Physics, Astrophysics, & Simulation of Gravitational Wave Sources

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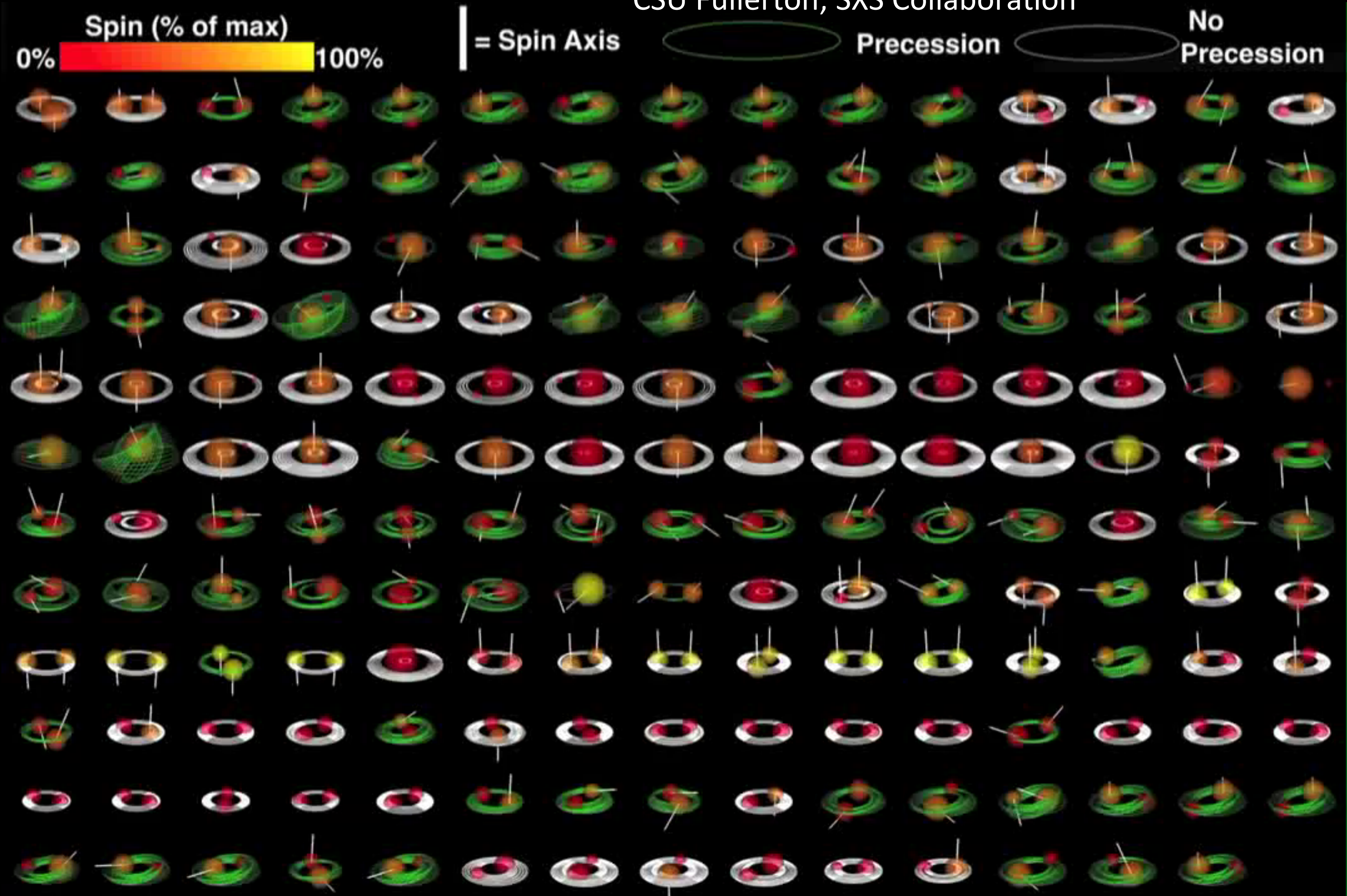
Lecture Plan

- Lecture 1 (now!)
  - (a) General Relativity & Gravitational Wave Refresher
  - (b) Overview of GW sources & phenomenology.
  - (c) Numerical relativity and general-relativistic (magneto-)hydrodynamics.
- Lecture 2 (Thursday)
  - (a) Continuation of Lecture 1, Part (c).
  - (b) Microphysics of neutron star mergers and stellar collapse.
  - (c) Neutron star mergers and Nucleosynthesis
- Lecture 3 (Friday)
  - (a) Massive star evolution, stellar collapse.
  - (b) Core-collapse supernovae and long gamma-ray bursts.
  - (c) Neutron star and black hole formation.

CSU Fullerton; SXS Collaboration



Numerical Relativity

MISNER summarized the discussion of this session: "First we assume that you have a computing machine better than anything we have now, and many programmers and a lot of money, and you want to look at a nice pretty solution of the Einstein equations. The computer wants to know from you what are the values of $g_{\mu\nu}$ and $\frac{\partial g_{\mu\nu}}{\partial t}$ at some initial surface, say at $t = 0$. Now, if you don't watch out when you specify these initial conditions, then either the programmer will shoot himself or the machine will blow up. In order to avoid this calamity you must make sure that the initial conditions which you prescribe are in accord with certain differential equations in their dependence on x, y, z at the initial time. These are what are called the "constraints." They are the equations analogous to but much more com-

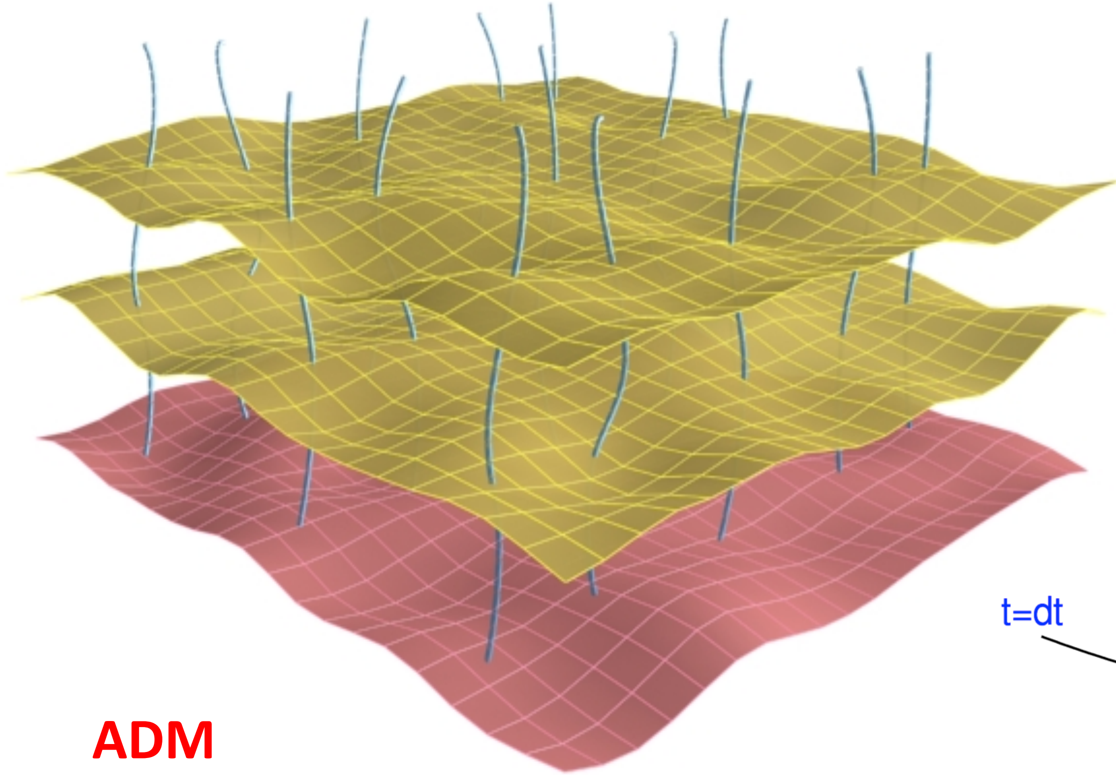
Proceedings of the GR1 Conference on the role of gravitation in physics
University of North Carolina, Chapel Hill [January 18-23, 1957]
(via Pablo Laguna & Deirdre Shoemaker)

