Criticality and multifractality in monitored quantum dynamics

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Jun 7, 2024@YITP, Kyoto

Outline

1. Introduction

---Localization, measurements, and phase transitions

2. Charge and entanglement transitions in monitored U(1)-symmetric circuit

Oshima and YF, Phys. Rev. B **107**, 014308 (2023)

3. Multifractality in monitored single-particle dynamics

Yajima, Oshima, Mochizuki, and YF, arXiv:2406.02386.

4. Summary

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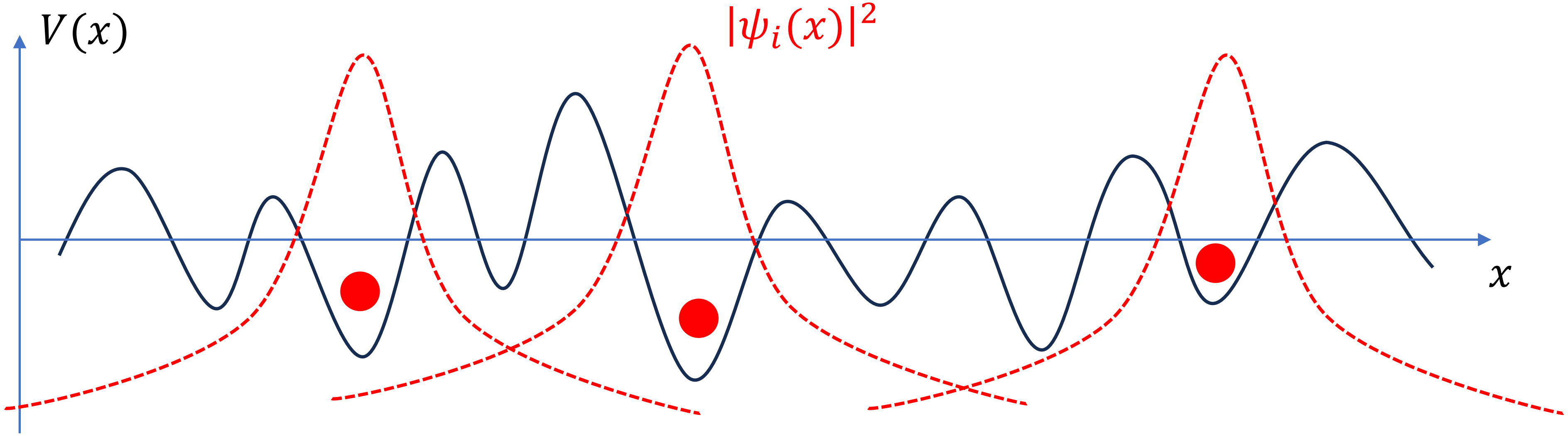
4. Summary

Anderson localization

Strong disorder induces localization of electrons.



$d < 2$: Disorder is always relevant → Any week disorder induces localization.

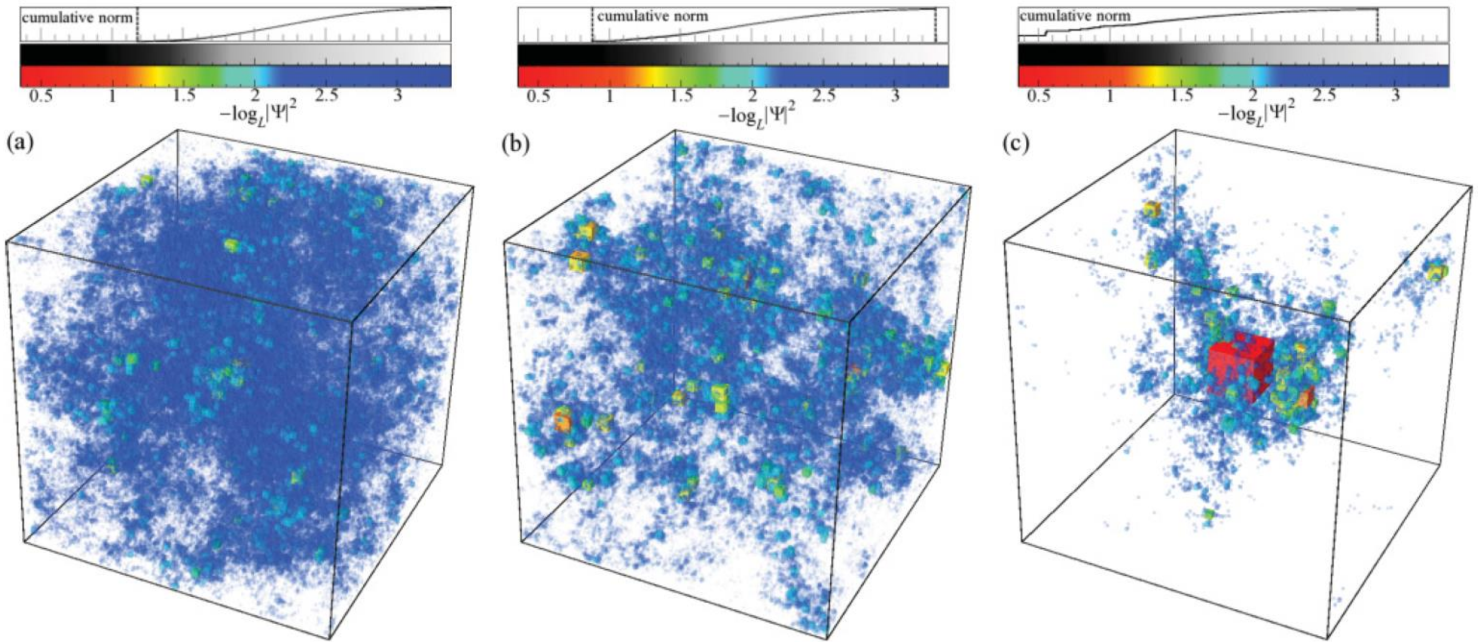
$d > 2$: *Anderson transition* (metal-insulator transition) for a certain disorder strength

Interaction enables many-body localization transition in 1D.

Anderson transition

Figure from Rodriguez, Vasquez, Slevin, & Römer, PRB **84**, 134209 (2011).

← Weak disorder



Strong disorder →

Metal
(Extended)

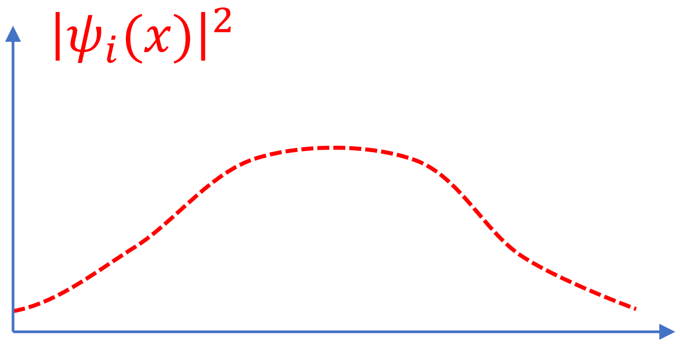
Anderson transition

Insulator
(Localized)

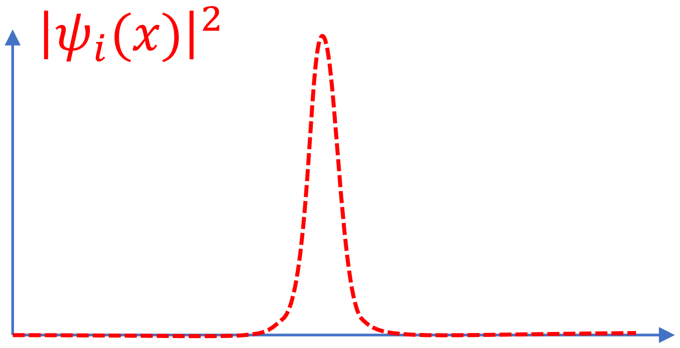
Anderson transitions exhibit critical phenomena with emergent scale invariance.

--- Consequence of *multifractality* of the wave function → Our second topic

Quantum measurements



Measure $\hat{x} = \int dx\, x|x\rangle\langle x|$



x' → Outcome

Collapse of wave function by quantum measurement

≀

Localization of wave function by disorder

Spreading of particles by unitary dynamics vs. localization of particles by measurements

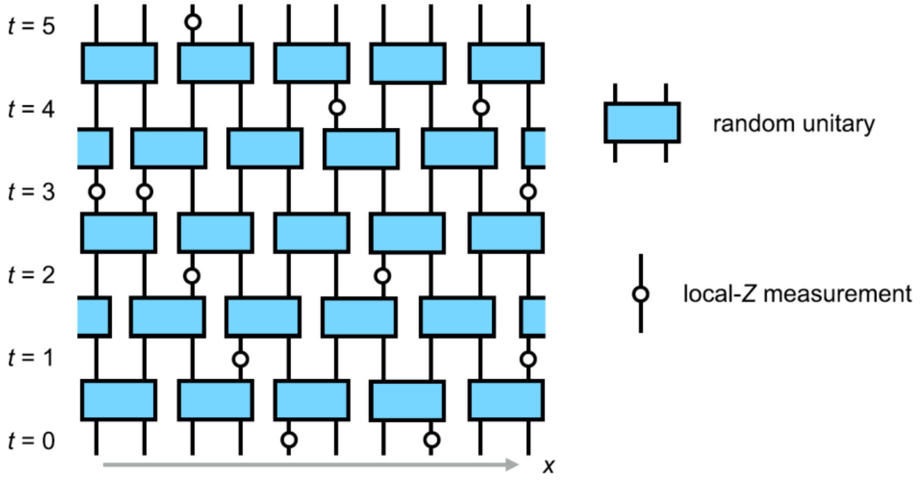
→ *Measurement-induced phase transitions*

Measurement-induced phase transitions

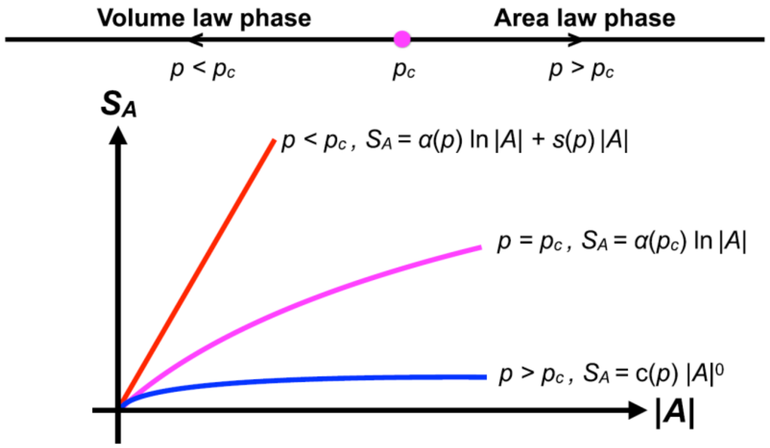
For many-body quantum systems

$|\psi_L\rangle = $ (ψ_L) —Measure Pauli Z_i→ $|\uparrow_i\rangle \otimes |\psi_{L-1}\rangle = $ ($\uparrow$ ψ_{L-1})

Entanglement by unitary dynamics vs. disentanglement by local measurements



Li, Chen, & Fisher, PRB **98**, 205136 (2018).
Chan, Nandkishore, Pretko, & Smith, PRB **99**, 224307 (2019).
Skinner, Ruhman, & Nahum, PRX **9**, 031009 (2019).
Li, Chen, & Fisher, PRB **100**, 134306 (2019).

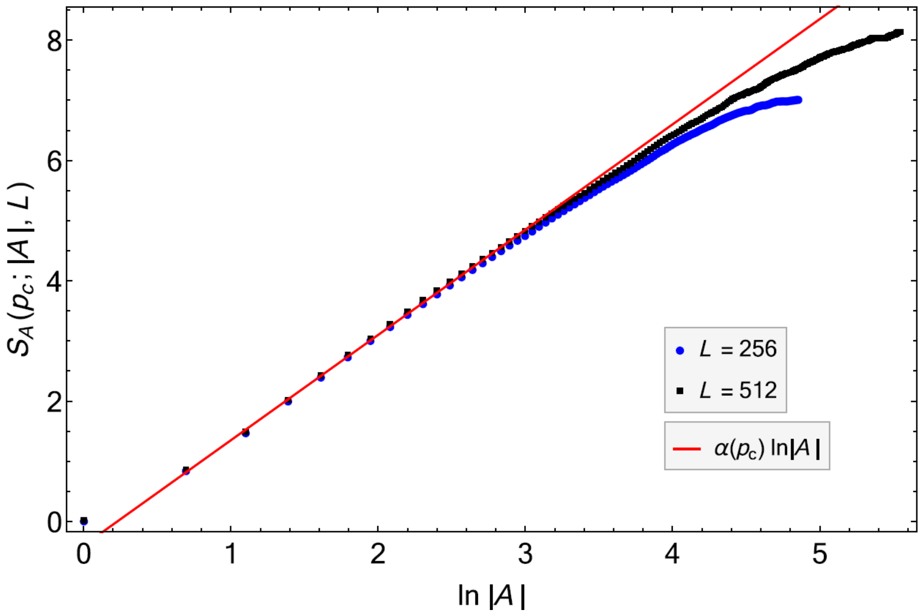


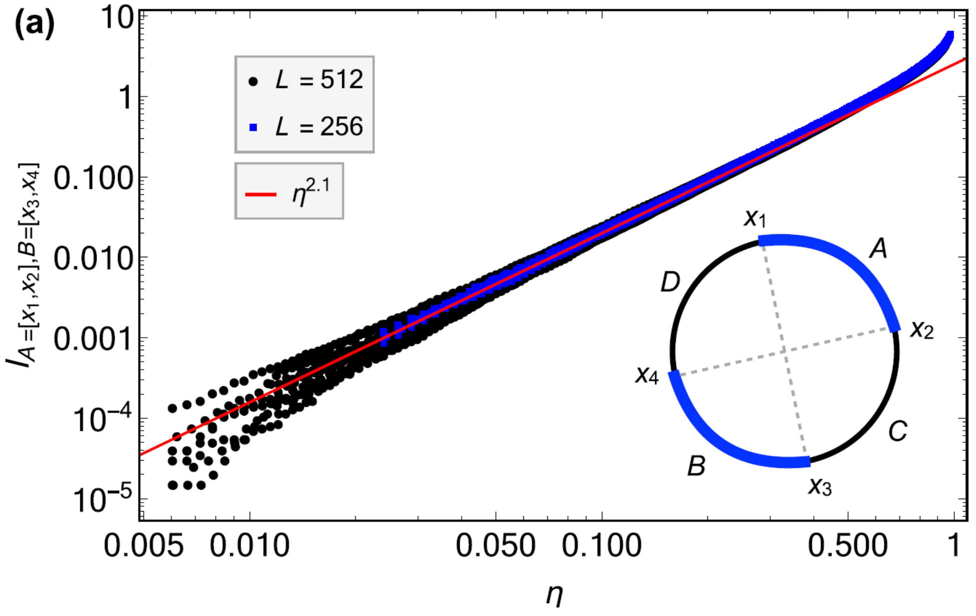
→ Entanglement transition from volume to area law

Measurement-induced phase transitions

Entanglement transition exhibits critical phenomena with emergent conformal invariance.

Li, Chen, & Fisher, PRB **100**, 134306 (2019).





As for equilibrium phase transitions, *symmetry* also plays a prominent role.

→ Our first topic

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Oshima and YF, Phys. Rev. B 107, 014308 (2023)

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Yajima, Oshima, Mochizuki, and YF, arXiv:2406.02386.

