Eigenvalue Distributions of Matrix Models for Chern-Simons-matter Theories

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Chern-Simons-matter matrix models

A family of matrix models are defined by the partition functions:

$$Z = \int \prod_{l=1}^{n} \prod_{i_l=1}^{N_l} du_{l,i_l} \, e^{-S}$$

$$S = S_{\text{tree}} + S_{\text{vector}} + S_{\text{matter}}$$

where

$$S_{\text{tree}} = \sum_{l,i_l} \frac{k_l}{4\pi i} \left(u_{l,i_l}\right)^2$$

$$S_{\text{vector}} = -\sum_{l} \sum_{i_l < j_l} \log\left[\sinh^2 \frac{u_{l,i_l} - u_{l,j_l}}{2}\right]$$

$$S_{\text{bi-fund}} = \sum_{i_l, j_{l'}} \log\left[\cosh \frac{u_{l,i_l} - u_{l'.j_{l'}}}{2}\right]$$

They are associated with $\mathcal{N}$=3 Chern-Simons-matter theories with gauge group $\prod_l \mathrm{U}(N_l)_{k_l}$ on S^3.

[Kapustin, Willet, Yaakov]

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$$S_{\text{bi-fund}} = \sum_{i_l, j_{l'}} \log\left[\cosh \frac{u_{l,i_l} - u_{l',j_{l'}}}{2}\right]$$ ← attractive

etc.

They are associated with $\mathcal{N}=3$ Chern-Simons-matter theories with gauge group $\prod_l \mathrm{U}(N_l)_{k_l}$ on S^3.

[Kapustin, Willet, Yaakov]

Interesting quatities in CSM matrix models:

- Free energy $F_{\mathrm{CSM}}(N_l, k_l) = F_{\mathrm{mm}}(N_l, k_l)$

This was calculated for various CSM including 1/N corrections.

[Marino, Putrov][Klebanov et al.][Fuji, Hirano, Moriyama] etc.

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- Wilson loop $\langle W[C] \rangle = \left\langle \frac{1}{N} \sum_{i=1}^{N} e^{u_i} \right\rangle_{\text{mm}}$

  BPS Wilson loops were constructed and valuated perturbatively.

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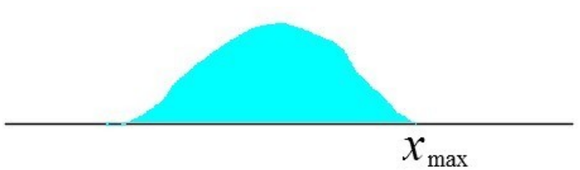
Note: Large 't Hooft coupling limit is interesting for AdS/CFT.

⇩

$$\langle W[C] \rangle = \int dx\, \rho(x) e^{x} \sim e^{x_{\max}}$$

if $x_{\max}$ is large.



Saddle-point equations

In the large N limit, the saddle-point approx. becomes exact.

$$\frac{k_1}{2\pi i}u_i = \sum_{j\neq i}^{N_1} \coth\frac{u_i - u_j}{2} - \sum_{a=1}^{N_2} \tanh\frac{u_i - v_a}{2},$$

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[ABJM]

[Gaiotto, Tomasiello]

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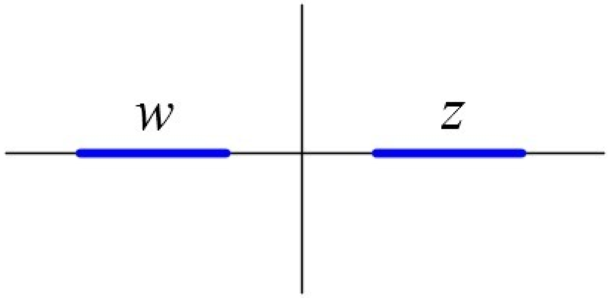
Introducing $z_i = e^{u_i}$ etc. makes these eqs. more familiar:

$$\coth\frac{u_i-u_j}{2} = 1-\frac{2z_j}{z_i-z_j}, \quad \tanh\frac{u_i-v_a}{2} = 1-\frac{2w_a}{z_i+w_a}.$$

⇨ Two-cut solution for log-type external force.