Recommended texts:

Baumgarte & Shapiro, ***Numerical Relativity***

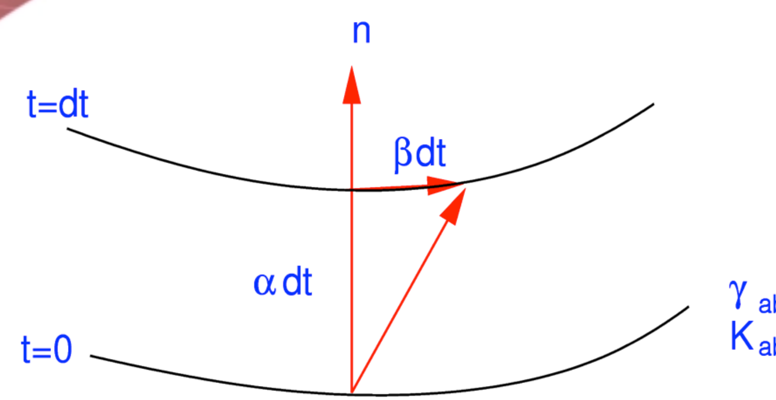
Alcubierre, ***Introduction to 3+1 Numerical Relativity***

Basic Idea of Numerical Relativity



3-hypersurface

Foliation of spacetime



ADM

3+1 split of spacetime

Figure: C. Reisswig

$$G^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu}$$

- 12 first-order hyperbolic *evolution* equations.
- 4 elliptic *constraint* equations
- 4 coordinate gauge degrees of freedom: α, β^i.

3+1 Split

3+1 split – key objects:

$$\begin{aligned} g_{\mu\nu} &= 4-\text{metric} \\ \gamma_{ij} &= 3-\text{metric} \\ \alpha &= \text{lapse function} \\ \beta^i &= \text{shift vector} \end{aligned}$$

$$\gamma = \det(\gamma_{ik})$$

$$\sqrt{-g} = \alpha\sqrt{\gamma}$$

$$g_{00} = -\alpha^2 + \beta_i\beta^i \qquad g_{0i} = \gamma_{ij}\beta^j \qquad g_{ij} = \gamma_{ij}$$

Extrinsic curvature: ≈ time derivative of 3-metric

$$\partial_t\gamma_{ij} = -2\alpha K_{ij} + \beta_{j;i} + \beta_{i;j}$$

4-velocity: u^μ ; $u^\mu u_\mu = -1$; $u^\mu = (-1, 0, 0, 0)$ in rest frame

3-velocity: Eulerian observer moving along time-like normal n^μ.
(at rest in a slice)

$$v^i = \frac{u^i}{W} + \frac{\beta^i}{\alpha}$$

$$W = (1 - v^i v_i)^{-1/2}$$

$$u^0 = \frac{W}{\alpha} \qquad u^i = W\left(v^i - \frac{\beta^i}{\alpha}\right)$$

Note: 3-velocity often defined

$$v^i = \frac{u^i}{u^0}$$

ADM Equations

(Historic: Arnowitt-Deser-Misner 1962; York 79)

Evolution System

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \beta_{j;i} + \beta_{i,j}$$

$$\partial_t K_{ij} = -\,\alpha_{;ij} + \alpha \Big[R_{ij} K K_{ij} - 2 K_{im} K^m{}_j - 8\pi \big(S_{ij} - \frac{1}{2}\gamma_{ij} S\big) - 4\pi \rho_{\mathrm{ADM}} \gamma_{ij} \Big] + \beta^m K_{ij;m} + K_{im}\beta^m{}_{;j} + K_{mj}\beta^m{}_{;i}$$

$$S^i = -\gamma^{i\mu} n^\nu T_{\mu\nu} \qquad \rho_{\mathrm{ADM}} = n_\mu n_\nu T^{\mu\nu}$$

$$S_{ij} = \gamma_{i\mu}\gamma_{j\nu} T^{\mu\nu} \qquad S, K - \text{traces of } S_{ij}, K_{ij}$$

Constraint Equations:

Hamiltonian $$R + K^2 - K_{ij}K^{ij} - 16\pi\rho_{\mathrm{ADM}} = 0$$

Momentum $$K^{ij}{}_{;j} - \gamma^{ij} K_{;j} - 8\pi S^i = 0$$

Cauchy Evolution

Initial boundary value problem.

Evolve
forward
in time &
monitor
constraints.