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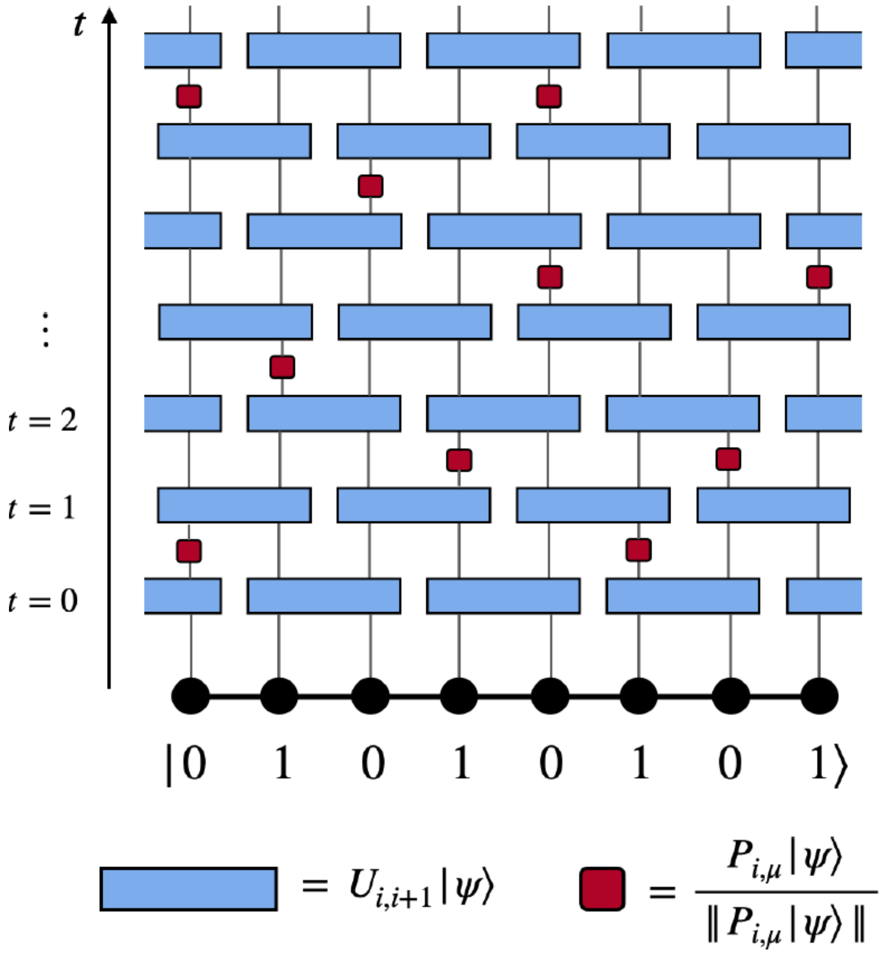
U(1)-symmetric monitored circuit



Haar-random unitary gate is block diagonal:

$$U_{i,i+1} = \begin{pmatrix} U_{1\times 1} & & \\ & U_{2\times 2} & \\ & & U_{1\times 1} \end{pmatrix} \begin{matrix} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{matrix}$$

Local charge measurement with probability p

$$P_{i,\mu} = \frac{\mathbb{I}_i + \mu Z_i}{2}.$$

Circuit evolution conserves total U(1) charge:

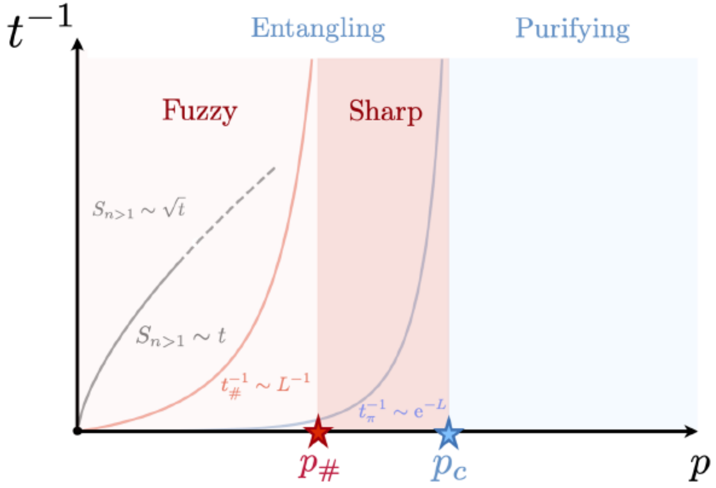
$$n_{\mathrm{tot}} = \sum_{i=1}^{L} n_i, \qquad n_i = \frac{\mathbb{I}_i - Z_i}{2},$$

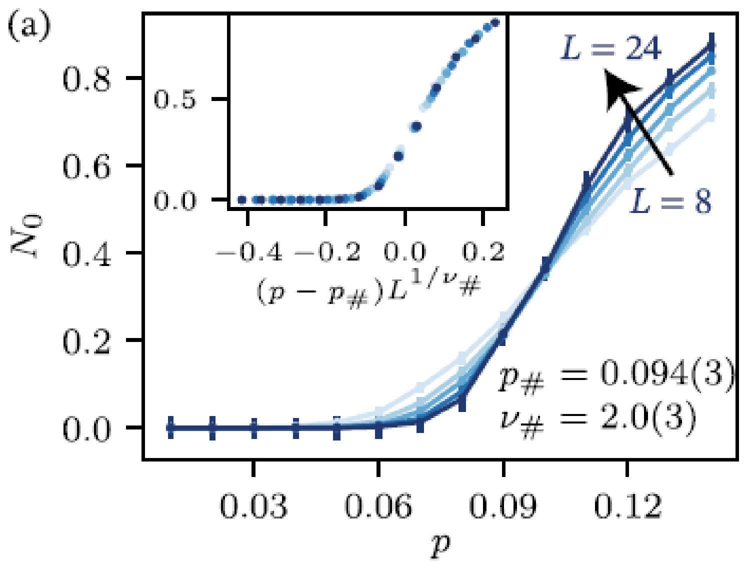
Charge-sharpening transition

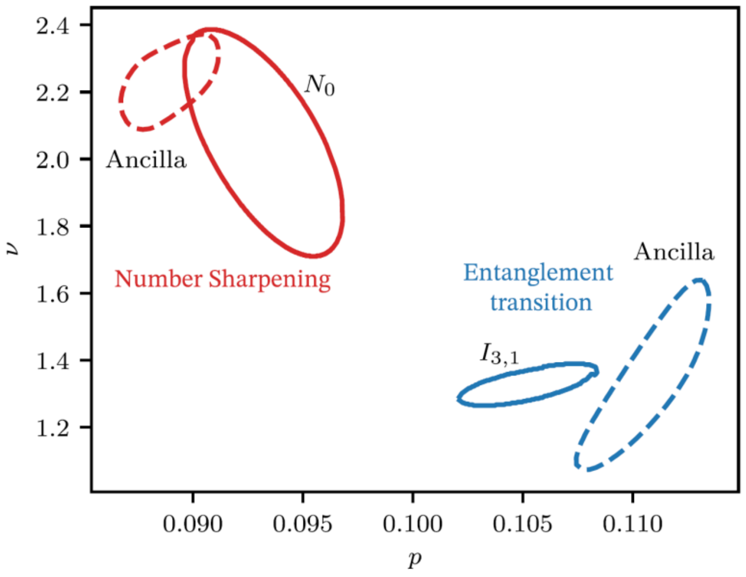
How fast do initially mixed charge sectors collapse into a single charge sector?

→ Another phase transition inside the volume-law phase

Agrawal, Zabalo, Chen, Wilson, Potter, Pixley, Gopalakrishnan, & Vasseur, PRX **12**, 041002 (2022).







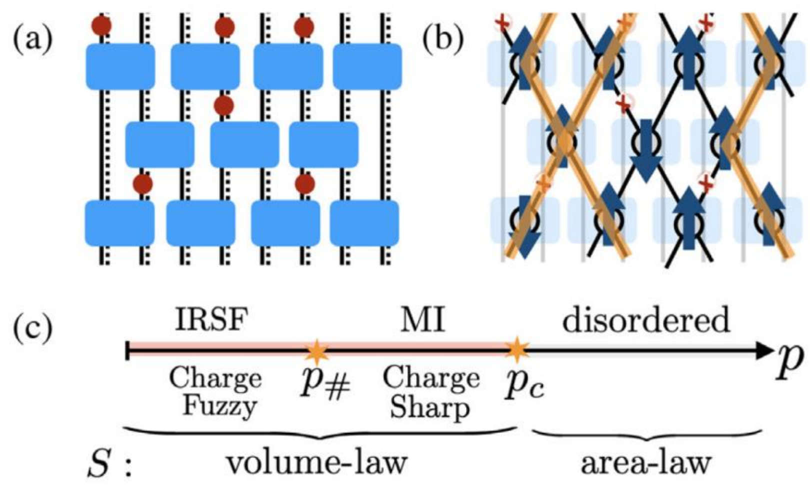
N_0: Number of trajectories collapsing into a fixed charger sector at $t \sim L$, evolved from $|\psi(0)\rangle = \bigotimes_{j=1}^{L} \frac{|0\rangle + |1\rangle}{\sqrt{2}}$

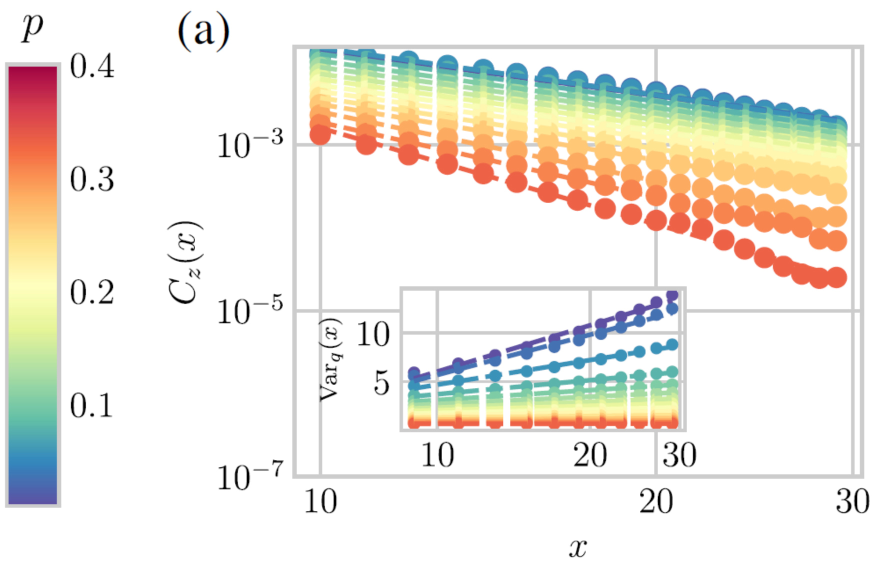
Charge-sharpening transition

Monitored circuit with charged qubits and large-d neutral qudits

→ Effective stat-mech model shows a BKT transition at the charge-sharpening transition.

Barratt, Agrawal, Gopalakrishnan, Huse, Vasseur, & Potter, PRL **129**, 120604 (2022).





Q1. Is there a steady-state probe in a fixed charge sector?

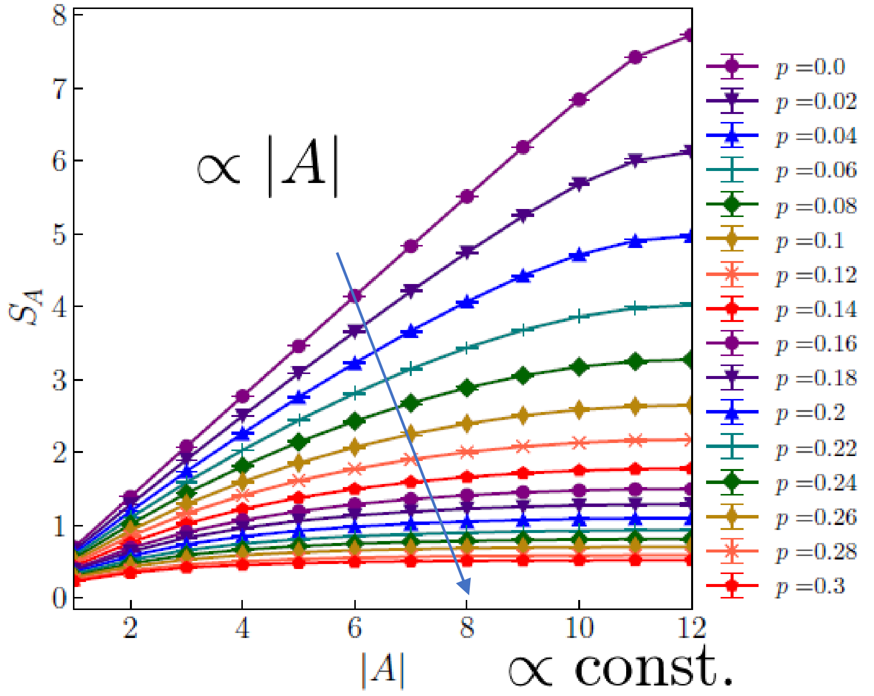
Q2. Are entanglement and charge-sharpening transitions separated in qubit systems?

Entanglement vs. Charge fluctuation

Oshima and YF, PRB **107**, 014308 (2023)

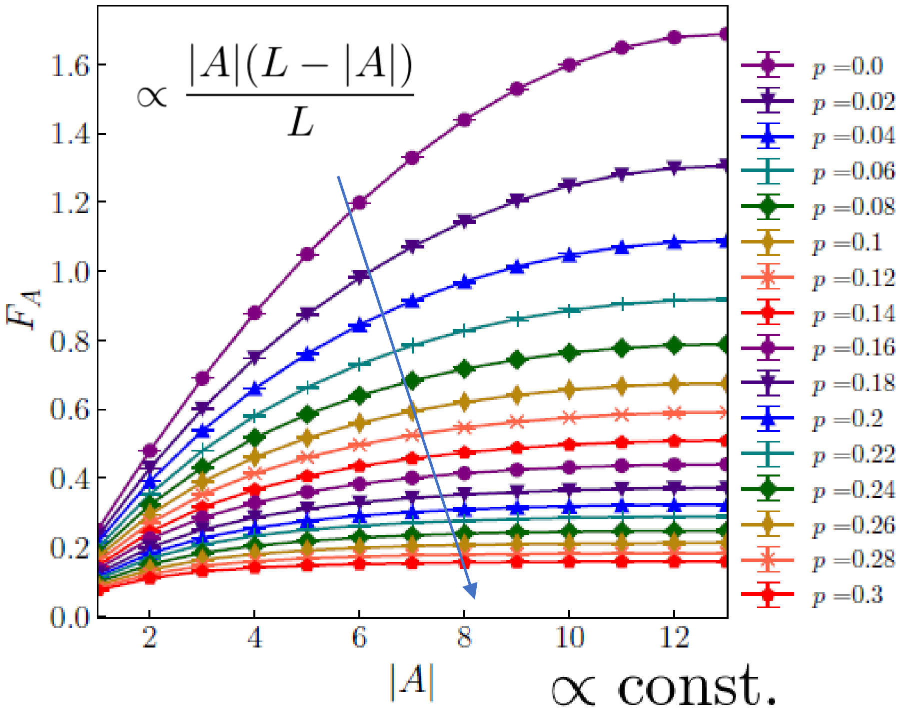
von Neumann entanglement entropy

$$S_A = -\mathrm{Tr}_A(\rho_A \ln \rho_A)$$



Bipartite charge fluctuation

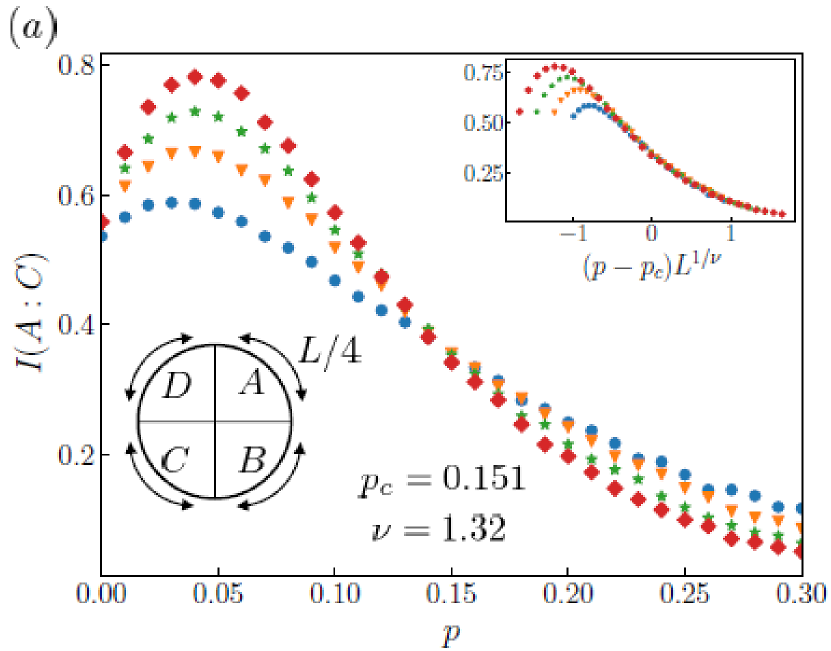
$$F_A = \mathrm{Tr}(\rho n_A^2) - [\mathrm{Tr}(\rho n_A)]^2$$



