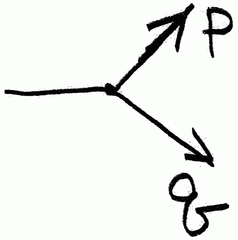
based on a work with

Masanori Hanada (KEK)

☆ Non-Commutative Field Theory

	(commutative) usual	NC
vertex	$g\int\phi^3 dx$	$g\int\phi*\phi*\phi\ d^4x$
	g	$g\,e^{iC^{\mu\nu}p_\mu q_\nu}$

$$f_1 * f_2 \overset{\text{def.}}{=} f_1\, e^{iC^{\mu\nu}\overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu} f_2$$

☆ Commutative limit

$$C \longrightarrow 0$$

Outline

1. $C \to 0$ is relevant for the use of NC space as regularisation method

2. $C \to 0$ is non-trivial

☆ $e^{i p_\mu C^{\mu\nu} q_\nu}$ oscillates alot in the UV

☆ $\log \Lambda$ -divergence $\longrightarrow$ $\log (k_\mu C^{\mu\nu})^2$; Λ : UV cut-off
k_μ : ext mom.
"UV/IR mixing"

☆ non-trivial for $D=4$, $\mathcal{N}=4$ NCSYM
(though no UV-div for $C=0$, $C \neq 0$ shoud be studied.)

3. Proof. $C \to 0$ for $D=4$, $\mathcal{N}=4$ NCSYM is continuous

similar to finiteness proof of
Brink-Lindgren-Nilsson, Mandelstam
for $C=0$
Ananth-Kovacs-HS
for similar deformation (β-defm.)

1. $C \to 0$ is relevant.

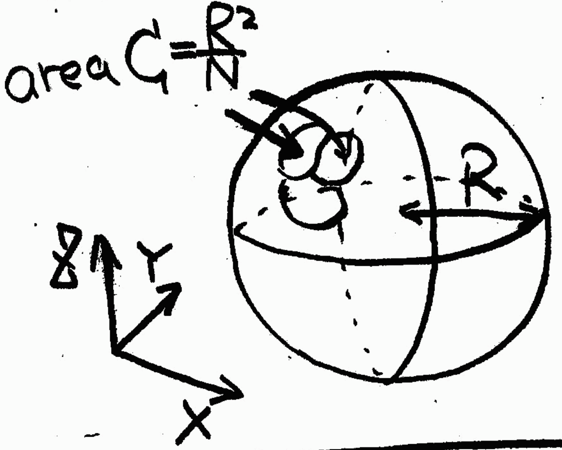
for application of NC space to
regularisation of QFT (with SUSY).

☆ NC sphere, NC plane
$N \to \infty$, $C \to 0$.

☆ use of NC space instead of lattice
for SYM.

☆ NC sphere, supermembrane, matrix model

deWit-Hoppe-Nicolai

Supermembrane	matrix model of M-theory
$H=\int p^{\alpha 2}+\{x^\alpha,x^\beta\}^2 + \theta^T\Gamma^\alpha\{x^\alpha,\theta\}\, d^2\sigma$	$H=\mathrm{Tr}(\hat{p}^{\alpha 2}-[\hat{x}^\alpha,\hat{x}^\beta]^2 + \hat{\theta}^T\Gamma^\alpha[\hat{x}^\alpha,\hat{\theta}])$
$x^\alpha(\sigma^1,\sigma^2)$	$\hat{x}^\alpha$: N × N matrices
 polynomial of X, Y, Z	polynomial of $\hat{X},\hat{Y},\hat{Z}$ $[\hat{X},\hat{Y}]=i\frac{1}{N}\hat{Z},\cdots$

☆ preserve SUSY

☆ becomes NC plane for $R\to\infty$ ($C=\frac{R^2}{N}$: fixed).