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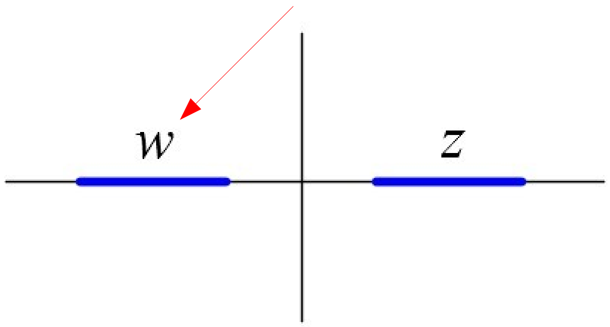


To solve the saddle-point eqs. define the resolvent:

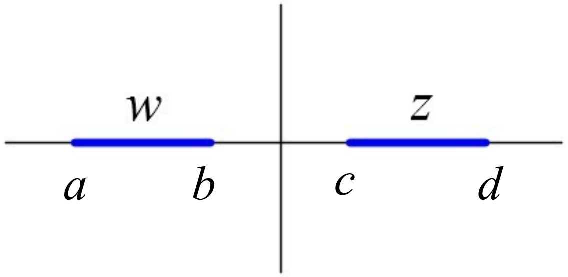
$$v(z) = t_1 \int_c^d dx\, \rho_1(x) \frac{x}{z-x} - t_2 \int_a^b dx\, \rho_2(x) \frac{x}{z-x}$$

where

$$t_1 = \frac{2\pi i N_1}{k}, \quad \rho_1(x) = \frac{1}{N_1} \sum_{i=1}^{N_1} \delta(x - z_i) \qquad \text{etc.}$$

Note: the planar limit is defined as

$$N_1, N_2, k_1, k_2 \propto k, \quad k \to \infty.$$



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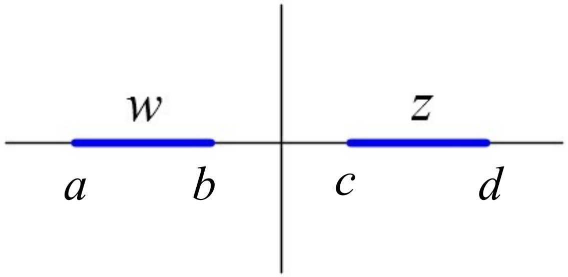

The resolvent satisfies

$$\kappa_1 \log y - t = v(y+i0) + v(y-i0), \quad (c<y<d)$$

$$\kappa_2 \log(-y) - t = v(y+i0) + v(y-i0), \quad (a<y<b)$$

where

$$t = t_1 + t_2, \quad \kappa_1 = \frac{k_1}{k}, \quad \kappa_2 = \frac{k_2}{k}.$$

An integral formula for the resolvent: [TS]

$$v(z) = \kappa_1 \int_c^d \frac{dx}{2\pi} \frac{\log\left(e^{-t/\kappa_1} x\right)}{z-x} \frac{\sqrt{(z-a)(z-b)(z-c)(z-d)}}{\sqrt{\left|(x-a)(x-b)(x-c)(x-d)\right|}}$$

$$- \kappa_2 \int_a^b \frac{dx}{2\pi} \frac{\log\left(-e^{-t/\kappa_2} x\right)}{z-x} \frac{\sqrt{(z-a)(z-b)(z-c)(z-d)}}{\sqrt{\left|(x-a)(x-b)(x-c)(x-d)\right|}}$$

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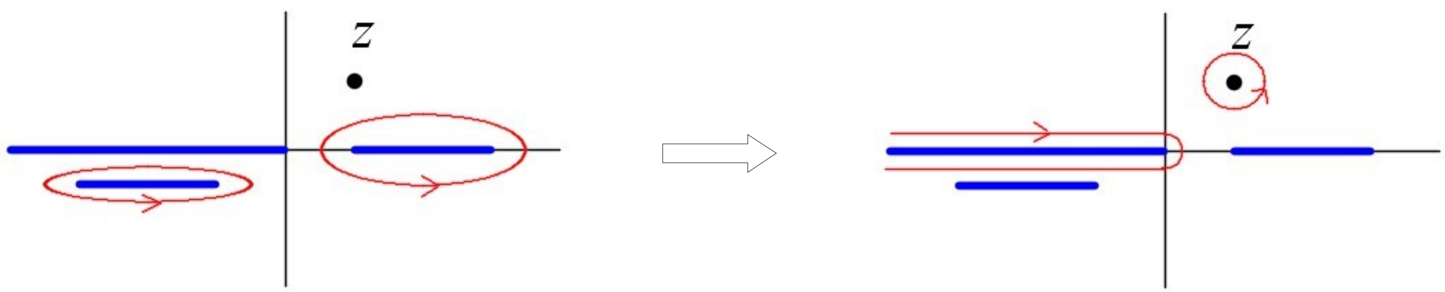
If $\kappa_1 = -\kappa_2$, then the following deformation of the contour