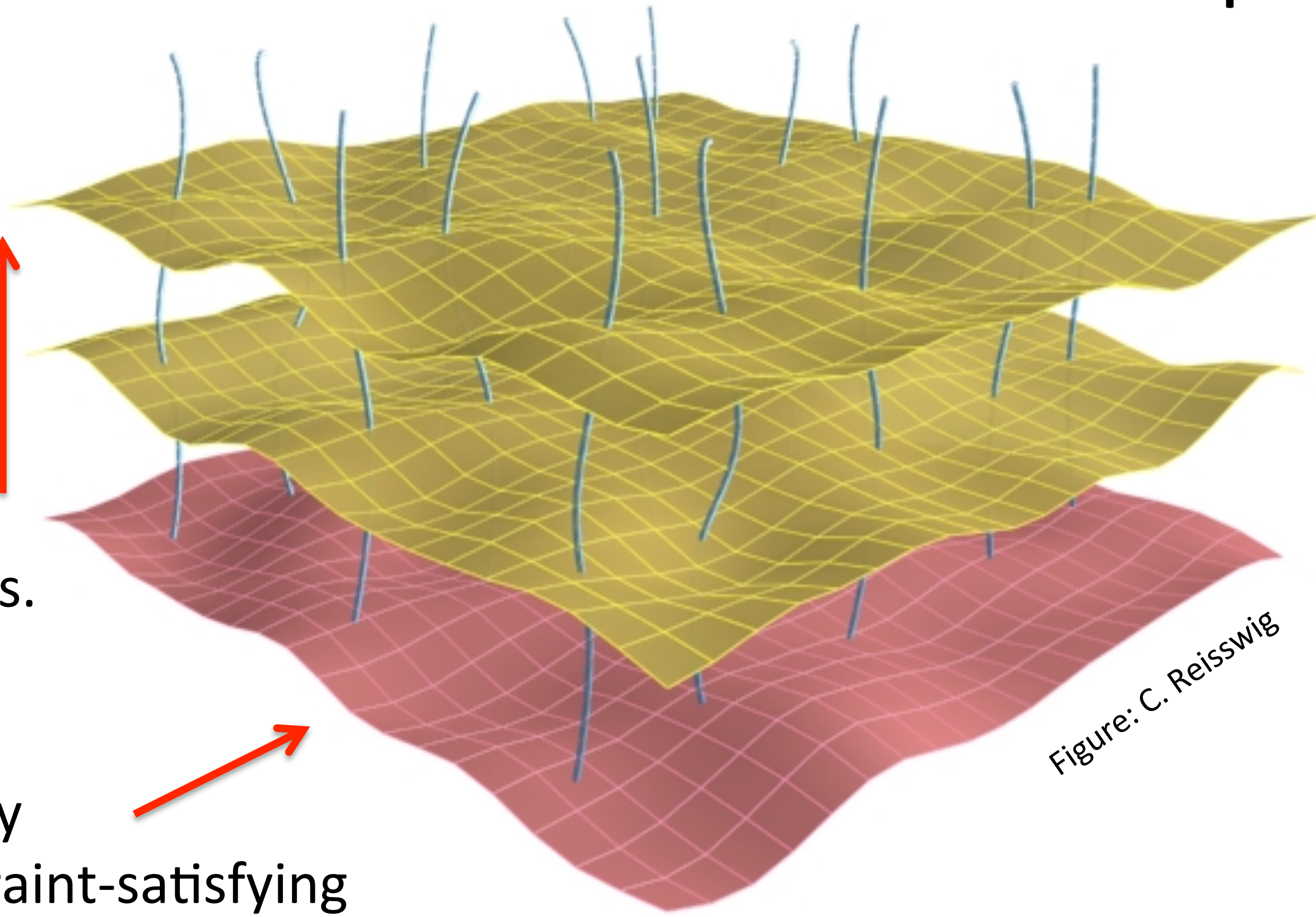
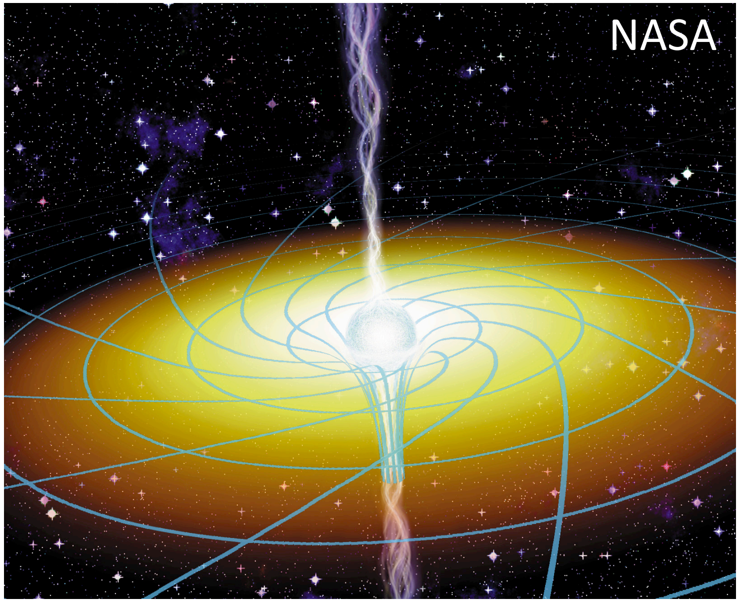


Figure: C. Reisswig

Specify
constraint-satisfying
initial data & boundary values.

Practical Numerical Relativity

Have not yet specified gauge conditions: Anything goes?



- GR dynamics will twist, squeeze, stretch coordinates.
- GR can develop coordinate singularities and physical singularities.
- For numerically stable evolution, must avoid singularities and control coordinate distortion.

- Spherical symmetry (1D):
  -> no radiative degrees of freedom, fully constrained evolution.
  -> ADM with simple gauge choices: no problem.

- **ADM form of the Einstein equations is unstable in 2D/3D**!
  -> ***well-posedness*** issues.
  -> small errors in constraints get amplified exponentially over time!

Well-Posedness (1)

Hadamard 1902:

A problem is well-posed if and only if:

- a solution exists;
- the solution is unique;
- the solution depends continuously on the initial and boundary data.

Example (1): Hadamard 1923

$$\partial_t^2 u - \partial_x^2 u = 0 \qquad x \in [0,1]$$

ID: $u|_{t=0} = 0 \quad \partial_t u|_{t=0} = \dfrac{\sin(2\pi n x)}{(2\pi n)^P} \quad P \geq 1$

BC: $u(x=0) = u(x=1) = 0$

Solution: $u(x,t) = \dfrac{\sin(2\pi n t)\sin(2\pi n x)}{(2\pi n)^{P+1}}$

For $n \to \infty \quad \mathrm{ID} \to 0 \quad u(x,t) \to 0$ well-posed

Well-Posedness (2)

Example (2): only change

$$\partial_t^2 u + \partial_x^2 u = 0 \qquad x \in [0,1]$$

ID: $u|_{t=0} = 0 \quad \partial_t u|_{t=0} = \frac{\sin(2\pi n x)}{(2\pi n)^P} \qquad P \geq 1$

BC: $u(x=0) = u(x=1) = 0$

recall:

$$\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$$

Solution: $u(x,t) = \frac{\sinh(2\pi n t)\sin(2\pi n x)}{(2\pi n)^{P+1}}$

For $n \to \infty \qquad \mathrm{ID} \to 0$

BUT $u(x,t) \to \infty \qquad$ ill posed

Given above eqn. + any ID and BC, one can introduce a small perturbation that leads to arbitrarily large solution at finite time.

Alternative Formulations that actually work:

- ADM is ill posed and boundary conditions unclear. (Kidder+01, Nagy+04) (-> ADM is called "weakly hyperbolic" in PDE theory).
- Want evolution system that is symmetric/strongly hyperbolic (well posed + clear boundary conditions)

BSSN Formulation

Nakamura+87, Shibata & Nakamura 95, Baumgarte & Shapiro 99

- Conformal-traceless reformulation of ADM.
- Additional evolution equations, conditionally **strongly hyperbolic**.
- Sensitive to gauge choice.
- Most widely used evolution system today.

Generalized Harmonic Formulation

Friedrich 1985, Pretorius 2005, Lindblom+ 2006

- Choice of coordinates that reduces Einstein equations to a set of inhomogeneous wave equations. **Symmetric hyperbolic**.
- Used primarily by Caltech/Cornell SXS code SpEC.

Baumgarte-Shapiro-**Shibata-Nakamura** (BSSN)

Nakamura+87, Shibata & Nakamura 95, Baumgarte & Shapiro 99

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Evolution of three-dimensional gravitational waves: Harmonic slicing case

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(Received 7 April 1995)

We perform numerical simulations of a three-dimensional (3D) time evolution of pure gravitational waves. We use a conformally flat and $K = 0$ initial condition for the evolution of the spacetime. We adopt several slicing conditions to check whether a long time integration is possible in those conditions. For the case in which the amplitude of the gravitational waves is low, a long time integration is possible by using the harmonic slice and the maximal slice, while in the geodesic slice ($\alpha = 1$) it is not possible. As in the axisymmetric case and also in the 3D case, gravitational waves with a sufficiently high amplitude collapse by their self-gravity and their final fates seem to be as black holes. In this case, the singularity avoidance property of the harmonic slice seems weak, so that it may be inappropriate for the formation problems of the black hole. By means of the gauge-invariant wave extraction technique we compute the waveform of the gravitational waves at an outer region. We find that the nonlinearity of Einstein gravity induces the higher multipole modes even if only a quadrupole mode exists initially.

Baumgarte-Shapiro-**Shibata-Nakamura** (BSSN)

Nakamura+87, Shibata & Nakamura 95, Baumgarte & Shapiro 99

Heuristic approach: rewrite ADM equations to find more stable evolutions, e.g., keep some terms that drop out in regular ADM.