Entanglement vs. Charge-fluctuation transition

Bipartite mutual information

$$I(A:B) = S_A + S_B - S_{A\cup B}$$

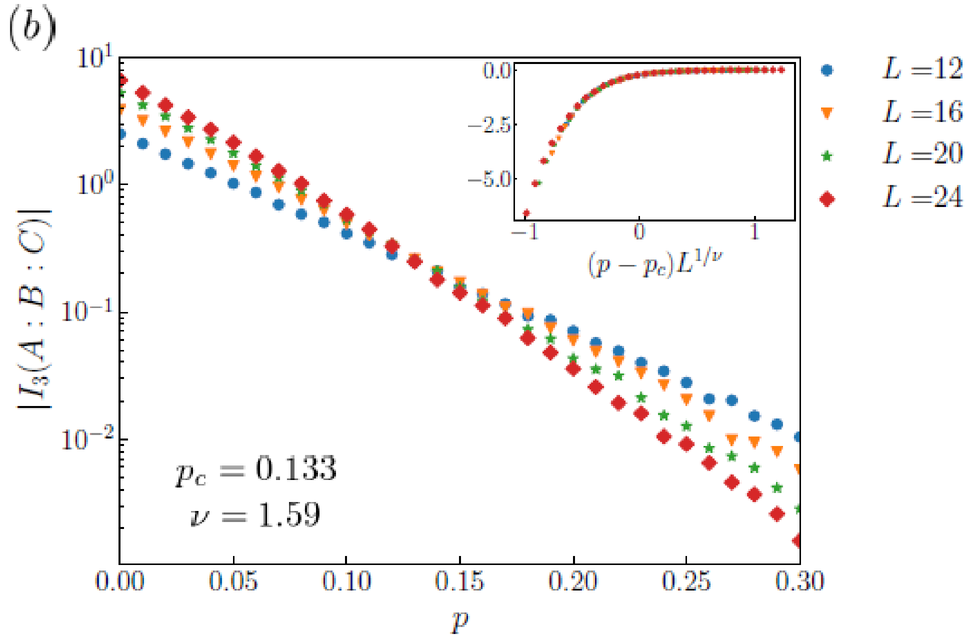


$$I(A:C) = f(\eta), \quad \eta = \frac{x_{ij}x_{kl}}{x_{ik}x_{jl}}$$ for CFT

Entanglement vs. Charge-fluctuation transition

Tripartite mutual information

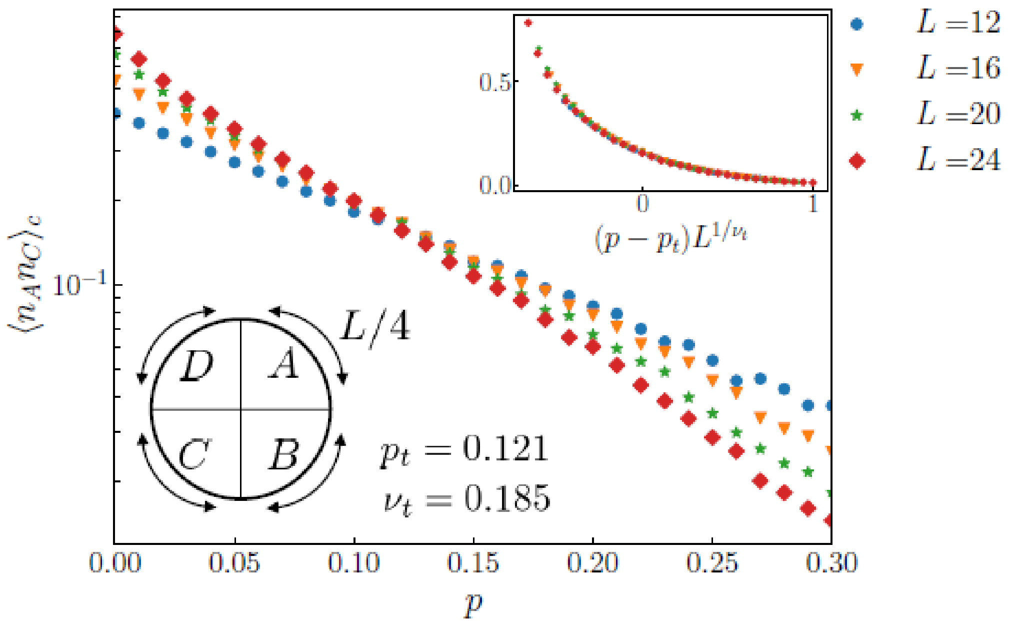
$$I_3(A:B:C) = I(A:B) + I(A:C) - I(A:B\cup C)$$



$$I(A:C) = f(\eta), \quad \eta = \frac{x_{ij}x_{kl}}{x_{ik}x_{jl}}$$ for CFT

Subsystem-charge correlation function

$$\langle n_A n_C\rangle_c = F_A + F_C - F_{A\cup C}$$

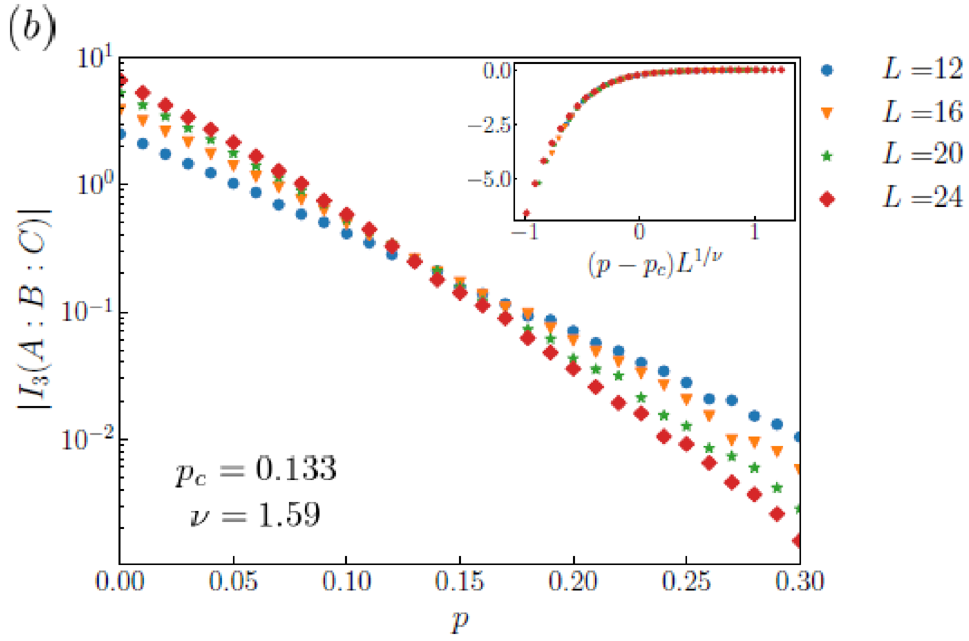


$$\langle n_A n_C\rangle_c \sim \frac{K}{\pi^2}\ln\frac{x_{il}x_{jk}}{x_{ik}x_{jl}} = \frac{K}{\pi^2}\ln(1-\eta)$$ for TLL theory

Entanglement vs. Charge-fluctuation transition

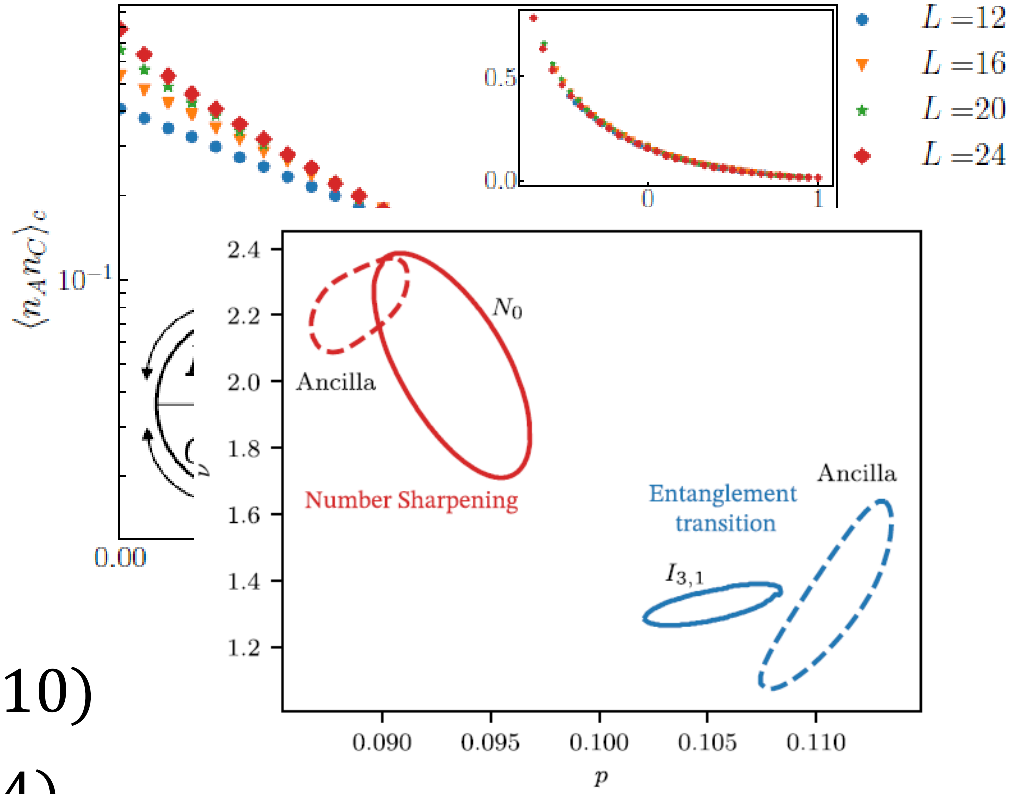
Tripartite mutual information

$$I_3(A:B:C) = I(A:B) + I(A:C) - I(A:B\cup C)$$



Subsystem-charge correlation function

$$\langle n_A n_C\rangle_c = F_A + F_C - F_{A\cup C}$$



For $N = L/2$, $p_c = 0.133(6)$ and $p_t = 0.121(10)$

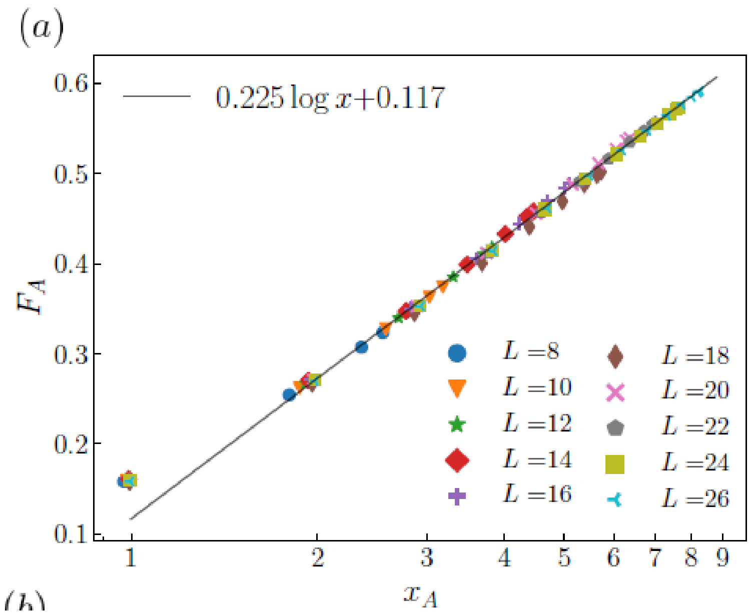
For $N = L/4$, $p_c = 0.095(7)$ and $p_t = 0.071(4)$

Agrawal *et al.*, PRX **12**, 041002 (2022).

TLL-like criticality

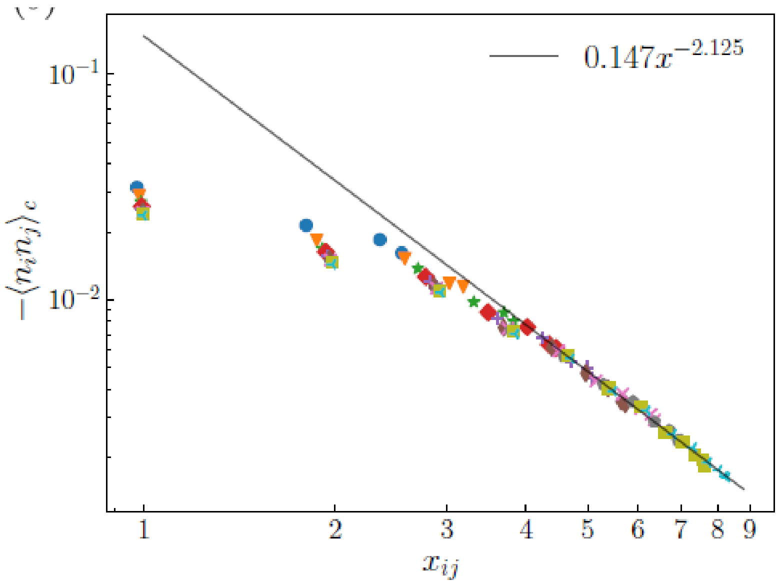
Close to the charge-fluctuation transition

Bipartite charge fluctuation



$$F_A \sim \frac{K}{\pi^2} \ln x_A$$

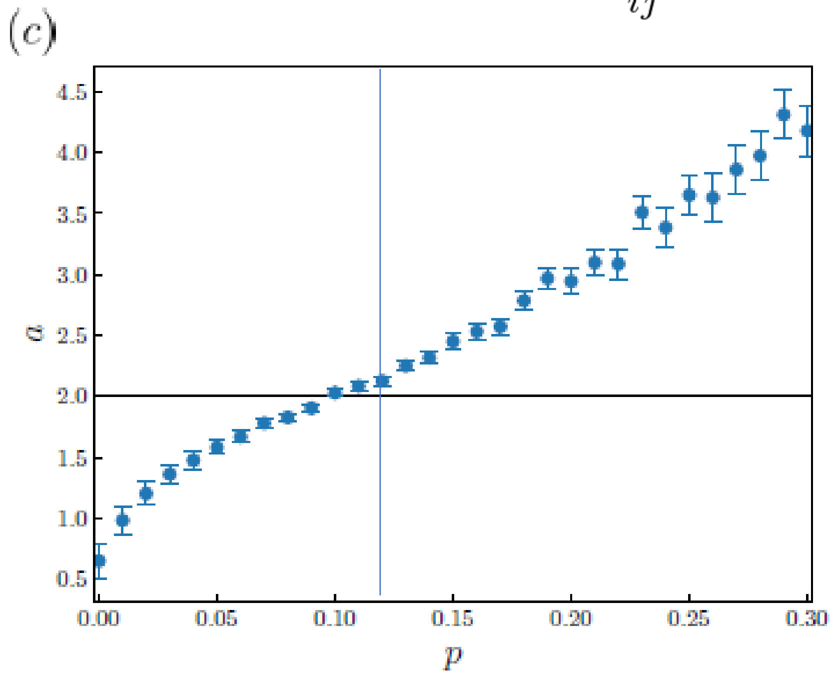
Connected charge correlation function



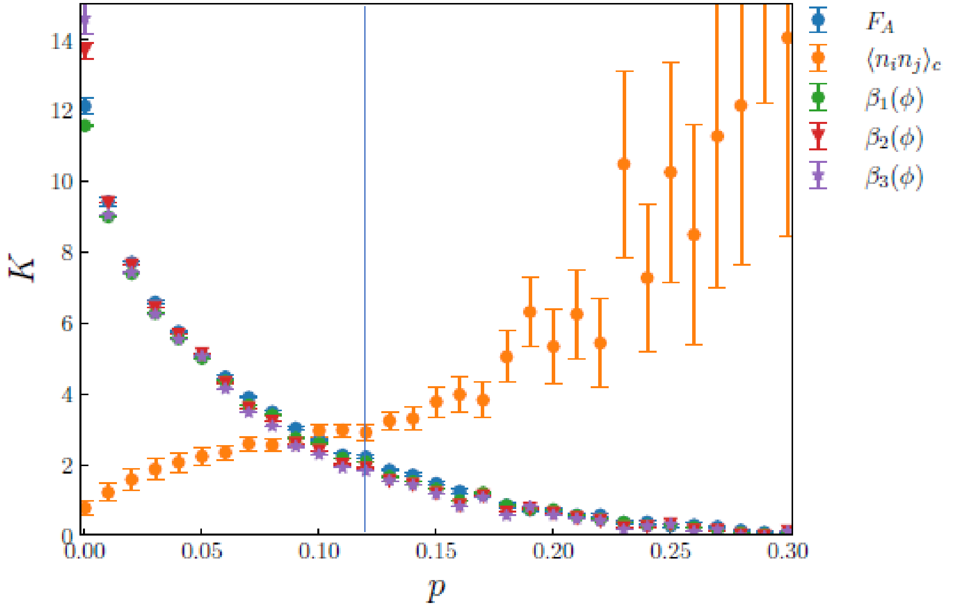
$$\langle n_i n_j \rangle_c \sim -\frac{K}{2\pi^2} \frac{1}{x_{ij}^2}$$

BKT transition?