enables us to obtain

[Marino, Putrov]

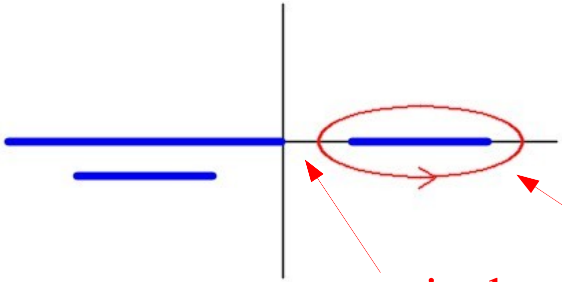
$$v(z) = \log\left[\frac{e^{-t/2}}{\sqrt{(c+d)-(a+b)}}\left(\sqrt{(z-a)(z-b)} - \sqrt{(z-c)(z-d)}\right)\right].$$

The 't Hooft couplings are derived from the resolvent as

$$t_1 = \oint_{C_{cd}} \frac{dz}{2\pi i} \frac{v(z)}{z}, \qquad t_2 = \oint_{C_{ab}} \frac{dz}{2\pi i} \frac{v(z)}{z}.$$

They are functions of a, b, c, d. t_1 will diverge iff $c \to 0$.

(simultaneously $d \to \infty$)



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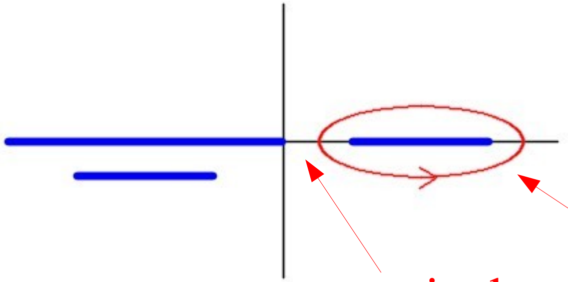

Therefore,

Large 't Hooft coupling ⟺ long branch cut

⇒ This observation enables us to derive qualitative results from the integral representation of the resolvent.

A simplification: In the limit $|a|,|d| \to \infty$,

$$\sqrt{|(x-a)(x-b)(x-c)(x-d)|} \to |x|\sqrt{|ad|},$$

for most of the range of integration.

simple!

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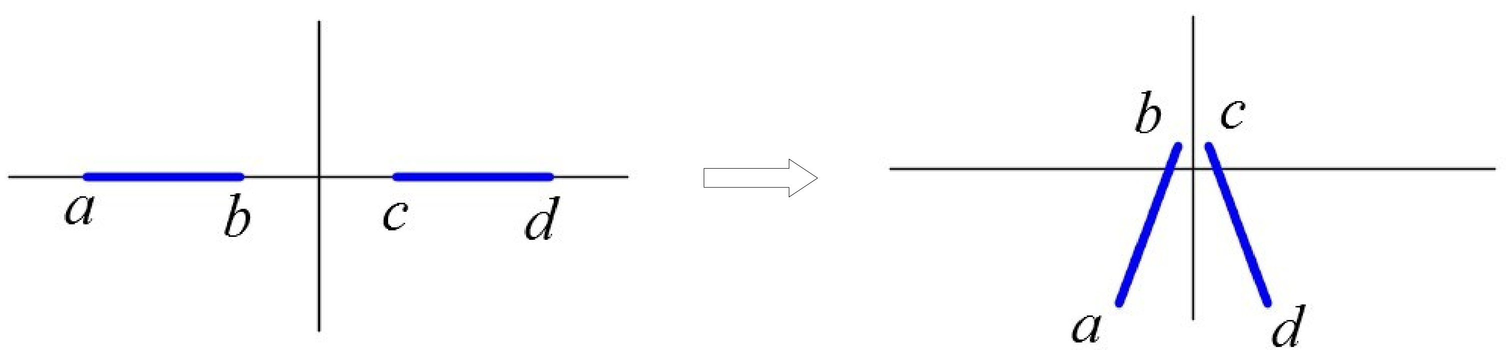
A subtlety: 't Hooft couplings must be purely imaginary while real a,b,c,d give real ones.