$C\to 0$ analogous to the important $N\to\infty$ of matrix model

☆ Use of NC space for SYM

☆ non-pertubative def. of super Yang-Mills

- Conceptually important
- possibly useful to study non-pert. properties of SYM numerically

☆ lattice is difficult

$\{Q, Q\} = P$ ← not realised on lattice

☆ some D=2 SYM possible to formulate on lattice. by using part of SUSY algebra

(Cohen-Katz-Unsal, Sugino)

☆ Recent proposal by Hanada-Matsuura-Sugino

(for D=3 Maldacena-Sheikh Jabbari - Raamsdonk)

$$\overset{0,1,2,3}{4D} = \underbrace{\overset{0,\ 3}{\text{2D lattice}}}_{\substack{\text{lattice const}\\ a}} + \underbrace{\overset{1,\ 2}{\text{NC sphere}}}_{\substack{\text{radius } R\\ N\times N \text{ matrix}}}$$

$$\Big\downarrow \quad \text{appropriate} \quad \begin{array}{l} a \to 0 \\ R \to \infty \\ N \to \infty \end{array}$$

def. of NC $\mathcal{N}=4$ SYM (on plane) with $C \neq 0$

$$\Big\downarrow \quad C \to 0$$

def. of usual $\mathcal{N}=4$ SYM (on plane)

Summary of 1.

$C \to 0$ limit is relevant

for appl. of NC space to regl. of QFT.

☆ for membrane / matrix model
  $C \to 0$ is like $N \to \infty$, continuum limit.

☆ $C \to 0$ also relevant for. non-pert. def.
  of $\mathcal{N}=4$ SYM

2. $C \to 0$ is non-trivial

☆ NC Field theory, generic

$\log \Lambda$-div. $\xrightarrow[\text{NC}]{}$ $\log (k_\mu C^{\mu\nu})^2$

↑ ext mom.

☆ In $N=4$ SYM, $C=0$, $\log \Lambda$-div. cancels.

→ does NOT guarantees non-singular behaviour (continuous) for $C \to 0$

☆ Math.

uniform convergence of Feynman integral

Physics.

Cancellation among integrals acquiring the same phase factor.

) should be shown

☆ 1-loop calculation in NC QFT

(Minwalla-Seiberg-Raamsdonk; Hayakawa; Matusis-Susskind-Toumbas)

$$\left(\begin{array}{lll} \Lambda^2\text{-div} & \longrightarrow & \dfrac{1}{(Ck)^2} \\ \log\Lambda\text{-div.} & \longrightarrow & \log (Ck)^2 \end{array}\right.$$

k: ext mom.

("UV/IR mixing")

☆ $C \to 0$

<u>integrand</u>

$$◎(p) \times e^{i(pCp'+\cdots)} \leftarrow$$ Smooth continuous

<u>integral</u>

$$\lim_{\Lambda\to\infty} \int^{\Lambda} ◎(p) \times e^{i(pCp'+\cdots)}\, dp \leftarrow$$ singular non-continuous

☆ order of $C \to 0$ and $\Lambda \to \infty$ matters.

☆ Matusis - Susskind - Toumbas suggested
" no singularities for $\mathcal{N}=4$ NCSYM "
on the ground that $\log\Lambda$- (and Λ^2-) div cancells at $C=0$

☆ This does NOT assure continuity for $C \to 0$.

☆ Mathematically, should show uniform convergence of $\Lambda \to \infty$ with respect to C

Physically,

☆ for finite C, phase factors might ruin the cancellation between diagramms which was $\log\Lambda$-div.

→ $\log(Ck)^2$ - like terms would appear

☆ should show cancel. among diagramms acquiring same phase factors.