Conformal-tracefree decomposition:

(1) $\tilde{\gamma}_{ij} = e^{-4\phi}\gamma_{ij}$ with requirement $\det \tilde{\gamma}_{ij} = \tilde{\gamma} = 1$

$$\longrightarrow \phi = \frac{1}{12}\ln\gamma$$

(2) Define $K_{ij} = e^{4\phi}\left(\tilde{A}_{ij} + \frac{1}{3}\tilde{\gamma}_{ij}K\right)$ $\quad \tilde{A}^i{}_i = \tilde{A} = 0$

Now evolve $\tilde{\gamma}_{ij}, \tilde{A}_{ij}, \phi, K$ instead of γ_{ij}, K_{ij}

-> additional evolution equations

Baumgarte-Shapiro-**Shibata-Nakamura** (BSSN)

Nakamura+87, Shibata & Nakamura 95, Baumgarte & Shapiro 99

$$\partial_t \phi = -\frac{1}{6}\alpha K + \frac{1}{6}\beta^i{}_{,i} + \beta^i \phi_{,i}$$

$$\partial_t K = -\tilde{\gamma}^{ij}\tilde{\nabla}_i\tilde{\nabla}_j\alpha + \alpha\left[\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2 + 4\pi(\rho_{\mathrm{ADM}} + S)\right] + \beta^i K_{,i}$$

covariant deriv. wrt. conformal 3-metric

$$\partial_t \tilde{\gamma}_{ij} = -2\alpha\tilde{A}_{ij} + \beta^k\tilde{\gamma}_{ij,k} + \tilde{\gamma}_{ik}\beta^k{}_{,j} + \tilde{\gamma}_{kj}\beta^k{}_{,i} - \frac{2}{3}\tilde{\gamma}_{ij}\beta^k{}_{,k}$$

$$\begin{aligned}\partial_t \tilde{A}_{ij} = e^{-4\phi}&\left[-(\tilde{\nabla}_i\tilde{\nabla}_j\alpha) + \alpha(R_{ij} - 8\pi S_{ij})\right]^{\mathrm{TF}} \\ &+ \alpha(K\tilde{A}_{ij} - 2\tilde{A}_{il}\tilde{A}^l{}_j) \\ &+ \beta^k\tilde{A}_{ij,k} + \tilde{A}_{ik}\beta^k{}_{,j} + \tilde{A}_{kj}\beta^k{}_{,i} - \frac{2}{3}\tilde{A}_{ij}\beta^k{}_{,k}\end{aligned}$$

tracefree

Baumgarte-Shapiro-**Shibata-Nakamura** (BSSN)

Nakamura+87, Shibata & Nakamura 95, Baumgarte & Shapiro 99

Not done yet: $$R_{ij} = R^{\phi}_{ij} + \tilde{R}_{ij}$$

$$R^{\phi}_{ij} = -2\tilde{\nabla}_i\tilde{\nabla}_j\phi - 2\tilde{\gamma}_{ij}\tilde{\nabla}^l\tilde{\nabla}_l\phi + 4\tilde{\nabla}_i\phi\tilde{\nabla}_j\phi - 4\tilde{\gamma}_{ij}\tilde{\nabla}^l\phi\tilde{\nabla}_l\phi$$

Define: $\tilde{\Gamma}^i := \tilde{\gamma}^{jk}\tilde{\Gamma}^i_{jk} = -\tilde{\gamma}^{ij}{}_{,j}$ "conformal connection functions"

(requires $\tilde{\gamma} = 1$)

$$U_{(ij)k} = \frac{1}{2}\left(U_{ijk} + U_{jik}\right)$$

$$\tilde{R}_{ij} = -\frac{1}{2}\tilde{\gamma}^{lm}\tilde{\gamma}_{ij,lm} + \tilde{\gamma}_{k(i}\tilde{\Gamma}^k{}_{,j)} + \tilde{\Gamma}^k\tilde{\Gamma}_{(ij)k} + \tilde{\gamma}^{lm}\left(2\tilde{\Gamma}^k_{l(i}\tilde{\Gamma}_{j)km} + \tilde{\Gamma}^k_{im}\tilde{\Gamma}_{klj}\right)$$

Only term containing explicit second derivatives of the metric. Essentially Laplacian; principal part in $\partial_t\tilde{A}_{ij}$

Now: advantageous for stability to promote $\tilde{\Gamma}^i$ to evolved variables.

Baumgarte-Shapiro-**Shibata-Nakamura** (BSSN)

Nakamura+87, Shibata & Nakamura 95, Baumgarte & Shapiro 99

Why is this a good idea? Consider wave equation:

$$U_{,tt} = U_{,xx} = \Delta U$$

OR:

$$\partial_t U = \Pi$$

$$\partial_t \Pi = \Delta \phi$$

With $\tilde{\Gamma}^i$ as independent variables, the BSSN equations become

$$\partial_t \tilde{\gamma}_{ij} \propto \tilde{A}_{ij}$$

$$\partial_t \tilde{A}_{ij} \propto \Delta \tilde{\gamma}_{ij}$$

-> more similar to symmetric hyperbolic wave equation!

Baumgarte-Shapiro-**Shibata-Nakamura** (BSSN)

Nakamura+87, Shibata & Nakamura 95, Baumgarte & Shapiro 99

So 3 more evolution equations:

$$\partial_t \tilde{\Gamma}^i = -2\tilde{A}^{ij}\alpha_{,j} + 2\alpha\left(\tilde{\Gamma}^i_{jk}\tilde{A}^{kj} - \frac{2}{3}\tilde{\gamma}^{ij}K_{,j} - 8\pi\tilde{\gamma}^{ij}S_j + 6\tilde{A}^{ij}\phi_{,j}\right)$$
$$+\beta^j\tilde{\Gamma}^i{}_{,j} - \tilde{\Gamma}^j\beta^i{}_{,j} + \frac{2}{3}\tilde{\Gamma}^i\beta^j{}_{,j} + \frac{1}{3}\tilde{\gamma}^{li}\beta^j{}_{,jl} + \tilde{\gamma}^{lj}\beta^i{}_{,lj}$$

(note that here a term involving $\partial_j \tilde{A}^{ij}$ was eliminated with the help of the momentum constraint.)

One can show that BSSN is strongly hyperbolic for fixed α, β^i.
(Sarbach+02, Nagy+04)

For non-static gauge, BSSN is strongly hyperbolic as long as shift stays "small" (not close to 1). (Gundlach & Martin-Garcia 06).

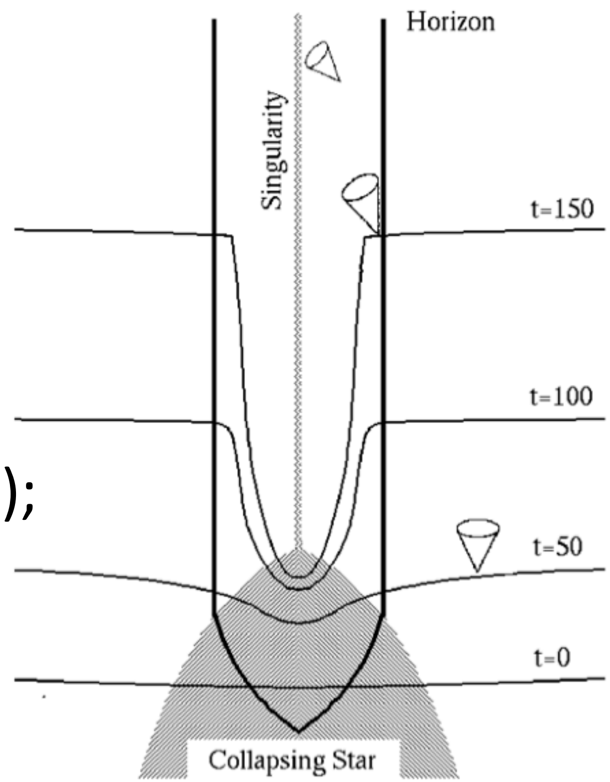
Price paid for stability: 5 additional evolution equations!