Ansatz: $\langle n_i n_j \rangle_c \propto \frac{1}{x_{ij}^a}$



Luttinger parameter K



BKT scenario:

--- Exponent $a = 2$ below the charge-fluctuation transition $p = p_t$

--- Universal Luttinger parameter $K = 2$ at $p = p_t$

Summary (Part 1)

U(1)-symmetric monitored systems are predicted to have a charge-sharpening transition, in addition to an entanglement transition.

Q1. Is there a steady-state probe in a fixed charge sector?

Yes: Bipartite charge fluctuation can probe it.

Q2. Are entanglement and charge-sharpening transitions separated in qubit systems?

Yes or no: Two transitions are too close.

On dynamical quantum trees, two transitions coincide for qubit systems.

Feng, Fishchenko, Gopalakrishnan, & Ippoliti, arXiv:2405.13894.

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Yajima, Oshima, Mochizuki, and YF, arXiv:2406.02386.

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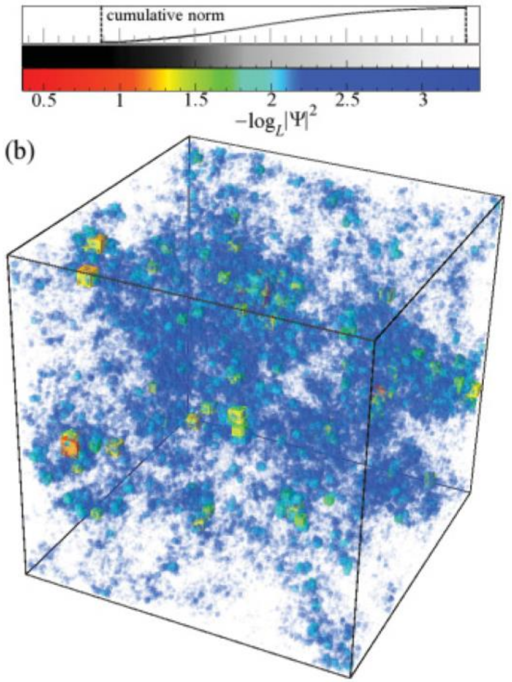
Multifractality at Anderson transition



Multifractal measure: Inverse participation ratio (IPR)

$$IPR(q) = \sum_x |\langle x|\psi\rangle|^{2q}$$ $x \in$ Box of the volume L^d

→ Asymptotic form $IPR(q) \sim L^{-\tau_q}$

Fractal dimension D_q defined by $\tau_q \equiv D_q(q-1)$ behaves as

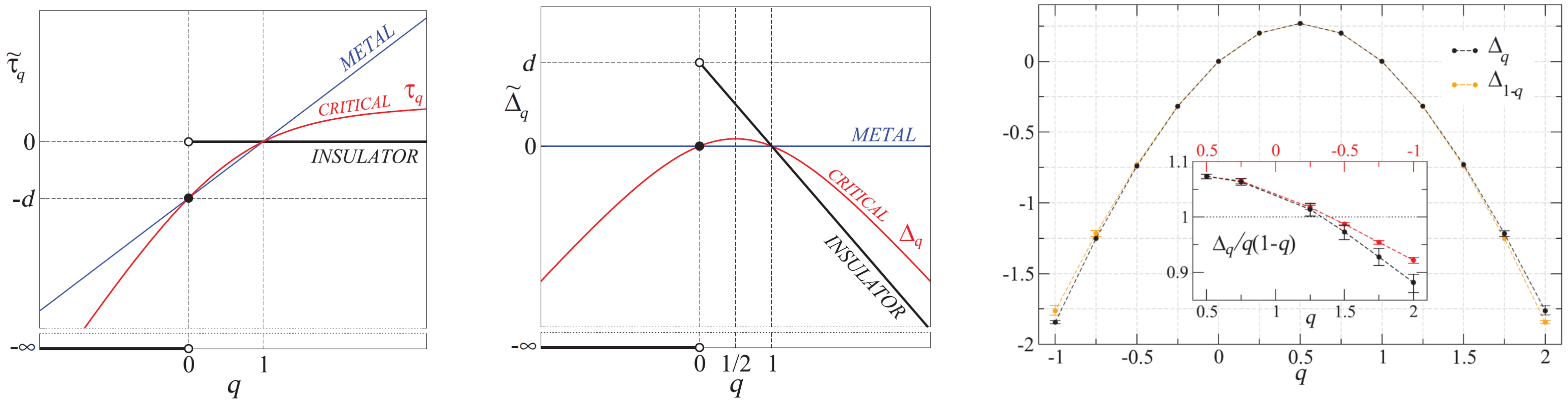
- Localized phase: $D_q = 0$
- Extended (metallic) phase: $D_q = d$
- Multifractal: D_q is a nonlinear function of q

Multifractality at Anderson transition

$$IPR(q) = \sum_x |\langle x|\psi\rangle|^{2q} \sim L^{-\tau_q}$$

Anomalous dimension Δ_q by $\tau_q \equiv d(q-1) + \Delta_q$

Figures from Rodriguez, Vasquez, Slevin, & Römer, PRB **84**, 134209 (2011).



Symmetry $\Delta_q = \Delta_{1-q}$ is expected at Anderson transition.

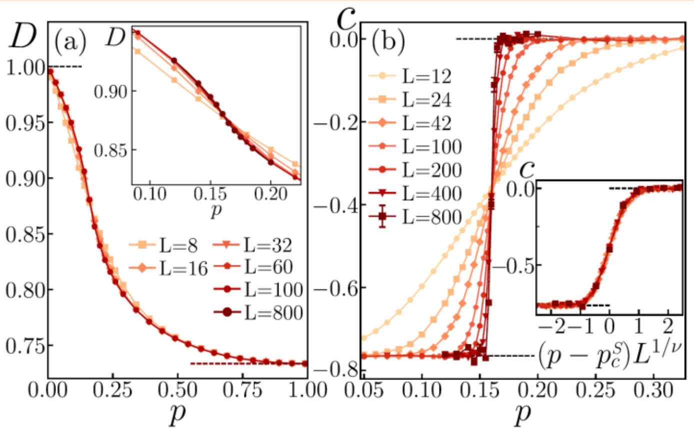
Mirlin, Fyodorov, Mildenberger, & Evers, PRL **97**, 046803 (2006).

Multifractality at MIPT

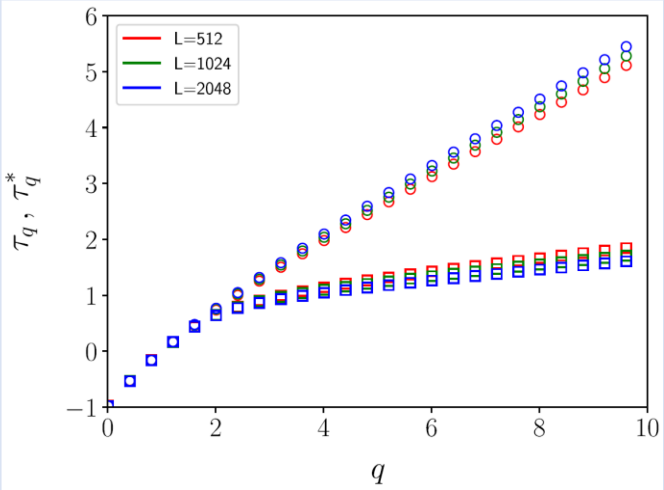
Monitored random circuits

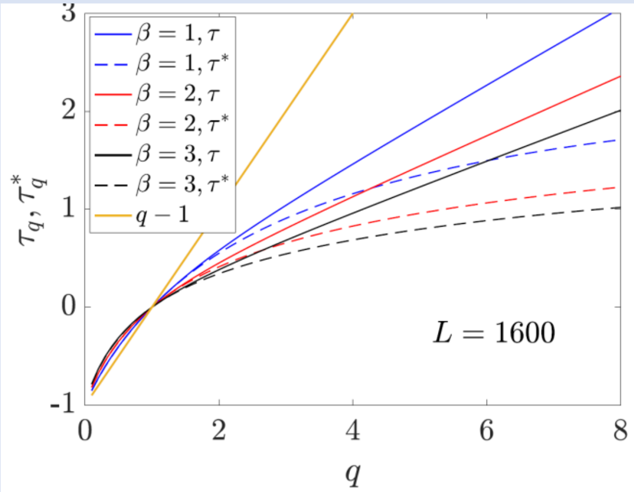
Sierant & Turkeshi, PRL **128**, 130605 (2022).

$$S_q = \frac{1}{1-q}\log_2 \sum_{\vec{\sigma}} |\psi_{\mathbf{m}}(\vec{\sigma})|^{2q} \equiv D_q L + c_q,$$



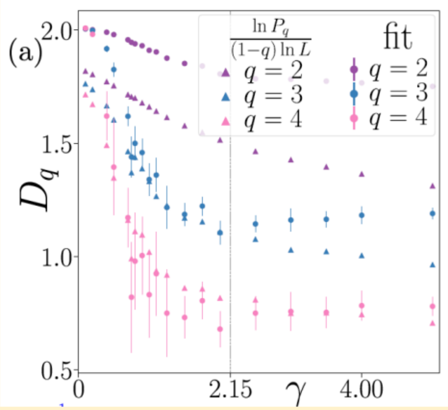
Monitored Clifford & Nonunitary free-fermion circuits





Iaconis & Chen, PRB **104**, 214307 (2021).

(2+1)D monitored free fermion circuit

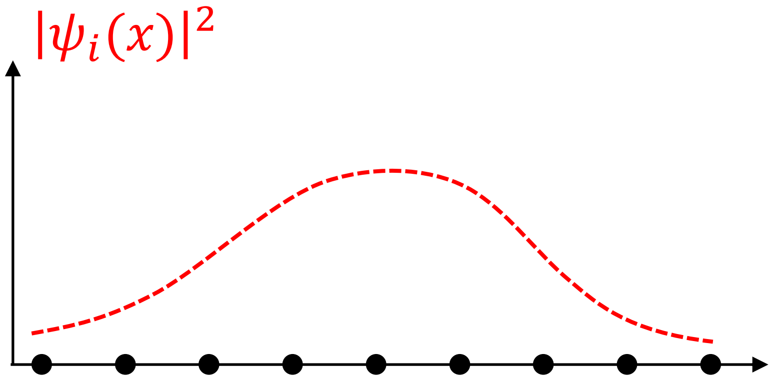


Chahine & Buchhold, arXiv:2309.12391.

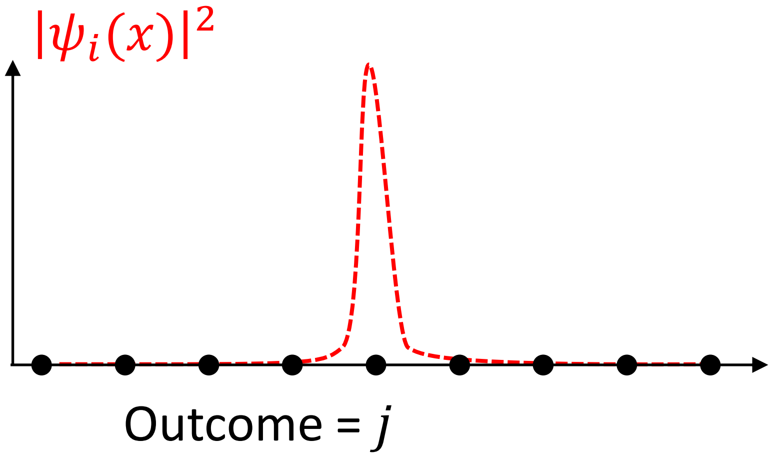
Mutlifractality in simplest monitored system

→ Single particle

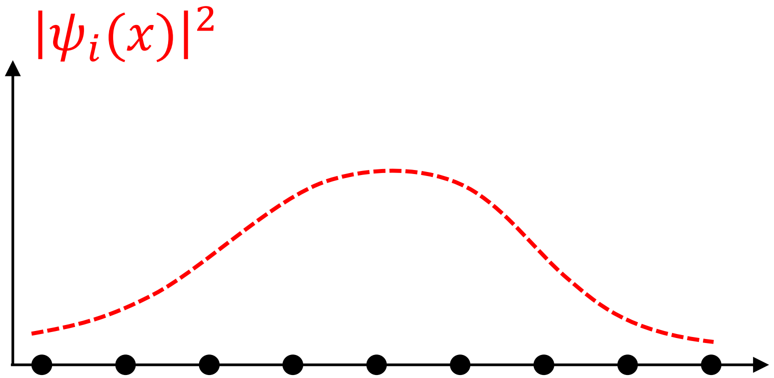
Monitored single particle



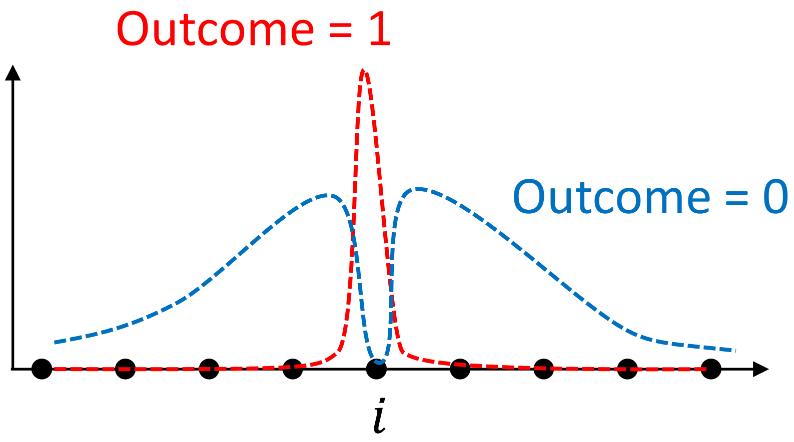
Measure $\hat{x} = \sum_i i|i\rangle\langle i|$



Global measurements: Effect of localization is too strong.