(cancel. among graphs with no phase factors were discussed by Jack-Jones)

☆ Some mathematics

☆. usual conv.

depend on q

$$\lim_{\Lambda\to\infty}\int^{\Lambda} f(p,q)\,dp \longrightarrow F(q)$$

Λ_0 ~ measures speed of convergence

$$\iff {}^{\forall}\varepsilon : {}^{\exists}\Lambda_0 \text{ s.t. } {}^{\forall}\Lambda \geqq \Lambda_0$$

$$\left|\int^{\Lambda} f(p,q)\,dp - F(q)\right| < \varepsilon$$

☆ uniform conv.

common for all q

$$\lim_{\Lambda\to\infty}\int^{\Lambda} f(p,q)\,dp \xrightarrow{\text{uniform}} F(q)$$

$$\iff {}^{\forall}\varepsilon : {}^{\exists}\Lambda_0 \text{ s.t. } {}^{\forall}\Lambda \geqq \Lambda_0,\ {}^{\forall}q$$

$$\left|\int^{\Lambda} f(p,q)\,dp - F(q)\right| < \varepsilon$$

☆ Thm. in case uniform conv.

$f(p,q)$: continuous in q $\Longrightarrow$ $F(q)$: continuous in q

☆ Simple integrals for illustration

☆ log-div tamed by phase factors

$$\lim_{\Lambda\to\infty}\int_a^{\Lambda}\frac{1}{p}e^{ik'cp}\,dp \underset{c\sim 0}{\sim} \log(ack')$$

☆ if cancellation are ruined by phase factors

$$\lim_{c\to 0}\lim_{\Lambda\to\infty}\int_a^{\Lambda}\left(\frac{1}{p}e^{ik'cp}-\frac{1}{p}e^{ik''cp}\right)dp$$

$$=\lim_{c\to 0}\int_{ack'}^{ack''}\frac{1}{u}e^{iu}\,du = \log\frac{k''}{k'}$$

while $\lim_{\Lambda\to\infty}\lim_{c\to 0}\int_a^{\Lambda}\left(\frac{1}{p}e^{ik'cp}-\frac{1}{p}e^{ik''cp}\right)dp = 0.$

Summary of 2.

☆ $C \to 0$ is non-trivial

$\log\Lambda$-div. $\xrightarrow[NC]{}$ $\log(Ck)^2$ ← ext mom.

☆ for $\mathcal{N}=4$ NCSYM

$\log\Lambda$-div cancel. at $C=0$. However,

in order to discuss continuity for $C \to 0$

☆ need to show uniform conv.

☆ need to show cancel. occurs among integrals which acquires same phase factors from $*$-product

3. proof of uniform conv. of Feynman integrals (in the UV) for $\mathcal{N}=4$ NC SYM

☆ use (lightcone (LG) superspace
power-counting in 2 steps

proof of finiteness of (commutative) $\mathcal{N}=4$ SYM
Mandelstam, Brink-Lindgren-Nilsson.

☆ $\ast$-product does not affect these methods
for similar but different $\ast$-product for β-defm.
Ananth-Kovacs-HS

☆ Lightcone (LC) gauge

$$\begin{cases} v^{\pm} = \frac{1}{\sqrt{2}}(x^0 \pm x^3) \\ v = \frac{1}{\sqrt{2}}(x^1 + i x^2) \end{cases}$$

$A_- = 0$: gauge fix. $\longrightarrow$ A_+ canbe solved.

☆ $\mathcal{N}=4$ SYM in LC gauge

A, $\bar{A}$: gauge field

χ^m : Fermion, $m = 1, 2, 3, 4$: $SU(4)$

φ^{mn} : 6 real scalar, $\overline{\varphi^{mn}} = \frac{1}{2}\epsilon^{mn}{}_{pq}\varphi^{pq}$

☆ LC superspace

- $$\phi(x^\mu,\theta,\bar{\theta}) = -\frac{1}{\partial_-}A(y) - \frac{i}{\partial_-}\theta^m\bar{\chi}_m(y) + \frac{i}{\sqrt{2}}\theta^m\theta^n\bar{\varphi}_{mn}(y)$$
$$+\frac{\sqrt{2}}{6}\theta^m\theta^n\theta^p\epsilon_{mnpq}\chi^q(y) - \frac{1}{12}\theta^m\theta^n\theta^p\theta^q\epsilon_{mnpq}$$
$$\times\partial_-\bar{A}(y)$$

$m=1,2,3,4$: $SU(4)$.