BSSN: Gauge Choices (1)

What about lapse function and shift vector?

Turns out: for BSSN, choice of gauge influences hyperbolicity of the evolution system -> after BSSN, took another 5-10 years to find good gauge.

Geodesic Slicing: $\alpha = 1, \beta^i = 0$ (SN95: bad idea! Coord. singularities!)



Can do better! Consider what learned in 1970s-90s about gauges for ADM.

A "good" gauge condition for BSSN should be:

- manifestly hyperbolic (in PDE nature);
- singularity avoiding ($\alpha \to 0$ near singularity);
- minimizing distortion/stretching of spatial coordinates.

(see, e.g., Alcubierre+03, Baumgarte & Shapiro book)

BSSN Gauge Choices (2)

(see, e.g., Alcubierre+03, Baumgarte & Shapiro book)

Maximal slicing: $K = 0$

$$\gamma^{ij}\nabla_i\nabla_j\alpha = \alpha\left(K_{ij}K^{ij} + 4\pi(\rho_{\mathrm{ADM}} + S)\right)$$

Condition on lapse, shift still free.
Leads to collapse of lapse inside BH horizon, but elliptic & late-time blow-up of γ_{rr} (grid stretching).

"1+log" slicing: $\partial_t\alpha = -\alpha^2\frac{2}{\alpha}K + \beta^k\alpha_{,k}$

(hyperbolic "driver" slicing condition; drives lapse towards 0 near singularity, not must give initial data for lapse -> e.g., via maximal slicing.)

Aside: "1+log" because: $\beta^i = 0 \quad \partial_t\alpha = -2\alpha K$

$$\rightarrow \alpha = 1 + \ln\gamma \ (\text{via ADM})$$

BSSN Gauge Choices (3)

(see, e.g., Alcubierre+03, Baumgarte & Shapiro book)

Shift conditions:

Look for hyperbolic variants of the elliptic "minimal distortion" shift.
(Smarr & York 78)

$$\nabla_i \Sigma^{ij} = 0 \quad \Sigma_{ij} := \frac{1}{2}\gamma^{1/3}\partial_t \tilde{\gamma}_{ij}$$

("distortion tensor")

BSSN: minimal distortion -> "Gamma-freezing": $\partial_t \tilde{\Gamma}^i = 0$
(but still elliptic)

Family of hyperbolic "Gamma-driver" shift conditions:

$$\partial_t \beta^i = \xi B_i$$

$$\partial_t B_i = \chi \partial_t \tilde{\Gamma}^i - \eta B^i - \zeta \beta^j \tilde{\Gamma}^j{}_{,j}$$

ξ, η, χ, ζ parameters, e.g., $\xi = \frac{3}{4}, \chi = 1, \eta = \frac{1}{2M}, \zeta = 0$

Simple, first-order parameter-free variant (Shibata 2003):

$$\partial_t \beta^i = \tilde{\Gamma}^i + \Delta t \cdot \partial_t \tilde{\Gamma}^i$$

Further Issues not addressed here:

- Initial data: finding solutions to the constraint equations.
- Boundary conditions.
- Gravitational wave extraction.
- (Apparent & Event) Horizon finding, spacetime diagnostics (measuring angular & linear momentum, mass-energy etc.).
- Singularity excision.
- Numerical methods for solving the Einstein equations: finite differences, pseudo-spectral, discontinuous Galerkin

Recommended texts:

Baumgarte & Shapiro, ***Numerical Relativity***
Alcubierre, ***Introduction to 3+1 Numerical Relativity***

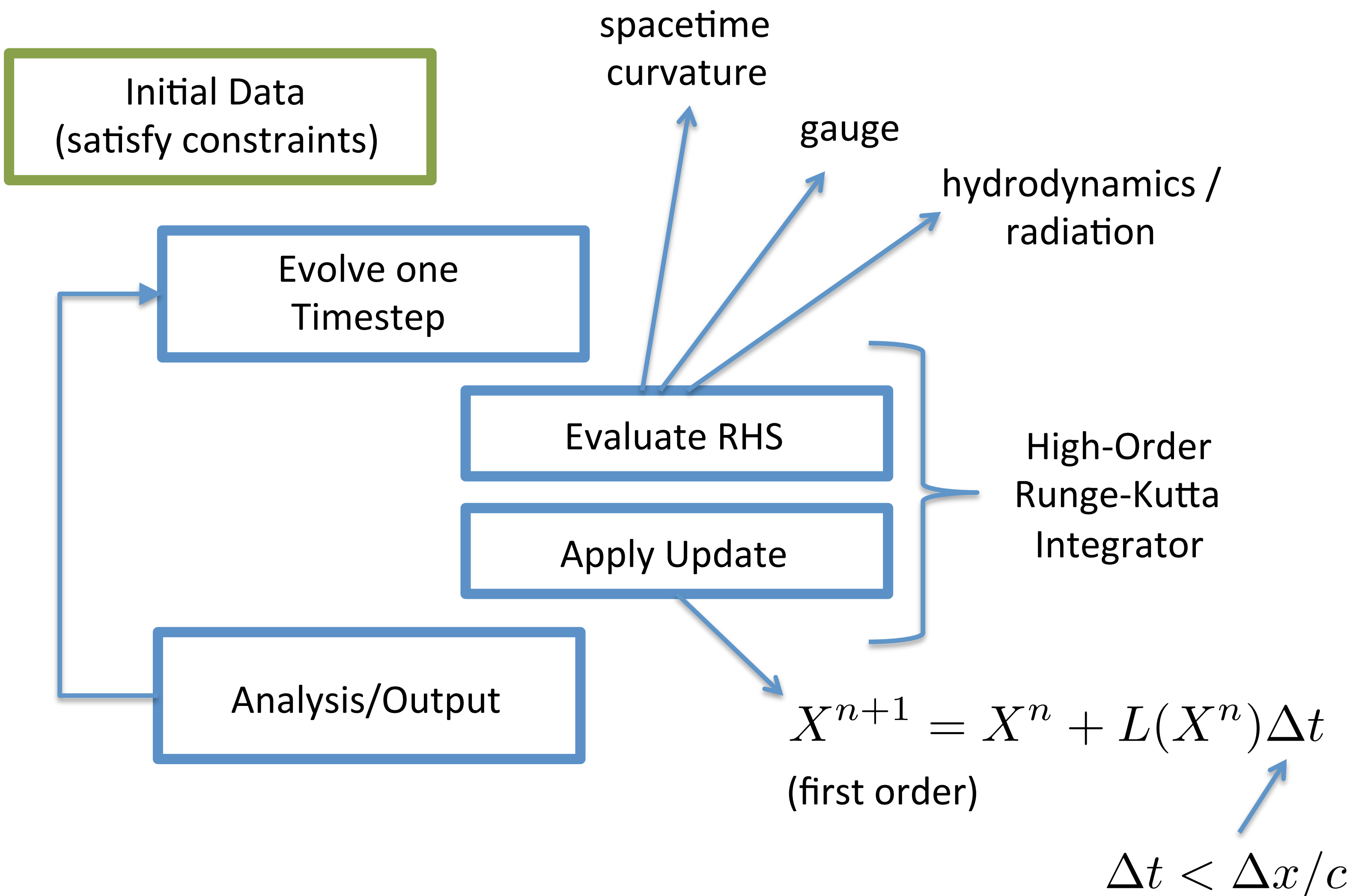
Putting it all together:

- Initial data
- Evolution system, gauge / gauge evolution, boundary conditions
- Discretization scheme for space and time – typically fourth order, central
- Common approach: **Method of Lines**
  Treat problem semi-discrete; discretize in space, then treat as ODE, integrate in time via Runge-Kutta (or similar)

$$\frac{d}{dt}L(q) = \mathrm{RHS}$$

- Analysis code (constraints, horizon finding etc.)