⇨ Integration contours have to be deformed,



while keeping $ab=1$, $cd=1$.

(Analytic continuation of the parameters.)

For a large $\alpha = \log d$,

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$$|\langle W \rangle| \sim \exp\left[\frac{\sqrt{3}}{2}\left(\frac{6\pi^3}{\kappa_1 + \kappa_2}\lambda\right)^{1/3}\right]$$

⟺ minimal surface in massive IIA

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(2) $\kappa_1 + \kappa_2 = 0$ ⇒ $c(\kappa_{1,}\kappa_2) = \frac{i}{\pi}$

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Massless/massive cases can be described uniformly.

- The large λ behavior has been determined.
- The perturbative behavior can be easily determined from saddle-point equations. [TS]
- A smooth interpolation is given by integral expression.

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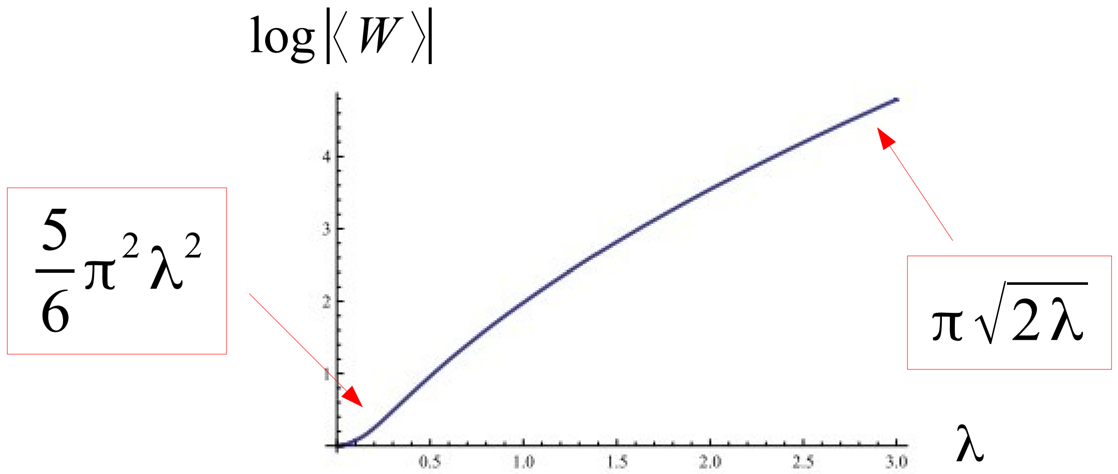
⇨ Enough information for physicists !

E.g. ABJM theory:



[Marino, Putrov]

Generalization of our method seems to be difficult...

For example,

$$\frac{k_1}{2\pi i} u_i = \sum_{j \neq i}^{N_1} \coth \frac{u_i - u_j}{2} - \frac{n_b}{2} \sum_{a=1}^{N_2} \tanh \frac{u_i - v_a}{2},$$

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corresponding to N=3 CS theory coupled to n_b bi-fund. matters.

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Note: Similarity to 2-dim. gravity coupled to $O(n)$ model,

$$V'(\phi_i) = \frac{2}{N}\sum_{j\neq i}\frac{1}{\phi_i - \phi_j} - \frac{n}{N}\sum_{j}\frac{1}{\phi_i + \phi_j}.$$

[Eynard, Kristjansen]

The case $n = 2$ is much easier than the other cases.

Summary

- Planar resolvent for a CSM theory is determined in an integral form.
- It is used to determine the large 't Hooft coupling limit which is relevant for AdS/CFT correspondence.
- Massless IIA/massive IIA are discussed in a uniform manner.
- Heavy machinery is not necessary.

Open issues:

- Generalization to more general CSM.
- Another large 't Hooft coupling behavior?
  (for models with long-range eigenvalue interactions?)
- etc.