$y^- = x^- - \frac{i}{\sqrt{2}}\theta^m\bar{\theta}_m$.

- **Action**

$$S = 72\int d^4x\int d^8\theta\,\mathrm{Tr}\Big\{-2\bar{\phi}\frac{\Box}{\partial_-^2}\phi$$
$$+i\frac{8}{3}g\Big(\frac{1}{\partial_-}\bar{\phi}[\phi,\bar{\partial}\phi] + \frac{1}{\partial_-}\phi[\bar{\phi},\partial\bar{\phi}]\Big)$$
$$+2g^2\times\Big(\frac{1}{\partial_-}[\phi,\partial_-\phi]\frac{1}{\partial_-}[\bar{\phi},\partial_-\phi] + [\phi,\bar{\phi}][\phi,\bar{\phi}]\Big)\Big\}$$

- $$\bar{\phi} = \frac{1}{48}\frac{\bar{d}^4}{\partial_-^2}\phi$$
$$\bar{d}_n = \frac{\partial}{\partial\theta^n} - \frac{i}{\sqrt{2}}\bar{\theta}_n\partial_-$$
$$\bar{d}^4 = \epsilon_{mnpq}\bar{d}^m\bar{d}^n\bar{d}^p\bar{d}^q$$

- derived by comparison to component form.

☆ LC superspace

$\underset{\substack{\shortparallel\\ \mathbb{C}}}{C^{12}} \neq 0$ gauge group $U(N)$

- $$\phi(x^\mu,\theta,\bar\theta) = -\frac{1}{\partial_-}A(y) - \frac{i}{\partial_-}\theta^m \bar\chi_m(y) + \frac{i}{\sqrt{2}}\theta^m\theta^n \bar\varphi_{mn}(y) + \frac{\sqrt{2}}{6}\theta^m\theta^n\theta^p\epsilon_{mnpq}\chi^q(y) - \frac{1}{12}\theta^m\theta^n\theta^p\theta^q\epsilon_{mnpq} \times \partial_-\bar A(y)$$

$m = 1, 2, 3, 4$: $SU(4)$.

$y^- = x^- - \frac{i}{\sqrt{2}}\theta^m\bar\theta_m$.

- **Action**

$$S = 72\int d^4x\int d^8\theta\, \mathrm{Tr}\Big\{-2\bar\phi\frac{\Box}{\partial_-^2}\phi + i\frac{8}{3}g\Big(\frac{1}{\partial_-}\bar\phi[\phi,\bar\partial\phi]_* + \frac{1}{\partial_-}\phi[\bar\phi,\partial\bar\phi]_*\Big) + 2g^2\cdot\Big(\frac{1}{\partial_-}[\phi,\partial_-\phi]_*\frac{1}{\partial_-}[\bar\phi,\partial_-\phi]_* + [\phi,\bar\phi]_*[\phi,\bar\phi]_*\Big)\Big\}$$

- $\bar\phi = \frac{1}{48}\frac{\bar d^4}{\partial_-^2}\phi$

$\bar d_n = \frac{\partial}{\partial\theta^n} - \frac{i}{\sqrt{2}}\bar\theta_n\partial_-$

$\bar d^4 = \epsilon_{mnpq}\bar d^m\bar d^n\bar d^p\bar d^q$

- $[A,B]_* = A*B - B*A$

- derived by comparison to component form

☆ Power-Counting 1st step

- propagator $\sim \underbrace{\frac{1}{p^2} d^4}_{\deg\, = 0} \delta^8(\theta_1 - \theta_2)$

$$\left(\begin{array}{l} \deg d = + \frac{1}{2} \\ \deg \partial = +1 \end{array} \right.$$