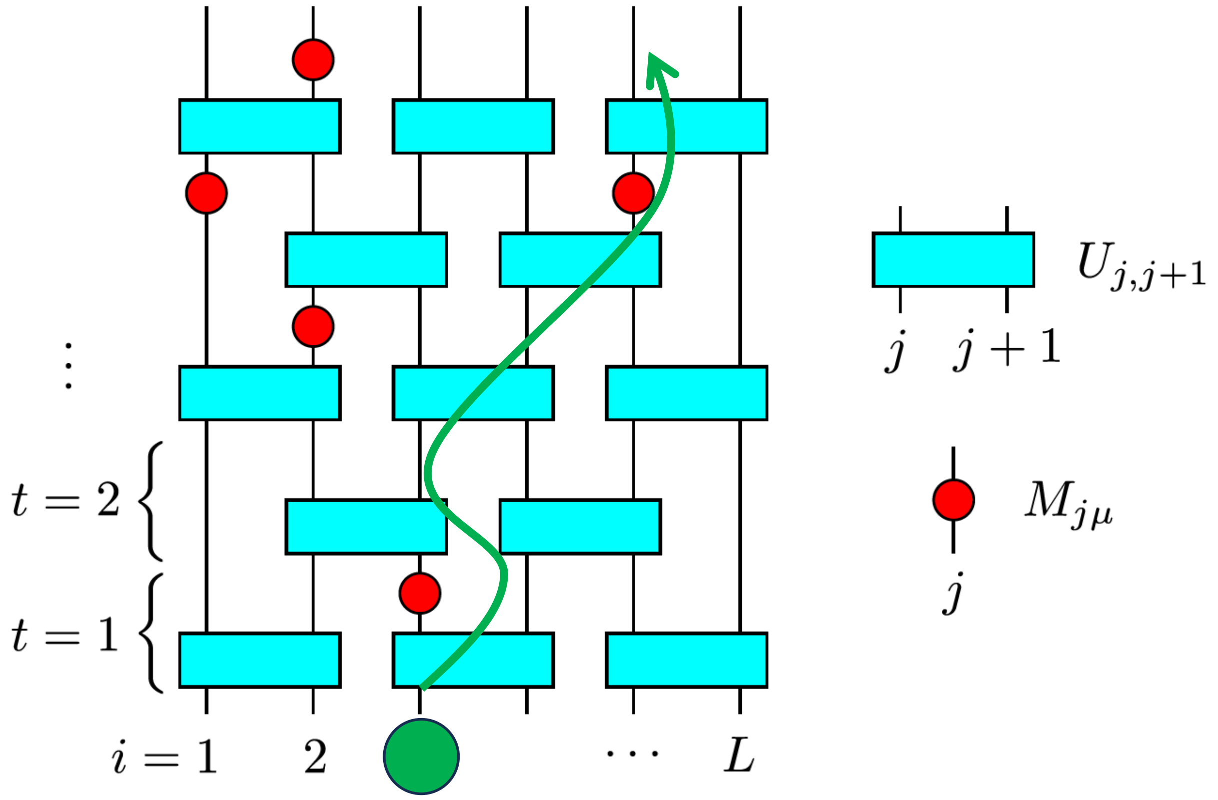
Measure $|i\rangle\langle i|$



We consider local measurements.

Quantum circuit

Yajima, Oshima, Mochizuki, and YF, arXiv:2406.02386.



Particle initially placed at $i = L/2$

--- $U_{j,j+1}$ are drawn randomly or fixed.

--- Measurements $M_{j\mu}$ are performed at every site with probability $p \sim O(1/L)$.

Projective measurements:

$$M_{i0} = I - |i\rangle\langle i|,$$
$$M_{i1} = |i\rangle\langle i|,$$

$\mu = 0{,}1$: Measurement outcome

Multifractal analysis

Inverse participation ratio: $IPR(q) = \sum_i |\langle i|\psi\rangle|^{2q}$

Taking average $\langle \cdots \rangle$ over random unitaries, measurement positions, & outcomes.

Mean IPR: $\langle IPR(q)\rangle \sim L^{-\tau_q}$

Typical IPR: $e^{\langle \ln IPR(q)\rangle} \sim L^{-\tau_q^*}$

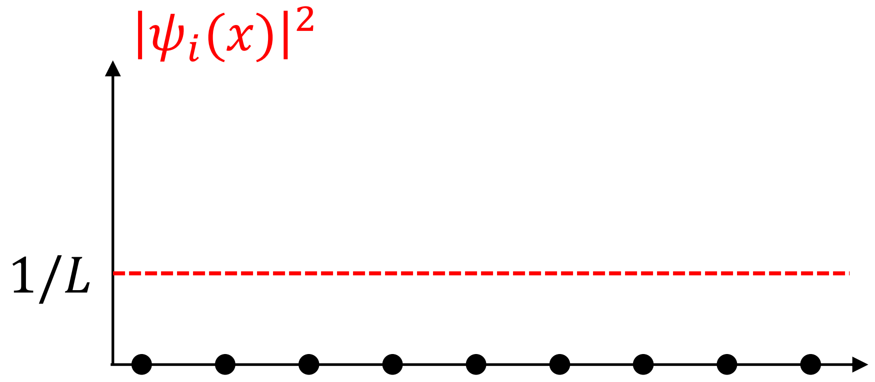
In general, τ_q and τ_q^* are different at Anderson transitions.

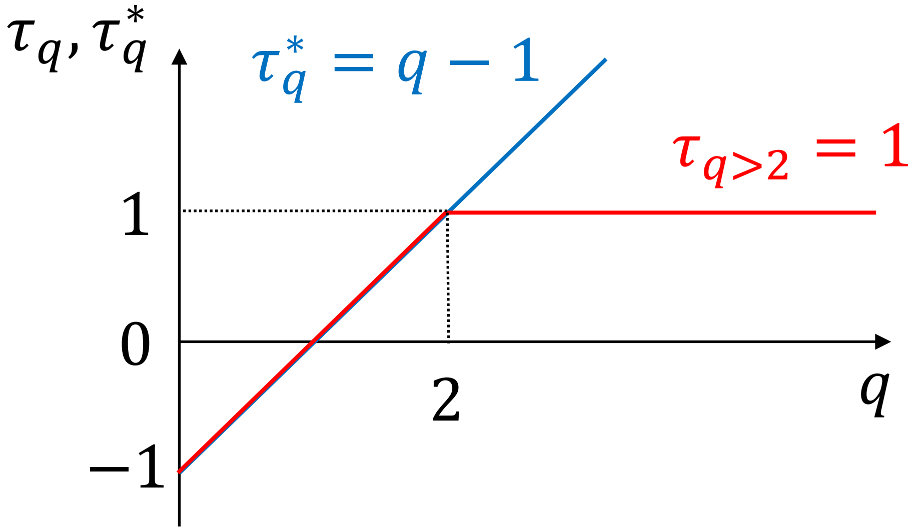
Variance of the position operator: $Var = \langle\psi|\hat{x}^2|\psi\rangle - \langle\psi|\hat{x}|\psi\rangle^2$

$\langle Var\rangle \sim L^{2\tau_{Var}}$

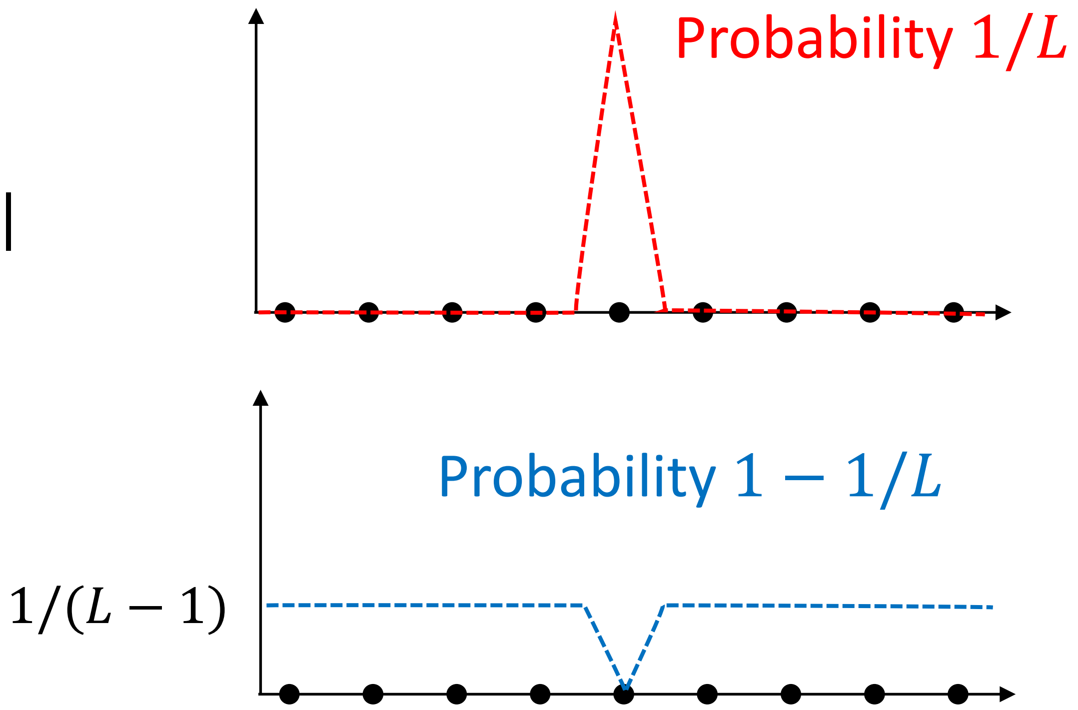
Single-shot measurement model

Uniform distribution





Measure $|i\rangle\langle i|$



τ_q for mean IPR saturates to 1 for $q > 2$.

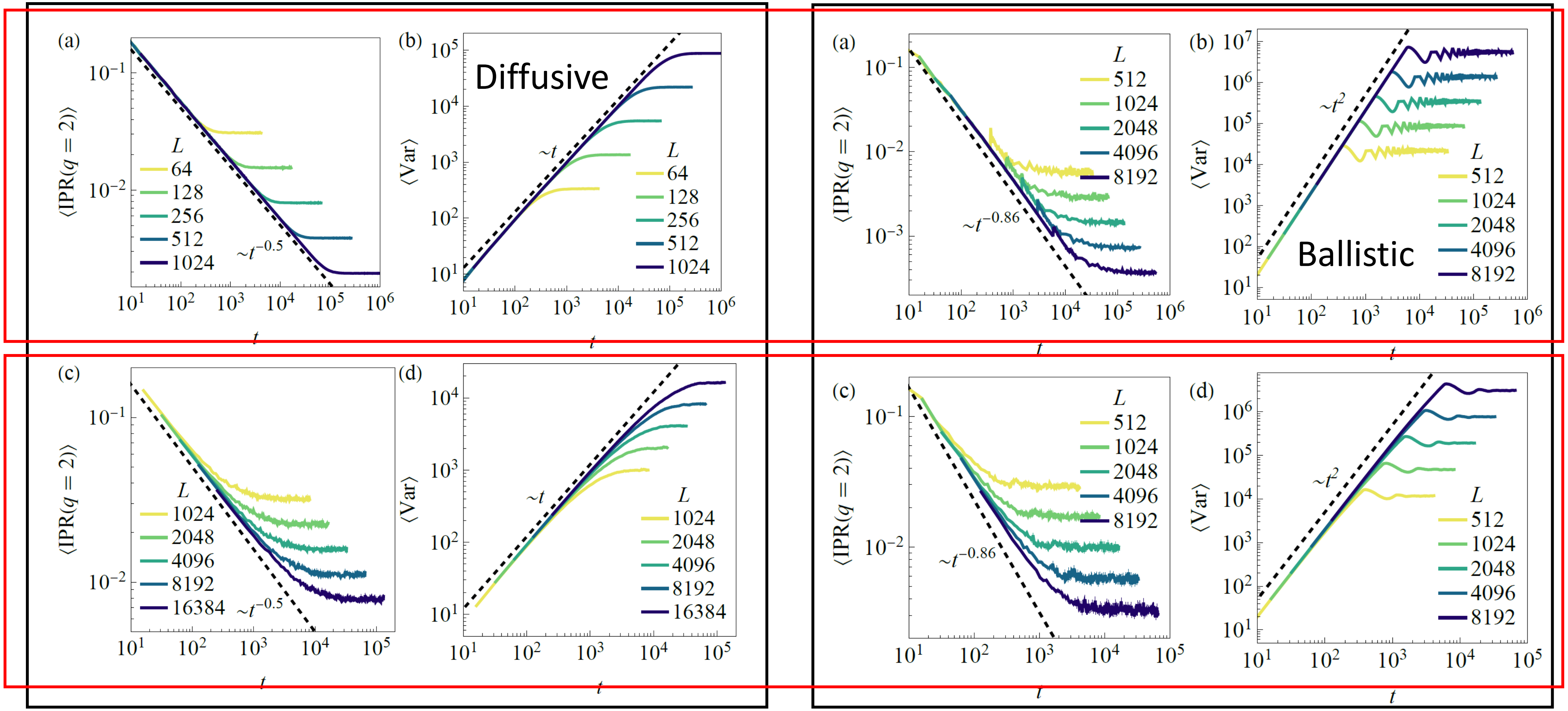
---Rare localized trajectories affect τ_q.