Schematic Numerical Relativity Simulation



Complication: Adaptive Mesh Refinement

Relativistic Fluid Dynamics (and MHD)

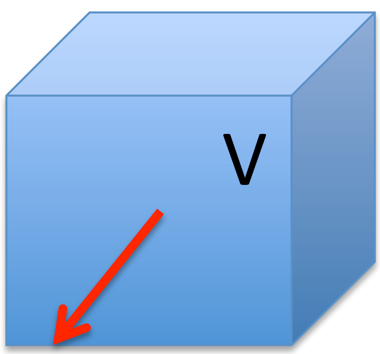
Reminder: Newtonian Fluid Dynamics

- Continuum assumption. System of size L

Can define macroscopic fluid element with size $l \ll L$
with averaged state variables $\rho, v, \epsilon, T, P, \ldots$
Mean free path of fluid particles $\lambda \ll l$, from which follows $\tau_{\mathrm{equil}} \ll \tau_{\mathrm{dyn}}$

- Conservation laws for mass, momentum, and energy

Mass conservation



$$d\mathbf{\Sigma} = \mathbf{n}dA$$

# of particles in V

$$N = \int_V n dV$$

density

$$\frac{d}{dt}N = \frac{d}{dt}\int_V n dV$$

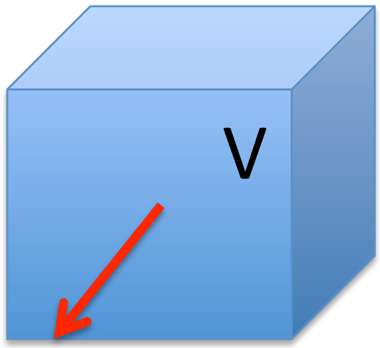
flux of particles $\mathbf{j} = n\mathbf{v}$

$$\frac{d}{dt}\int_V n dV + \int_{\partial V} \mathbf{j} d\mathbf{\Sigma} = 0$$

change of N

# of particles passing through surface

Reminder: Newtonian Fluid Dynamics



$$d\mathbf{\Sigma} = \mathbf{n} dA$$

$$\frac{d}{dt}\int_V n dV + \int_{\partial V} \mathbf{j} d\mathbf{\Sigma} = 0$$

Divergence Theorem

$$\int_V (\nabla \cdot \mathbf{A}) dV = \int_{\partial V} \mathbf{A} d\mathbf{\Sigma}$$

$$\int_V \frac{\partial}{\partial t} n \, dV + \int_V \nabla \cdot \mathbf{j} \, dV = 0$$

$$\frac{\partial}{\partial t} n + \nabla \cdot \mathbf{j} = 0 \qquad \rho = mn$$

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla(\rho \mathbf{v}) = 0}$$

continuity equation

Reminder: Newtonian Fluid Dynamics

Continuity Equation
(mass conservation)

$$\frac{\partial \rho}{\partial t} + \nabla(\rho \mathbf{v}) = 0$$

Euler Equation
(momentum cons.)

$$\frac{\partial(\rho \mathbf{v})}{\partial t} + \nabla(\rho \mathbf{v}\mathbf{v} + P) = \rho \mathbf{g}$$

Energy Equation
(energy cons.)

$$\frac{\partial}{\partial t}\mathcal{E} + \nabla((\mathcal{E} + P)\mathbf{v}) = \rho \mathbf{v}\mathbf{g}$$

$$\mathcal{E} = \rho\epsilon + \frac{1}{2}\rho v^2$$

Equation of State (EOS)

$$P = P(\rho, \epsilon, X_i)$$

introduce
a 3-stress tensor

$$\underline{\mathbf{T}} = \rho \mathbf{v} \otimes \mathbf{v} + P\mathbb{1} \quad \longrightarrow \quad \frac{\partial(\rho \mathbf{v})}{\partial t} + \nabla \cdot \underline{\mathbf{T}} = 0$$

GR Hydrodynamics

(neglecting magnetic fields for now)

Recall Einstein equation:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu}$$

curvature

source of curvature

(due to Bianchi identity)

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \qquad \nabla^\mu G_{\mu\nu} = G_{\mu\nu}{}^{;\mu} = 0$$

$$\nabla^\mu T_{\mu\nu} = 0$$

equations of motion

GR Hydrodynamics

(neglecting magnetic fields for now)

$$j^\mu = \rho u^\mu$$ mass flux

$$T^{\mu\nu} = \rho h u^\mu u^\nu + P g^{\mu\nu}$$

Stress Energy Tensor of an ideal fluid (inviscid, no magnetic field)

$$h = 1 + \epsilon + P/\rho$$

relativistic specific enthalpy

Conservation of mass, momentum, and energy:

$$j^\mu{}_{;\mu} = (\rho u^\mu)_{;\mu} = 0$$

Mass conservation

$$T^{\mu\nu}{}_{;\mu} = (\rho h u^\mu u^\nu + P g^{\mu\nu})_{;\mu} = 0$$

Energy-momentum conservation

GR Hydrodynamics

$$\gamma = \det(\gamma_{ik})$$

$$\sqrt{-g} = \alpha\sqrt{\gamma}$$

Flux-conservative Formulation:

e.g., Banyuls+97, Font 08

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}^{\,i}}{\partial x^{\,i}} = \mathbf{S}$$

$$W = (1 - v^i v_i)^{-1/2}$$

$$\mathbf{U} = [\hat{D}, \hat{S}_j, \hat{\tau}] \quad \hat{D} = \sqrt{\gamma}\rho W,$$

conserved mass

$$\hat{S}^i = \sqrt{\gamma}\rho h W^{\,2} v^{\,i},$$

conserved momenta

$$\hat{\tau} = \sqrt{\gamma}\left(\rho h W^{\,2} - P\right) - D$$

conserved energy

$$\mathbf{F}^{\,i} = \alpha\left[\hat{D}\tilde{v}^{\,i}, \hat{S}_j\tilde{v}^{\,i} + \delta^{\,i}_j P, \hat{\tau}\tilde{v}^{\,i} + P v^{\,i}\right]$$

fluxes

$$\tilde{v}^i = v^i - \beta^i/\alpha$$

$$\mathbf{S} = \alpha\left[0, T^{\mu\nu}\left(\frac{\partial g_{\nu j}}{\partial x^{\,\mu}} - \Gamma^{\lambda}_{\mu\nu} g_{\lambda j}\right),\right.$$

$$\left.\alpha\left(T^{\mu 0}\frac{\partial \ln\alpha}{\partial x^{\,\mu}} - T^{\mu\nu}\Gamma^{\,0}_{\mu\nu}\right)\right]$$

curvature source
"gravitational
acceleration"
+ any interaction
terms etc.

+ Equation of state (EOS): $P = P(\rho, \epsilon, X_i)$

GR Hydrodynamics

Primitive and Conserved Variables

$\rho, T, \epsilon, Y_e, v^i$ primitive

$\hat{D}, \hat{\tau}, \hat{D}Y_e, \hat{S}_i$ conserved

Why worry about primitive vars at all?