- vertices $\deg = 0$

  for ex. $\mathrm{Tr} \frac{1}{\underset{\sim}{\partial_-}} \phi \left[\frac{\bar{d}^4}{\underset{\sim}{\partial_-^2}} \phi , \underset{\sim\sim\sim}{\partial \frac{\bar{d}^4}{\partial_-^2}} \phi \right]$

- First do θ-integrals (cf. Grisaru-Siegel-Rocek)

  For each loop, $\neq 0$ contribution comes after using

$$\delta^8(\theta_1 - \theta_2) \underset{\sim\sim}{d^4 \bar{d}^4} \delta^8(\theta_1 - \theta_2) = \delta^8(\theta_1 - \theta_2)$$

  This compensates $\underset{\sim\sim}{\int d^4 p}$ → <u>deg for loop</u> $= 0$

- $\rightsquigarrow$ deg. of div ≈ 0

☆ Power-Counting 1st step

- propagator $\sim \underbrace{\frac{1}{p^2} d^4}_{\deg\,=0} \delta^8(\theta_1-\theta_2)$ $\left(\begin{array}{l}\deg d = +\frac{1}{2}\\ \deg \partial = +1\end{array}\right.$

- vertices deg = 0.

  for ex. $\mathrm{Tr}\, \underset{\sim}{\frac{1}{\partial_-}} \phi \left[\frac{\bar{d}^4}{\underset{\sim}{\partial_-^2}} \phi, \underset{\sim}{\partial \frac{\bar{d}^4}{\partial_-^2}} \phi \right]_*$

  (we omit the phase factor when counting power)

- First do θ-integrals (cf. Grisaru-Siegel-Rocek)

  For each loop, $\neq 0$ contribution comes after using

  $$\delta^8(\theta_1-\theta_2)\, \underset{\sim}{d^4}\, \bar{d}^4 \delta^8(\theta_1-\theta_2) = \delta^8(\theta_1-\theta_2)$$

  This compensates $\underset{\sim}{\int d^4p}$ → deg for loop = 0

  (non-trivial for β-defm. in Ananth-Kovacs-HS where $*$- interfere with $d, \bar{d}$)

- ⇝ deg. of div ≈ 0

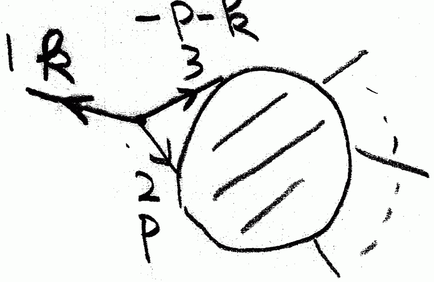
☆ Power-Counting 2nd step

- focus on a vertex connected to external line

  (☆ distinguish external and internal momentum + partial integ.

  ☆ use cancellation in the leading order)

- ex.



$$\mathrm{Tr}\Bigg(\frac{1}{\underset{\overset{\uparrow}{3}}{\partial_-}}\phi\Big(\frac{\bar\partial^4}{\underset{\overset{\uparrow}{1}}{\partial_-^2}}\phi\ \ \partial\frac{\bar\partial^4}{\underset{\overset{\uparrow}{2}}{\partial_-^2}}\phi\Big)$$

$$-\frac{1}{\underset{\overset{\uparrow}{2}}{\partial_-}}\phi\Big(\partial\frac{\bar\partial^4}{\underset{\overset{\uparrow}{3}}{\partial_-^2}}\phi\ \ \frac{\bar\partial^4}{\underset{\overset{\uparrow}{1}}{\partial_-^2}}\phi\Big)\Bigg)$$

$$\Rightarrow \frac{1}{-p_- - \cancel{k_-}}\ \frac{1}{k_-^2}\ \frac{p}{p_-^2}$$

$$-\frac{1}{p_-}\ \frac{-p-\cancel{k}}{(-p_- - \cancel{k_-})^2}\ \frac{1}{k_-^2}$$