Unitary + projective measurements

Random unitary | Fixed unitary

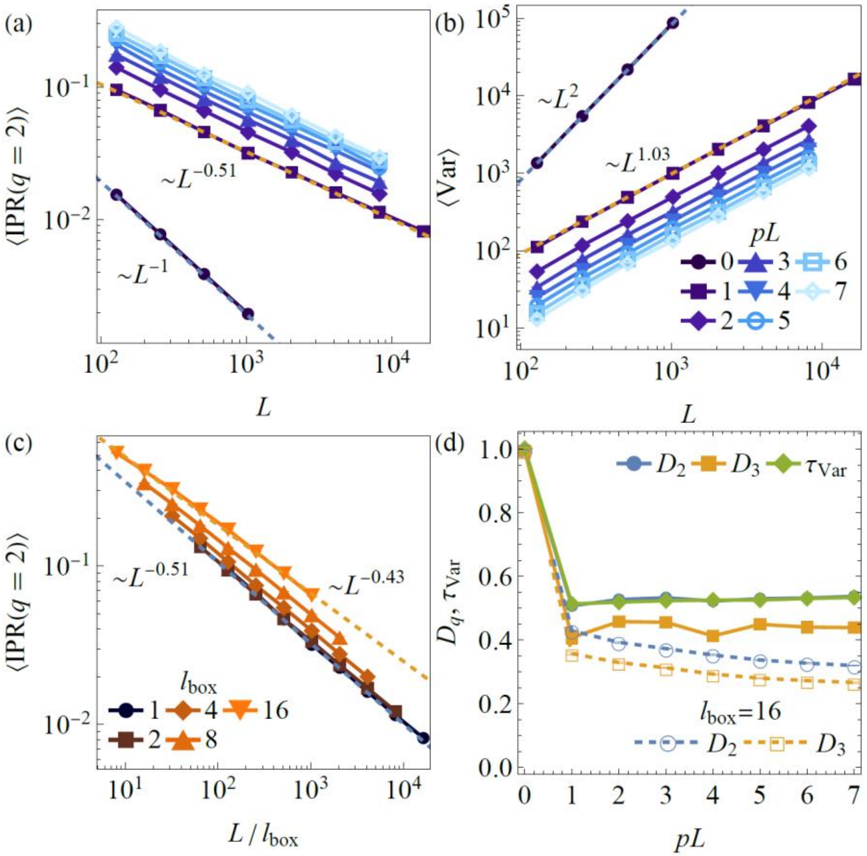
$p = 0$

$p = 1/L$

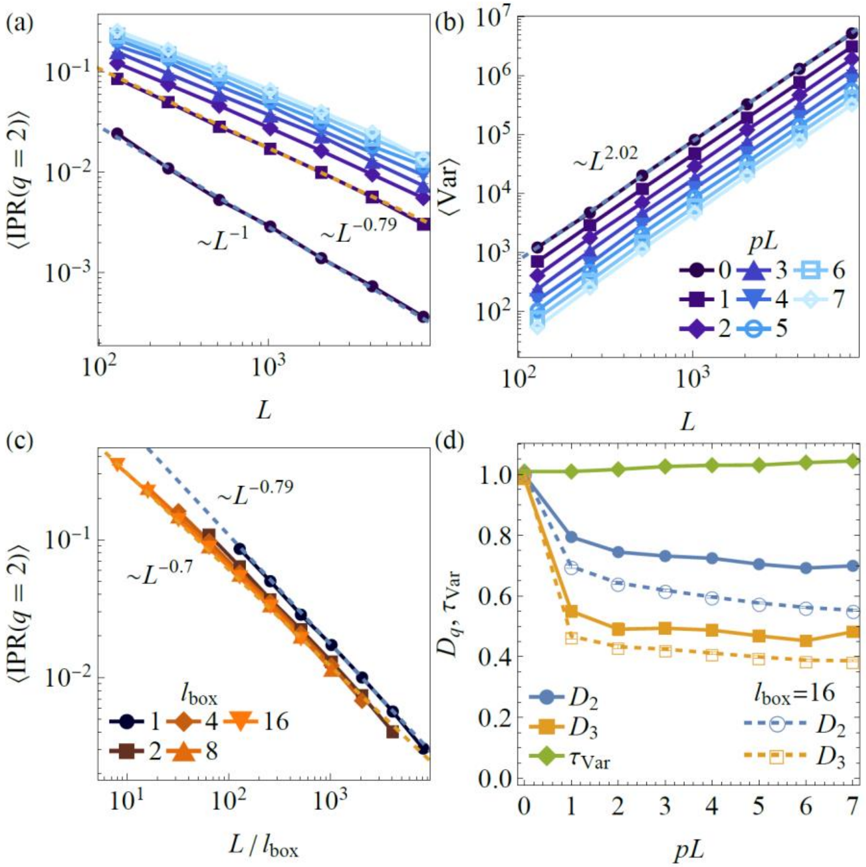


Unitary + projective measurements

Random unitary



Fixed unitary

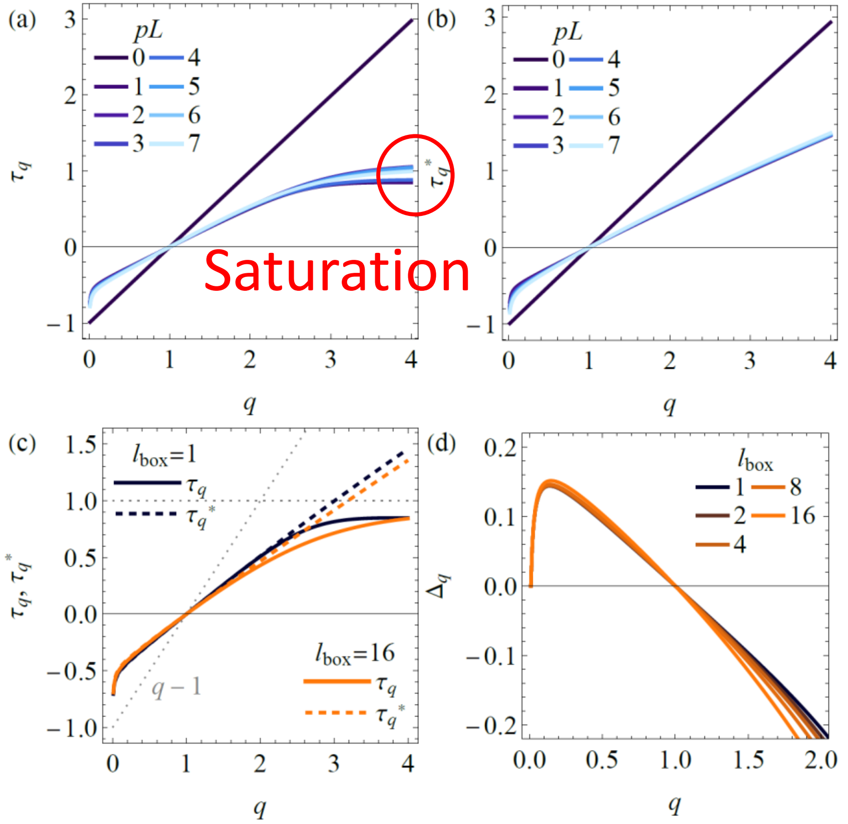


---Fractal dimension D_q takes nontrivial values between 0 and 1. → Multifractal?

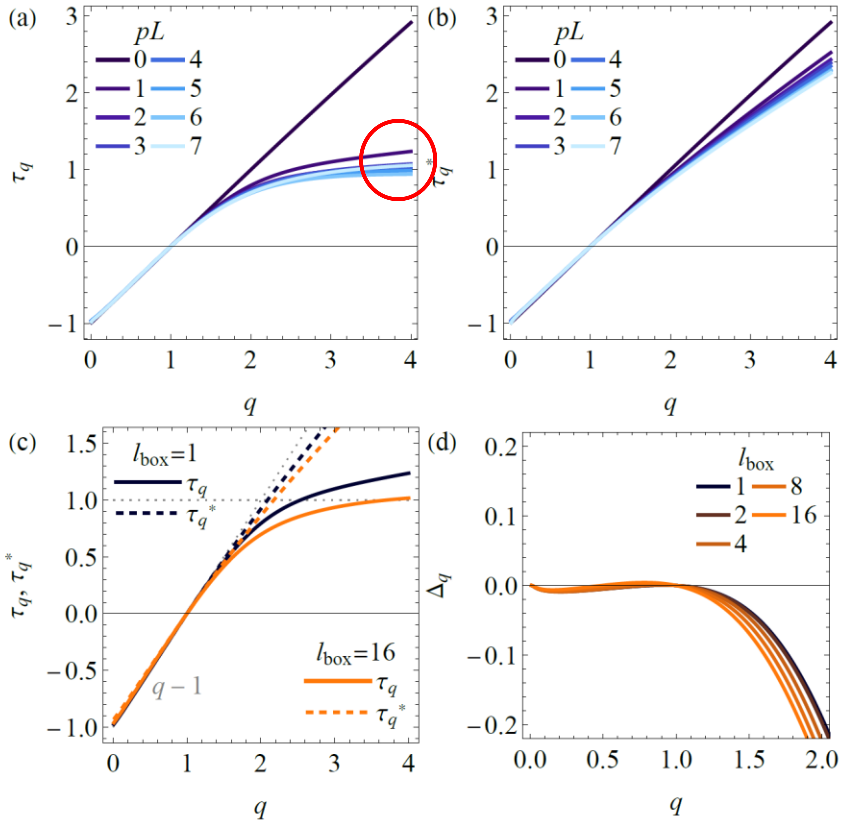
---No scale invariance under coarse-graining in boxes of size l_{box}.

Unitary + projective measurements

Random unitary



Fixed unitary

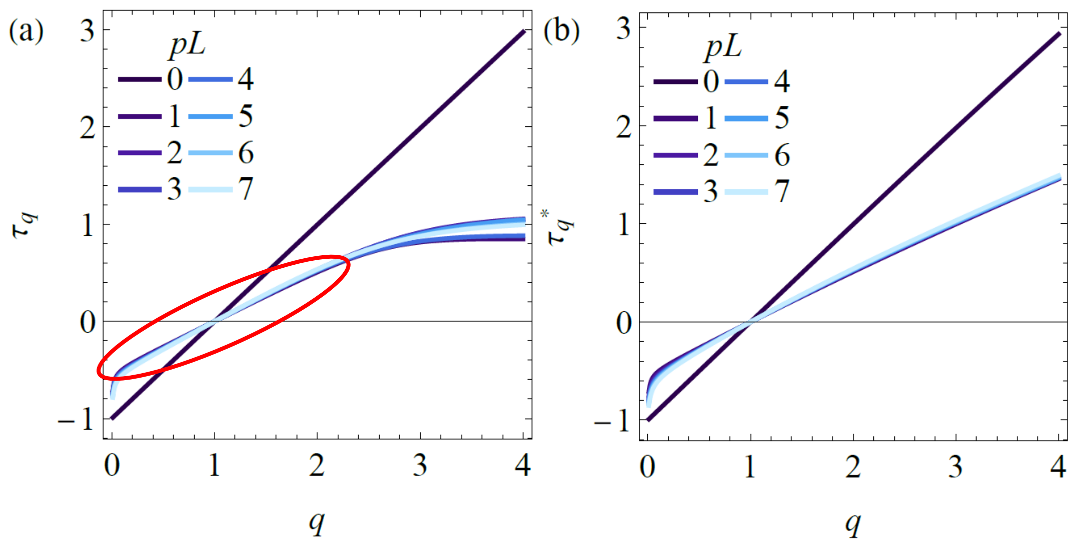


---Both τ_q and τ_q^* are nonlinear functions of q. → Multifractal

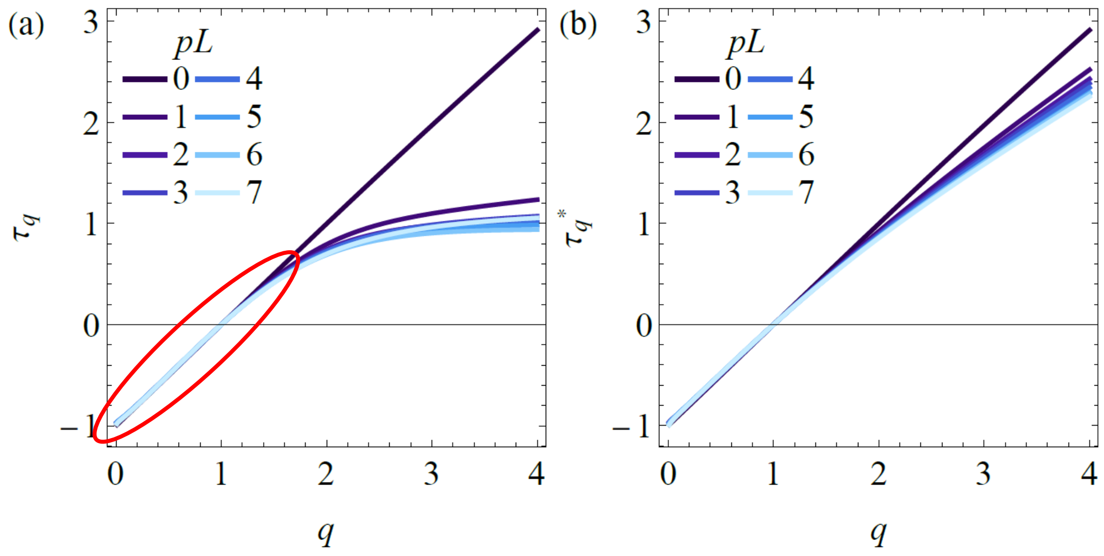
---No symmetry for anomalous dimension Δ_q

Multifractality for monitored quantum particle

Random unitary Diffusive



Fixed unitary Ballistic



---Strong deviation from $q - 1$

---Agreement with $q - 1$ for small q

Particle transport (diffusive or ballistic) strongly matters to multifractality.

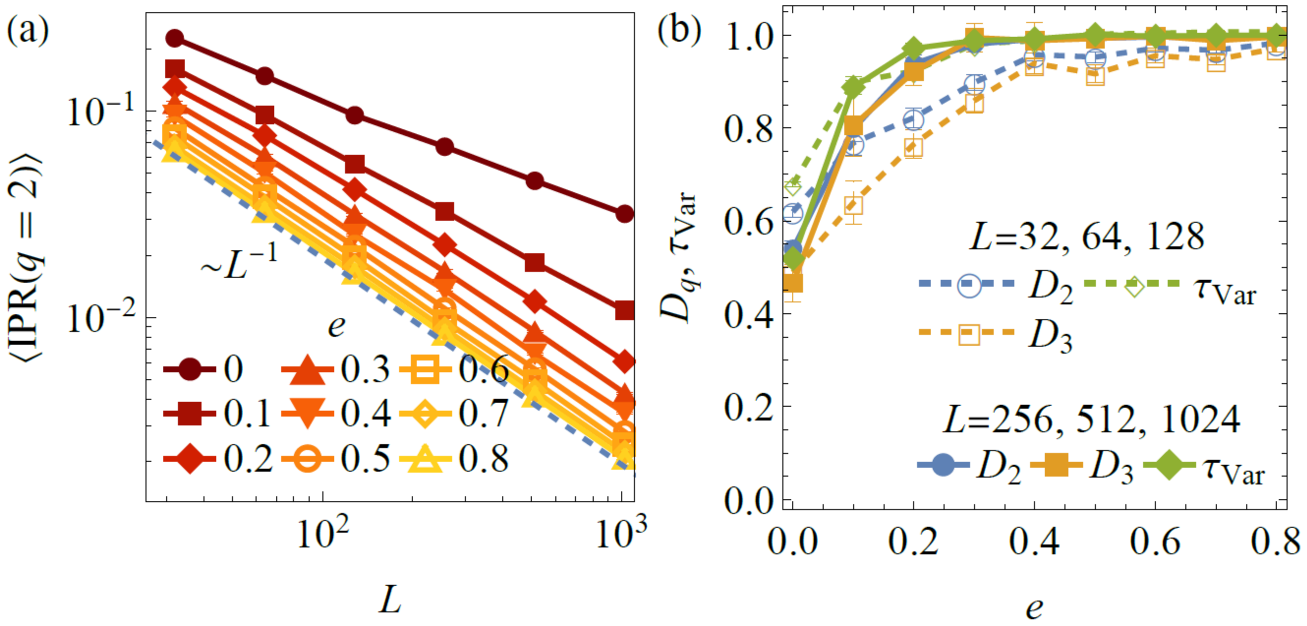
Generalized measurements

Generalized measurements:

$$M_{i0} = \sqrt{1-\frac{e}{2}}\,(I - |i\rangle\langle i|) + \sqrt{\frac{e}{2}}|i\rangle\langle i|,$$

$$M_{i1} = \sqrt{\frac{e}{2}}\,(I - |i\rangle\langle i|) + \sqrt{1-\frac{e}{2}}|i\rangle\langle i|.$$

$e \in [0,1]$: Error rate



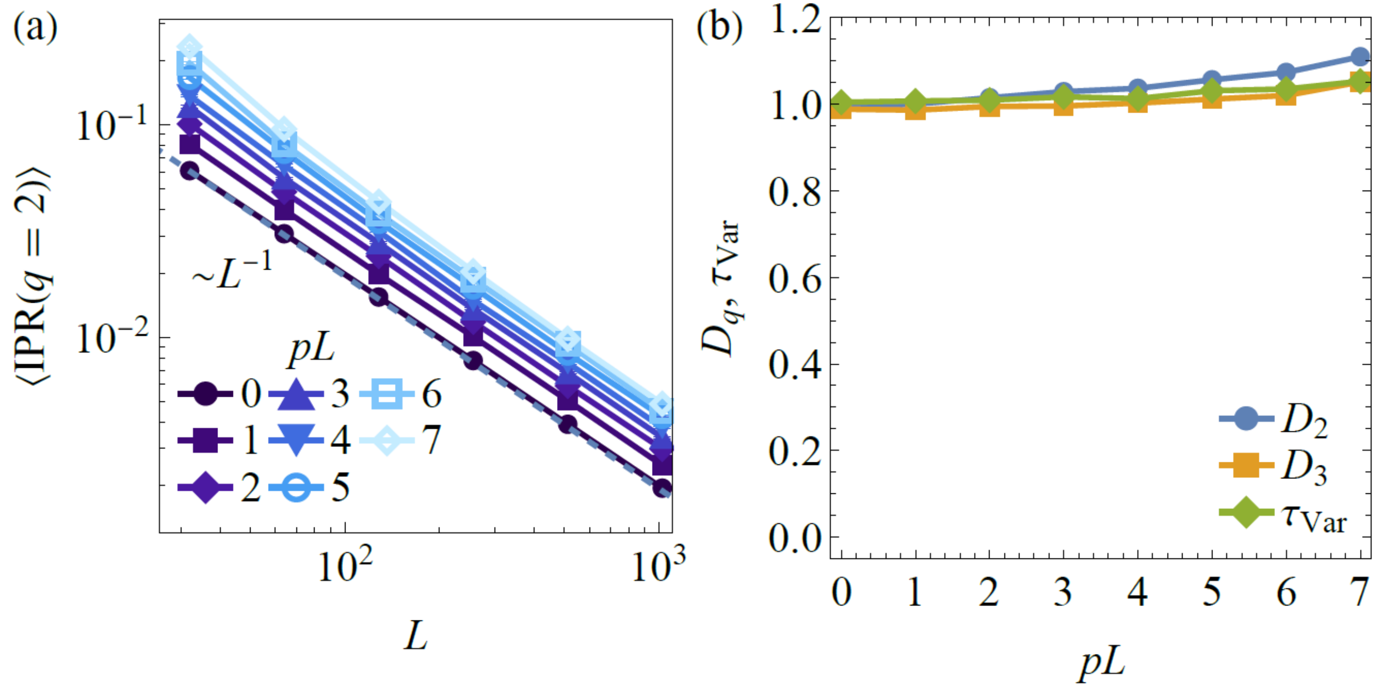
Single-shot measurement model predicts $\tau_q = \tau_q^* = q - 1$.

→ Absence of multifractality for any finite error rate

No-click measurements

Postselection of no-click outcomes

---Particle is never detected.