-> EOS is a function of the prim. vars!

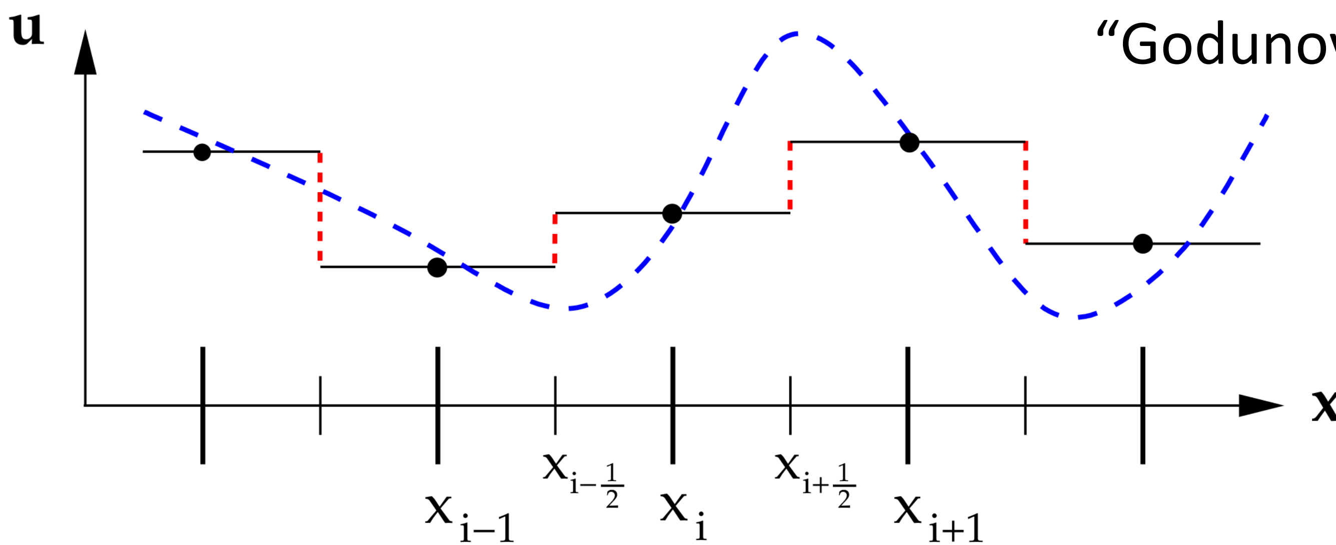
Consequence:

Must compute primitive variables from evolved conserved variables after each step!

There is no closed expression
-> must use Newton iteration to solve for primitive variables.

(This makes GR Hydro (and SR hydro) more expensive than Newtonian hydro)

Numerical Approach: Finite Volume

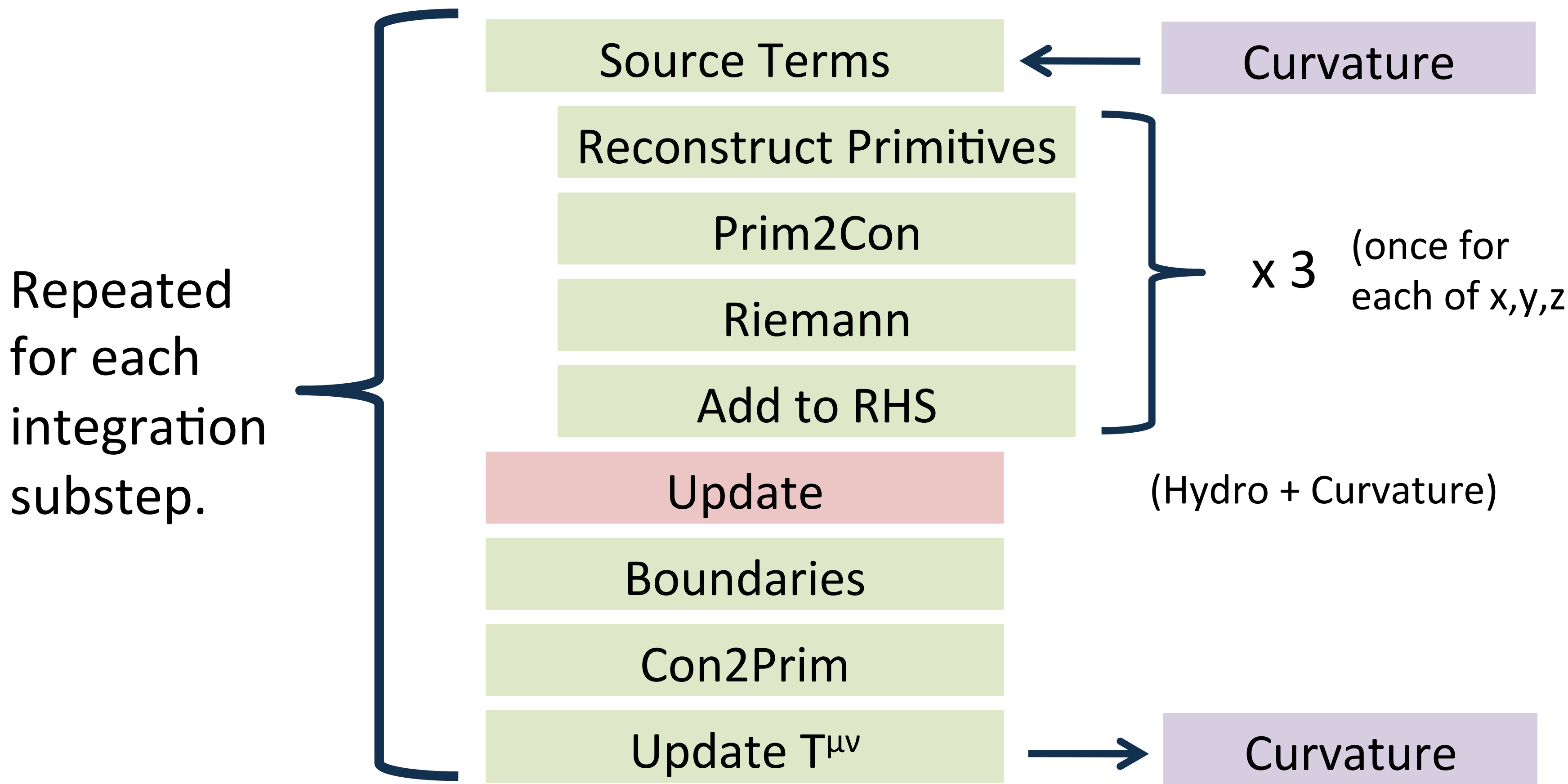


"Godunov method"

- Represent data as cell averages at cell centers.
- Reconstruct data to cell interfaces.
- Compute physical fluxes by solution of local Riemann shock problems.
- Update cell center values by difference of left and right fluxes.