$p \gg k$

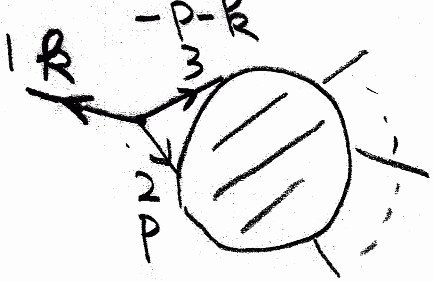
☆ Power-Counting 2nd step

- focus on a vertex connected to external line

(☆ distinguish external and internal momentum

☆ use cancellation in the leading order + partial integ.

- ex.



$$\mathrm{Tr}\left(\underset{\uparrow 3}{\frac{1}{\partial_-}}\phi\left(\underset{\uparrow 1}{\frac{\partial^4}{\partial_-^2}}\phi * \partial\underset{\uparrow 2}{\frac{\partial^4}{\partial_-^2}}\phi\right)\right.$$

$$\left.-\underset{\uparrow 2}{\frac{1}{\partial_-}}\phi\left(\partial\underset{\uparrow 3}{\frac{\partial^4}{\partial_-^2}}\phi * \underset{\uparrow 1}{\frac{\partial^4}{\partial_-^2}}\phi\right)\right)$$

$$\Rightarrow \frac{1}{-p_- - \cancel{k}_-}\frac{1}{k_-^2}\frac{p}{p_-^2}e^{ik_\mu C^{\mu\nu}p_\nu}$$

$$-\frac{1}{p_-}\frac{-p-\cancel{k}}{(-p_- - \cancel{k}_-)^2}\frac{1}{k_-^2}e^{i(-p-k)_\mu C^{\mu\nu}k_\nu}$$

$p \gg k$

$C^{\mu\nu} = -C^{\nu\mu}$

cancel. preserved !

☆ Weinberg's th.

- After Powercounting step 1, step 2. (for all subgraphs)

$$\int \underbrace{f(p)} e^{i(pCp' + \cdots)} \underbrace{dp}$$

deg. of div < 0

- Weinberg's th. assures absolute conv.

$\longrightarrow$ the integral converges uniformly (with respect to C)

- $\longrightarrow$ $C \to 0$ limit. Continuous.

Summary & Discussion

1. $C \to 0$: relevant , use NC to regularise QFT.

2. $C \to 0$: non-trivial

($\log \Lambda$ -div $\longrightarrow$ $\log \, (C \overset{\text{ext. mom.}}{k})^2$

($N=4$ NCSYM finite for $C=0$

necessary to check e^{ipCq} do not affect cancellation.

3. Proven uniform convergence.

LC superfield method : direct & powerful.

$\longrightarrow$ UV effects do not rain

continuity for $C \to 0$

- ## More on $C \to O$

☆ studied Green func. of fundamental field

$\longrightarrow$ gauge inv. operator ?

☆ Gauge independence for NC gauge theory ?

☆ Have we proven absence of "UV/IR mixing"?

No. Our focus : $C \to 0$

UV/IR mixing : $k \to 0$ (k : external mom.)

although known examples of "UV/IR mixing" such as $\log(k)$ are excluded.

☆ $C \to 0$ differentiable?

Should take $\frac{\partial}{\partial C}$ → extra p^2- factor from e^{ipCq}

→ probably not.

☆ Can we study similar aspects of membrane world volume theory?

$$H = \int (p^{\alpha})^2 + \{x^{\alpha}, x^{\beta}\}^2 + \theta^T \Gamma^{a} \{x^{\alpha}, \theta\} \, d^2\sigma$$

- non-renormalisable in usual power-counting sense.
- highly supersymmetric
- relevant for $N \to \infty$ problem of matrix model.