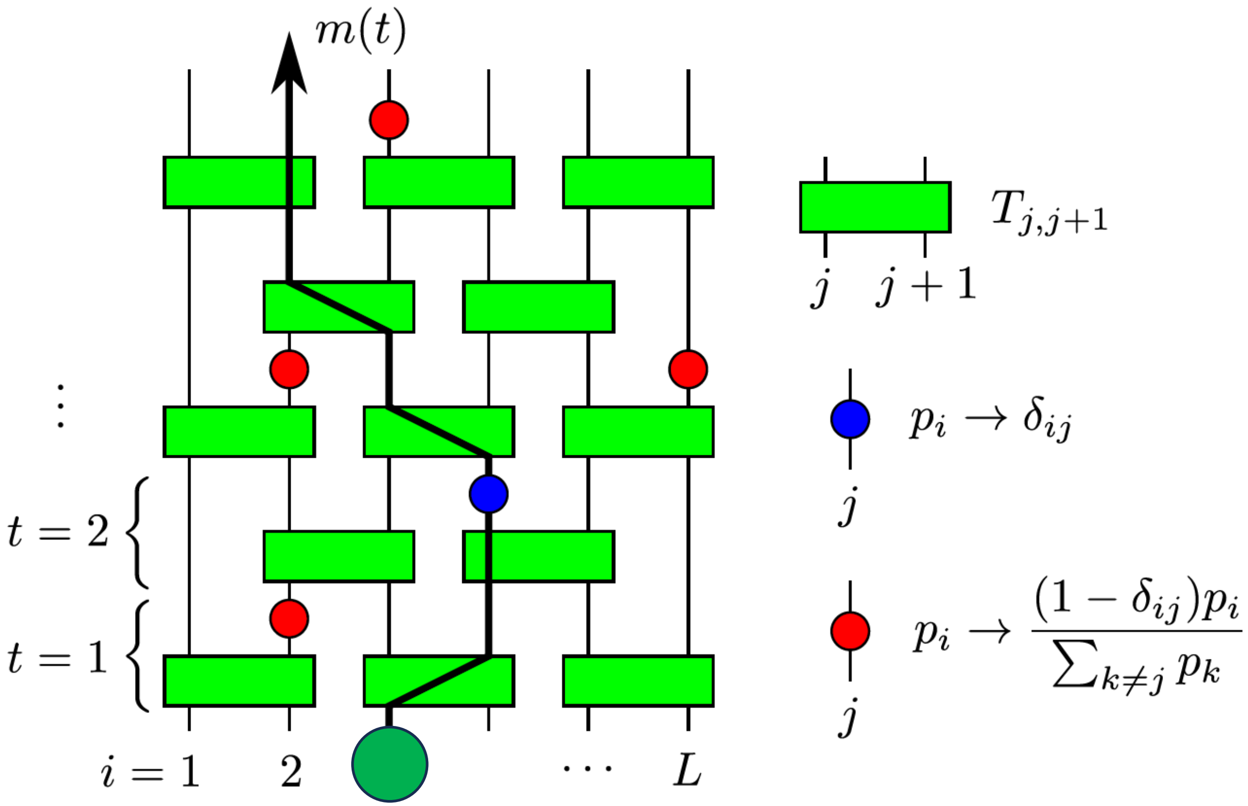


Absence of multifractality under no-click measurements

→ Only click outcomes matter to multifractality.

Classical circuit

Multifractality is not unique to quantum systems.



Particle initially placed at $i = L/2$

--- $T_{j,j+1}$ are drawn randomly or fixed.

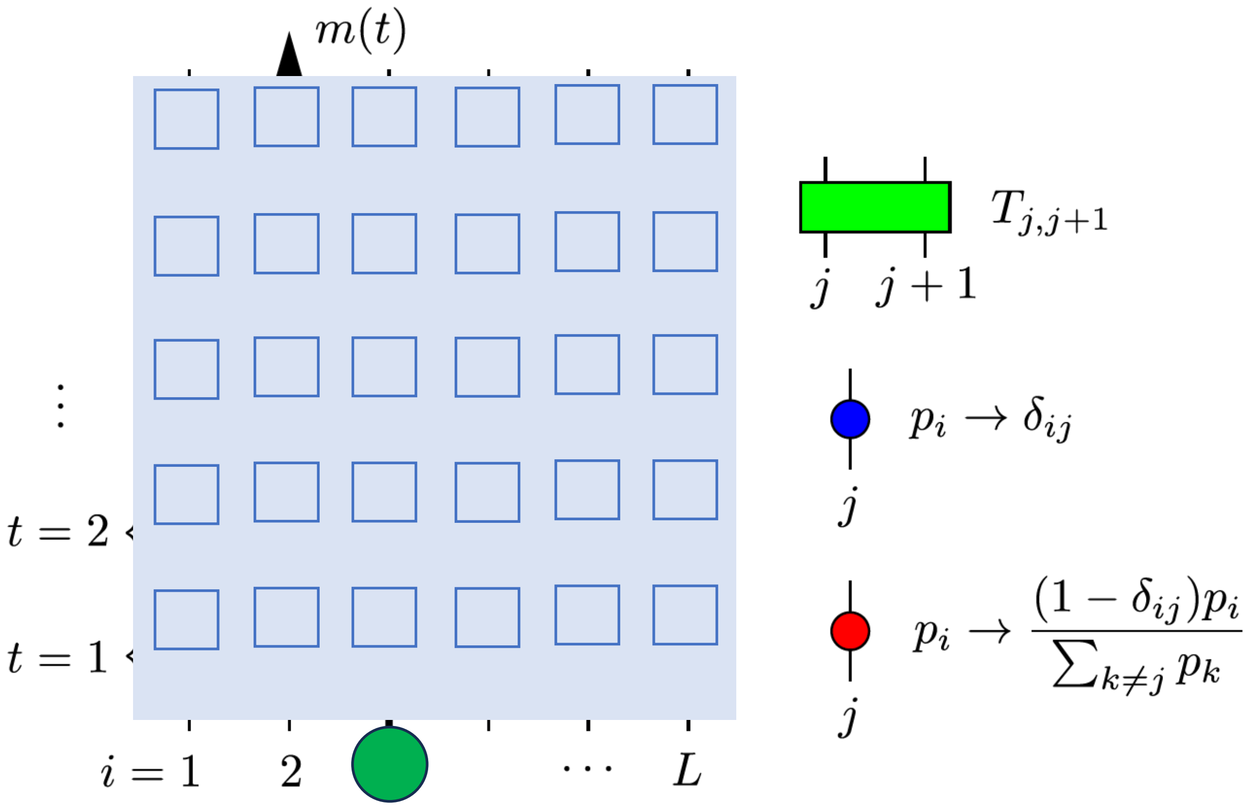
--- Estimate particle trajectory $m(t)$

$$\tilde{\boldsymbol{p}}(t) = \left(\bigoplus_{j \in \mathcal{S}_t} T_{j,j+1} \right) \boldsymbol{p}(t),$$

--- Measurements are performed at every site with probability $p \sim O(1/L)$.

Classical circuit

Multifractality is not unique to quantum systems.



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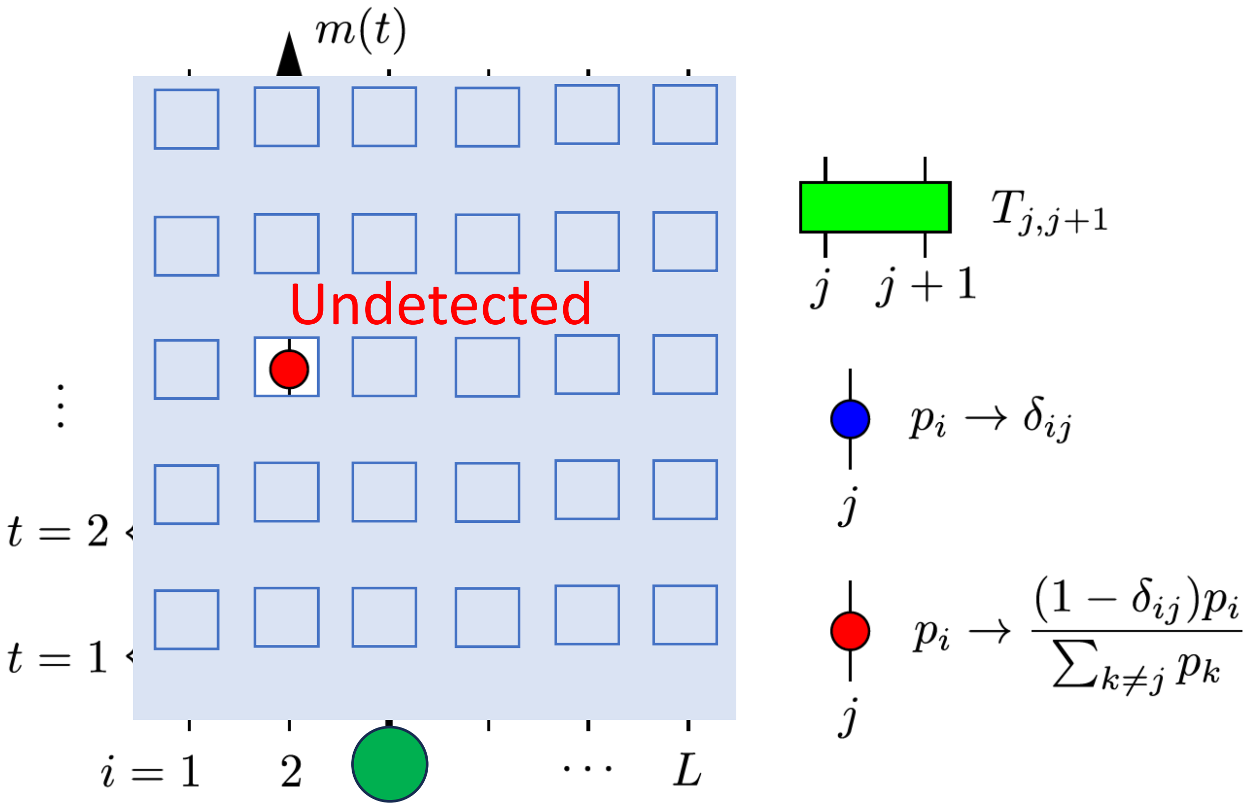
At each time, you are allowed to open one window.

Classical circuit

Multifractality is not unique to quantum systems.



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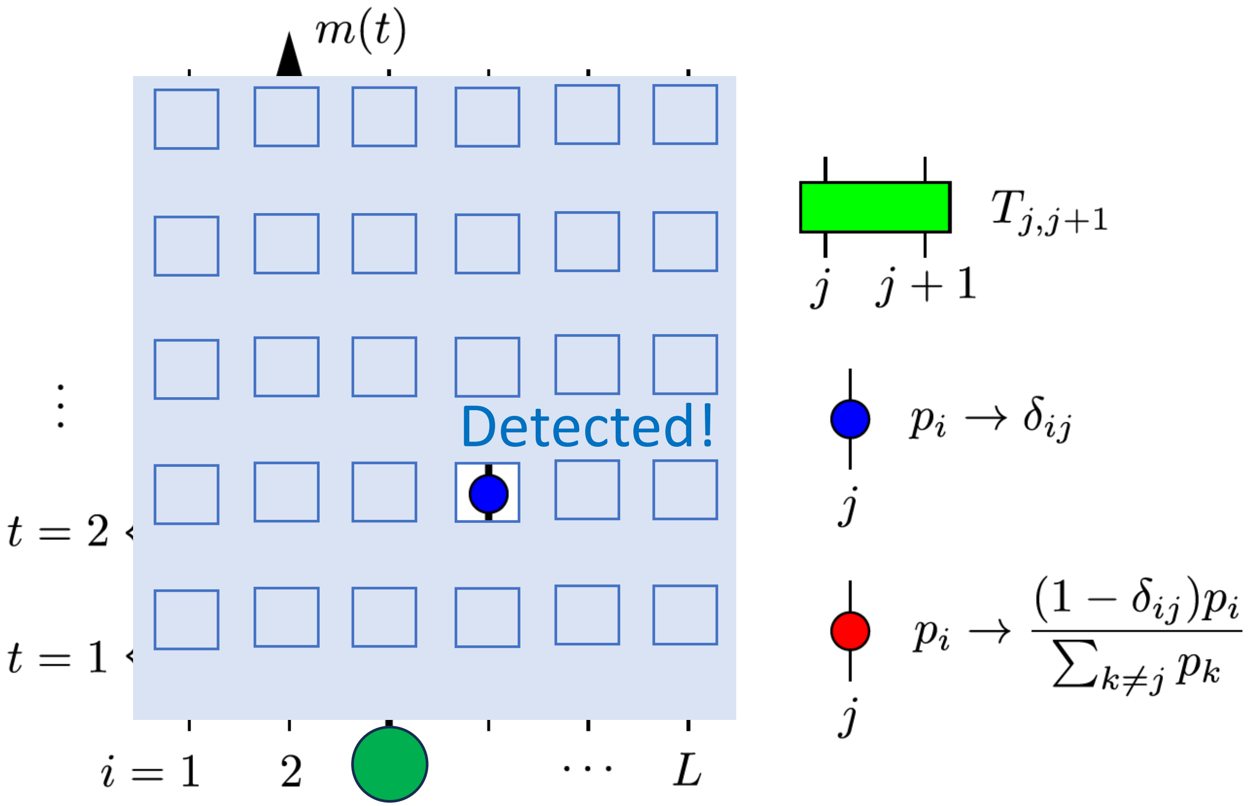

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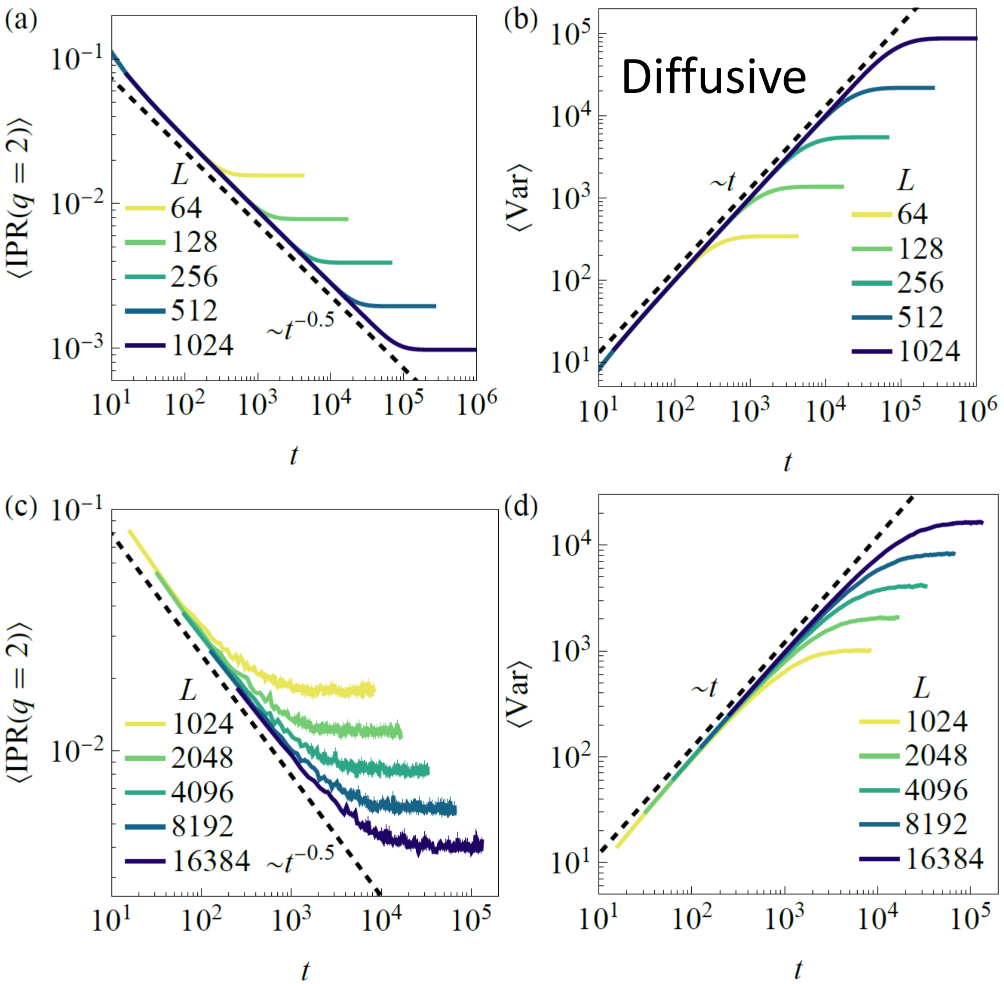
At each time, you are allowed to open one window.