More details on formulations and numerics:
Font 2008, Liv. Rev. Rel. 2008-7
Rezzolla & Zanotti, *Relativistic Hydrodynamics*

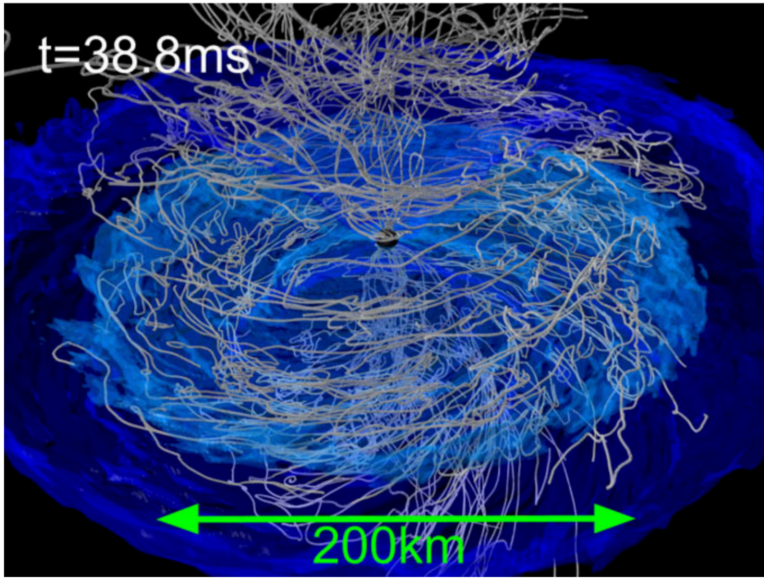
GR Hydrodynamics Flowchart



Magnetohydrodynamics (MHD)

- Magnetic (B) fields are everywhere in astrophysics.
- Non-uniform fluid motion will always amplify field.
- B-field likely dynamically relevant in NSNS & BHNS, and core-collapse supernovae.

Kiuchi+14



Newtonian MHD: (no net charge, E ~ v/c B in lab frame, E = 0 in rest frame, no displacement currents, $v/c \ll 1$)

Induction Equation

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{B} \times \mathbf{v}) = -\nabla \times (\eta \nabla \times \mathbf{B})$$

Constraint

$$\nabla \cdot \mathbf{B} = 0$$

resistivity; 0 in **ideal** MHD

Lorentz Force

$$\mathbf{f}_\mathrm{L} = \frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B} = \frac{1}{4\pi}(\mathbf{B} \cdot \nabla)\,\mathbf{B} - \frac{1}{8\pi}\nabla\left(B^2\right)$$

(Ideal) General-Relativistic MHD

Font 08, Shibata & Font 05, Kiuchi+12, Etienne+12, Palenzuela+12, Moesta+14, ...

Faraday Tensor

$$F^{\mu\nu} = u^\mu E^\nu - u^\nu E^\mu - \epsilon^{\mu\nu\lambda\delta} u_\lambda B_\delta \qquad {}^*F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\delta} F_{\lambda\delta}$$

MHD approx: E-field in fluid rest frame vanishes: -> $F^{\mu\nu} u_\nu = 0$

$$T^{\mu\nu} = T^{\mu\nu}_{\mathrm{fluid}} + T^{\mu\nu}_{\mathrm{EM}}$$

$$T^{\mu\nu}_{\mathrm{fluid}} = \rho h u^\mu u^\nu + P g^{\mu\nu} \qquad h = 1 + \epsilon + P/\rho$$

$$T^{\mu\nu}_{\mathrm{EM}} = F^{\mu\lambda} F^\nu_\lambda - \frac{1}{4} g^{\mu\nu} F^{\lambda\delta} F_{\lambda\delta} = b^2 u^\mu u^\nu + \frac{1}{2} g^{\mu\nu} b^2 - b^\mu b^\nu$$

$b^\mu = u_\nu {}^*F^{\mu\nu}$ magnetic field in fluid rest frame

Eulerian observer B-field: $B^i = n_\mu {}^*F^{i\mu} = -\alpha {}^*F^{i0}$ $\quad n^\mu$ unit normal on slice

Equations of motion: $\nabla_\mu j^\mu = 0 \qquad \nabla_\mu T^{\mu\nu} = 0 \qquad \nabla_\nu {}^*F^{\mu\nu} = 0$

(Ideal) General-Relativistic MHD

Font 08, Shibata & Font 05, Kiuchi+12, Etienne+12, Palenzuela+12, **Moesta+14**, ...

Conserved variables:

$$\hat{D} = \sqrt{\gamma}\rho W$$

$$\hat{S}_j = \sqrt{\gamma}(\rho h^* W^2 v_j - \alpha b^0 b_j)$$

$$\hat{\tau} = \sqrt{\gamma}(\rho h^* W^2 - P^* - (\alpha b^0)^2) - \hat{D}$$

$$\hat{B}^k = \sqrt{\gamma} B^k$$

$$h^* = 1 + \epsilon + (P + b^2)/\rho$$

$$P^* = P + P_m = P + b^2/2$$

Evolution equations: $$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}^{\,i}}{\partial x^{\,i}} = \mathbf{S} \qquad \mathbf{U} = [\hat{D}, \hat{S}_j, \hat{\tau}, \hat{B}^k]$$

$$\mathbf{F}^i = \alpha \begin{bmatrix} \hat{D}\tilde{v}^i \\ S_j \tilde{v}^i + \sqrt{\gamma} P^* \delta^i_j - b_j \hat{B}^i / W \\ \tau \tilde{v}^i + \sqrt{\gamma} P^* v^i - \alpha b^0 \hat{B}^i / W \\ \hat{B}^k \tilde{v}^i - \hat{B}^i \tilde{v}^k \end{bmatrix}$$

$$\mathbf{S} = \text{as for GRHD}$$

(no source for B)

(Ideal) General-Relativistic MHD

Font 08, Shibata & Font 05, Kiuchi+12, Etienne+12, Palenzuela+12, **Moesta+14**, ...

Constraint:

$$\nabla \cdot \mathbf{B} = \frac{1}{\sqrt{\gamma}}(\sqrt{\gamma}B^i)_{,i} = 0 \quad \rightarrow \hat{B}_{,i} = 0$$

- Enforcing constraint numerically -> highly non-trivial.
- Various numerical approaches:
  - Constrained transport (CT; Toth 00),
  - flux-CT (Kuroda&Umeda 10; Kiuchi+12),
  - divergence cleaning (Liebling+10),
  - vector-potential evolution (e.g., Etienne+12,15).

$$\mathbf{B} = \nabla \times \mathbf{A} \longrightarrow \nabla \cdot \mathbf{B} = 0$$

The Einstein Toolkit Project

http://einsteintoolkit.org

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The Einstein Toolkit

http://einsteintoolkit.org

- Collection of open-source software components for the simulation and analysis of general-relativistic astrophysical systems.

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The Einstein Toolkit

http://einsteintoolkit.org

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- Collection of open-source software components for the simulation and analysis of general-relativistic astrophysical systems.
- Supported by NSF via collaborative grant to Georgia Tech, LSU, RIT, and Caltech.
- ~110 users, 53 groups; ~10 active maintainers.
- Goals:
  - Reproducibility.
  - Build a community codebase for numerical relativity and computational relativistic astrophysics.
  - Enable new science by lowering technological hurdles for researchers with new ideas. Enable code verification/validation, physics benchmarking, regression testing.
  - Make it easy for users to take advantage of new technologies.
  - Provide cyberinfrastructure tools for code and data management.

The Einstein Toolkit

- Regular releases of stable code versions.
  Most recent: "Herschel" release, November 2014
- Support via mailing list and weekly open conference calls.
- Working examples for BH mergers, NS mergers, isolated NSs, rotating, magnetized core collapse.

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Available Components:

- Cactus (framework), Carpet (adaptive mesh refinement)
- GRHydro – GRMHD solver
- McLachlan – BSSN/Z4c spacetime solver
  (code auto-generated based on Mathematica script, GPU-enabled)
- Initial data solvers / importers
- Analysis tools (wave extraction, horizon finders, etc.)
- Visualization via VisIt (http://visit.llnl.gov)