Local transition process + measurements

Random transition matrix



Fixed transition matrix

||

Discrete random walk

→ Diffusive spreading of the particle

Fixed unitary evolution

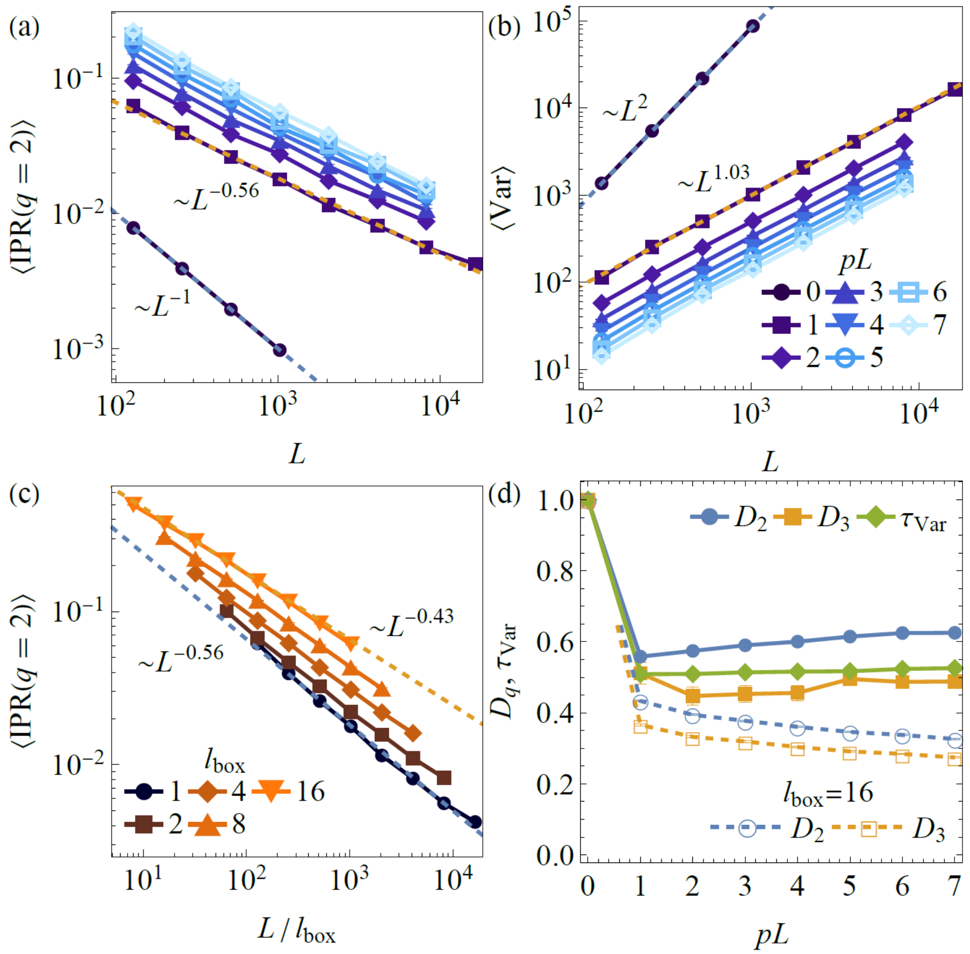
||

Quantum walk

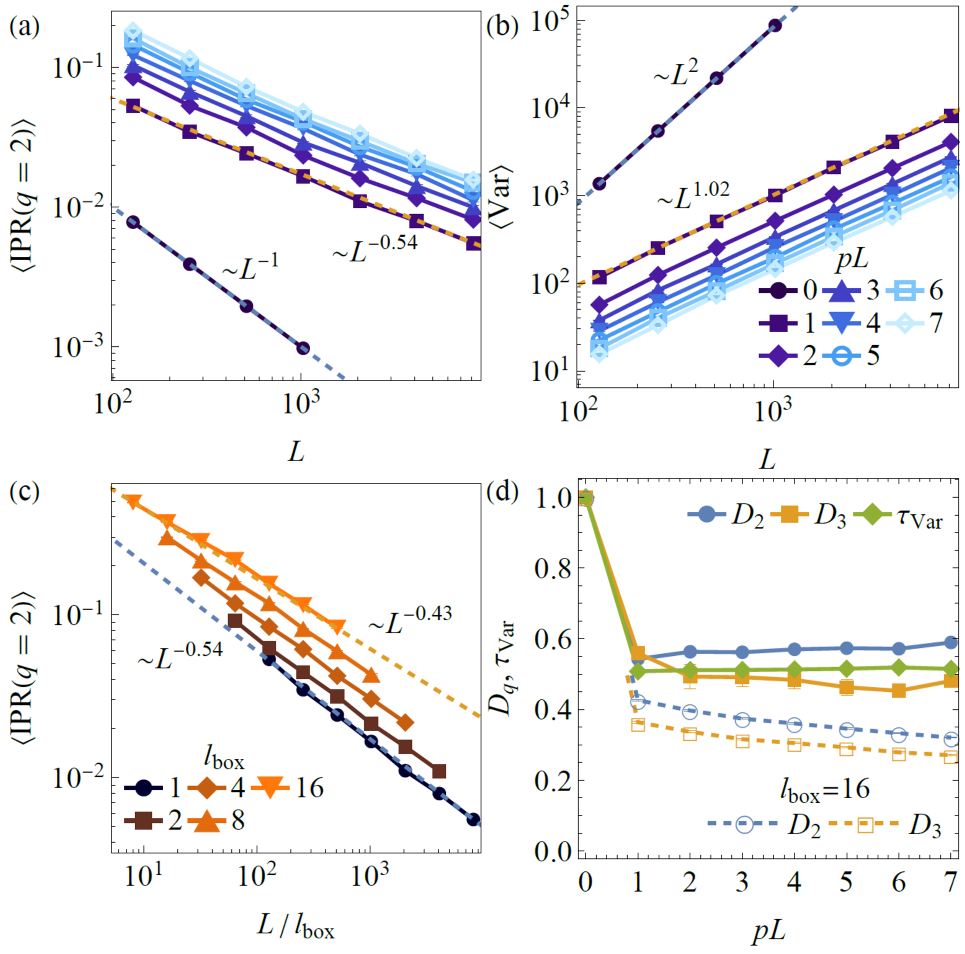
→ *Ballistic* spreading of the particle

Local transition process + measurements

Random transition matrix



Fixed transition matrix

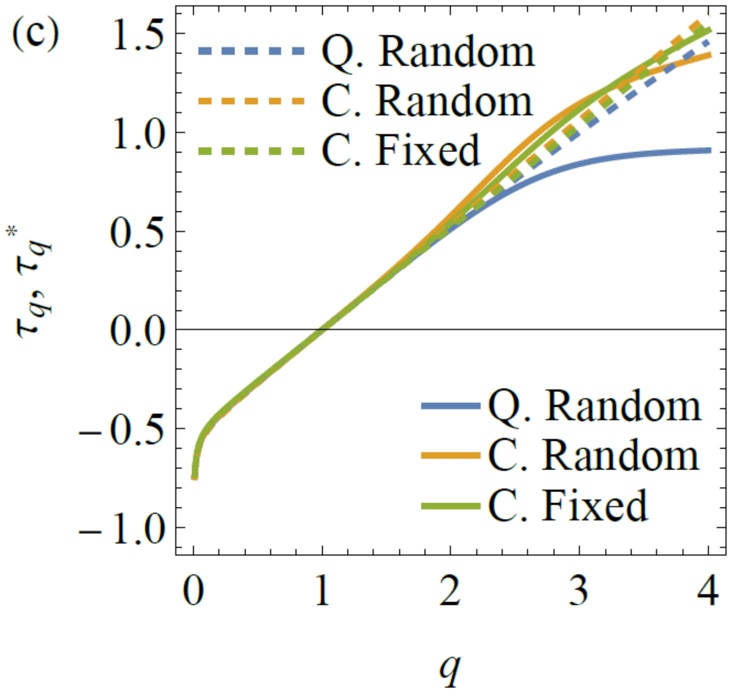


---Exponents similar to random unitary + projective measurements (diffusive) case

Multifractality for monitored particle

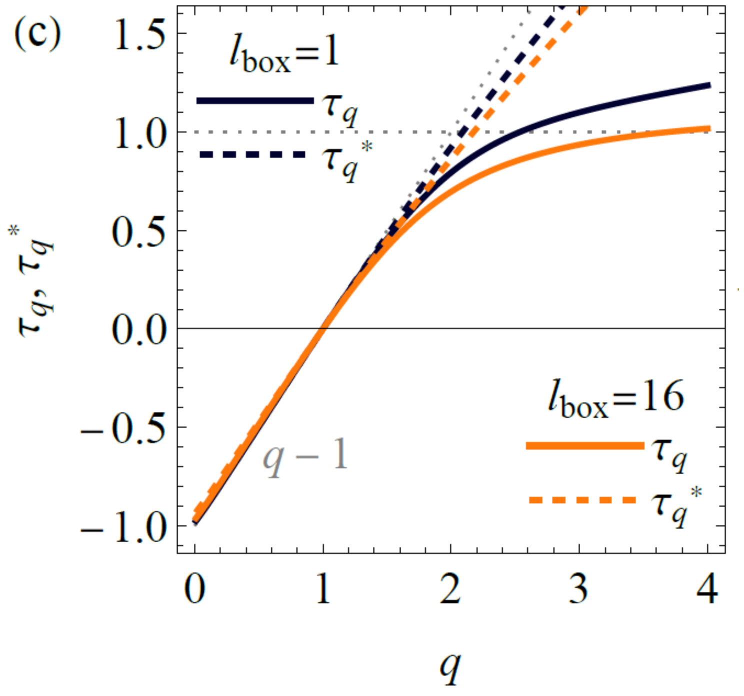
Diffusive cases:

-Random unitary (quantum)
-Random transition matrix (classical)
-Fixed transition matrix (classical)



Ballistics case:

-Fixed unitary (quantum)



Random walk with stochastic resetting

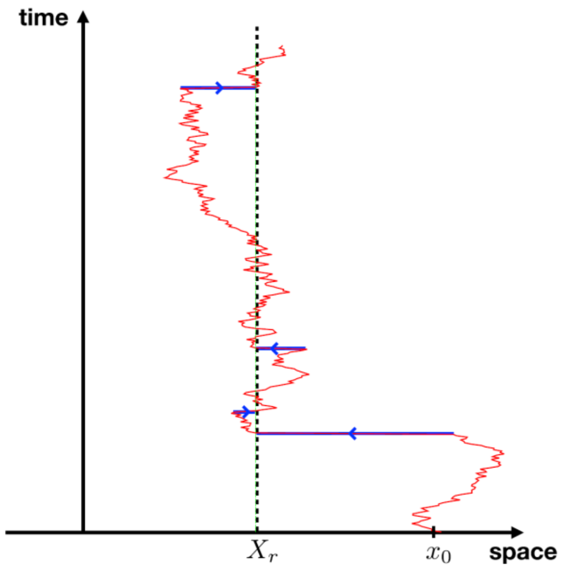
We consider two simplification for monitored dynamics:

- No-click measurements are irrelevant. → Keeping only click measurements
- Measurements occur at a constant rate $\sim 1/L$.

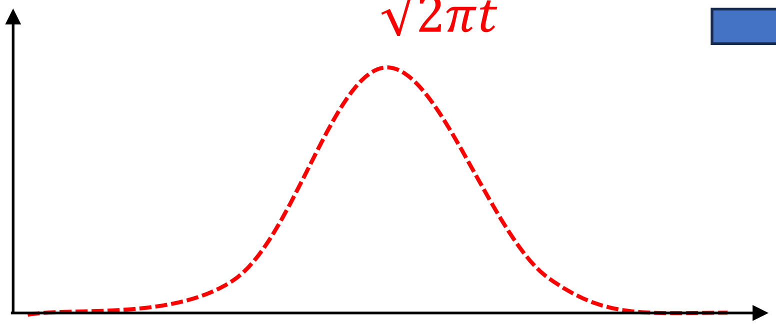
This reduces to Poissonian stochastic resetting to the initial state.

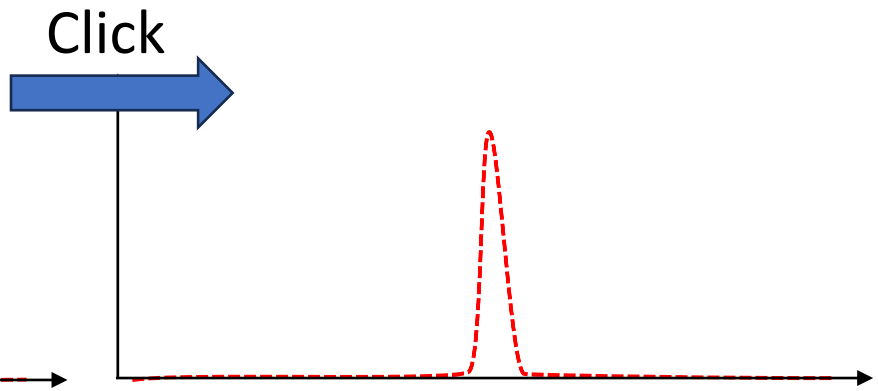
Evans & Majumdar, PRL **106**, 160601 (2011).

Evans, Majumdar, & Schehr, J. Phys. A **53**, 193001 (2020).



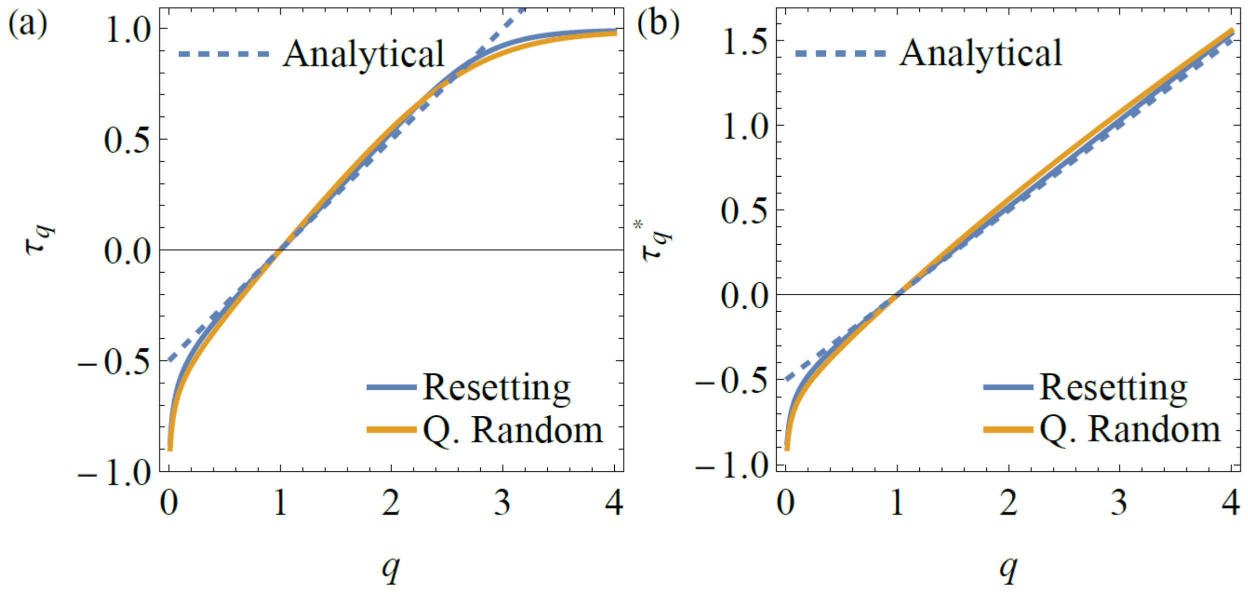
$$p(x,t) \sim \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$$





Random walk with stochastic resetting

Random walk subject to Poissonian stochastic resetting with rate $\lambda = 1/L$



Analytical solutions:

$$\langle \mathrm{IPR}(q) \rangle = \sqrt{\frac{1}{q}\left(\frac{2\pi}{L}\right)^{1-q}} \Gamma\left(\frac{3-q}{2}\right) \text{ if } q < 3,$$

$$e^{\langle \ln \mathrm{IPR}(q) \rangle} = e^{\gamma(q-1)/2} \frac{1}{\sqrt{q(2\pi L)^{q-1}}},$$

$$\tau_q = (q-1)/2 \quad (q < 3),$$
$$\tau_q^* = (q-1)/2.$$

Strong deviation from $q - 1$ for monitored diffusive particles

Summary (Part 2)

Multifractality appears in monitored single particle subject to projective measurements.

- Particle transport (e.g., diffusive or ballistic) strongly affects multifractal scaling.
- Diffusive model reduces to stochastic resetting of a random walker.

--- Monitoring $\sim$ Stochastic resetting always hold?

--- Any implication for elusive free-fermion MIPT?

--- Many-particle generalizations like simple exclusion processes?