Topological Aspects of Quantum Cellular Automata in One Dimension

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UTokyo

MPQ

ZG, C. Sünderhauf, N. Schuch, and J. I. Cirac, PRL **124**, 100402 (2020)

ZG and T. Guaita, arXiv:2106.05044

ZG, L. Piroli, and J. I. Cirac, PRL **126**, 160601 (2021)

ZG, A. Nahum, and L. Piroli, PRL **128**, 080602 (2022)

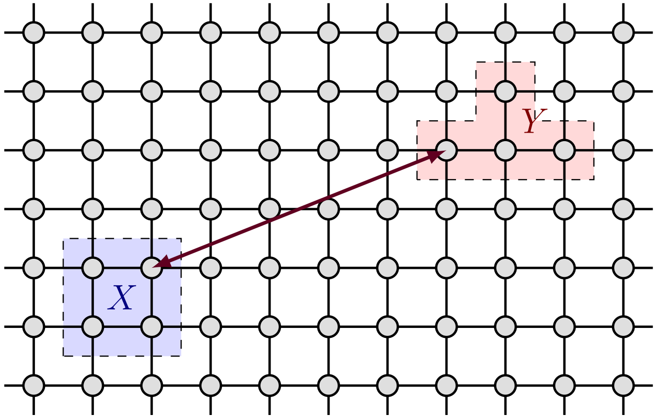
Quantum cellular automata

- Lieb-Robinson bound

"Soft" light cone from locality:

$$\|[O_X(t), O_Y]\| \leq Ce^{-\kappa[\text{dist}(X,Y)-vt]}$$

E. H. Lieb and D. W. Robinson, CMP **28**, 251 (1972)

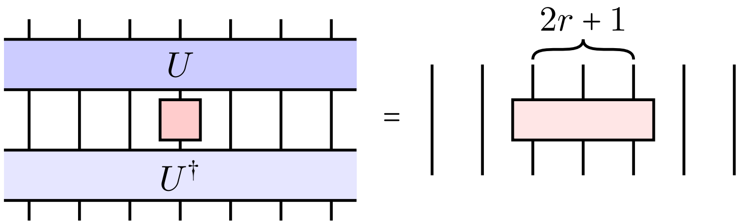


- Definition of quantum cellular automata (QCA)

Unitary that ***strictly*** preserves locality:

$$O_{\bar{A}} = UO_AU^\dagger$$

B. Schumacher and R. F. Werner, arXiv:quant-ph/0405174



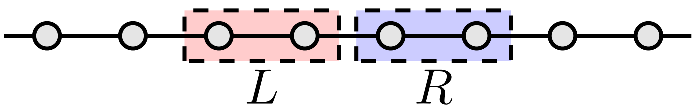
In 1D, QCA = matrix-product unitary (MPU) J. I. Cirac *et al.*, JSM (2017) 083105

Index for 1D QCA

- Definition of the (chiral) index

D. Gross *et al.*, Commun. Math. Phys. **310**, 419 (2012)

$$\mathrm{ind} = \log \frac{\eta(\mathcal{A}_L^U, \mathcal{A}_R)}{\eta(\mathcal{A}_L, \mathcal{A}_R^U)} \in \log \mathbb{Q}^+$$



$$\eta(\mathcal{A}_L^U, \mathcal{A}_R) = \frac{\sqrt{d_L d_R}}{d_\Lambda} \sqrt{\sum_{i,j=1}^{d_L} \sum_{m,n=1}^{d_R} |\mathrm{Tr}_\Lambda[U e_{ij}^{L\dagger} U^\dagger e_{mn}^R]|^2} \qquad e_{ij} = |i\rangle\langle j|$$

Finite-depth quantum circuits are of zero index

Nontrivial example: right translation T on a qudit lattice

$$L = x, R = x + 1$$

$\eta(\mathcal{A}_x^T, \mathcal{A}_{x+1}) = \eta(\mathcal{A}_{x+1}, \mathcal{A}_{x+1}) = d$ (maximal overlap) $\Rightarrow \boxed{\mathrm{ind} = \log d}$

$\eta(\mathcal{A}_x, \mathcal{A}_{x+1}^T) = \eta(\mathcal{A}_x, \mathcal{A}_{x+2}) = 1$ (zero overlap)

Part I

Classification of 1D QCA with finite unitary symmetries

Collaborators: Christoph Sünderhauf, Norbert Schuch,

J. Ignacio Cirac

Phys. Rev. Lett. **124**, 100402 (2020)

Floquet SPT phases (review)

- A heuristic classification

D. V. Else and C. Nayak, PRB **93**, 201103(R) (2016)

Künneth formula

$$H^{d+1}(\mathbb{Z} \times G, \mathrm{U}(1))$$

bulk — Classification of static *d*-dim. SPT phases

edge — Classification of static (*d*-1)-dim. SPT phases

Time-translation symmetry U_{F}

NOT complete! At least in d=2 dim. H. C. Po *et al*., PRX **6**, 041070 (2016)

∵ 1D edges, which are locality-preserving unitaries well described by ***MPUs***, are characterized by the ***chiral index even without symmetry protection***

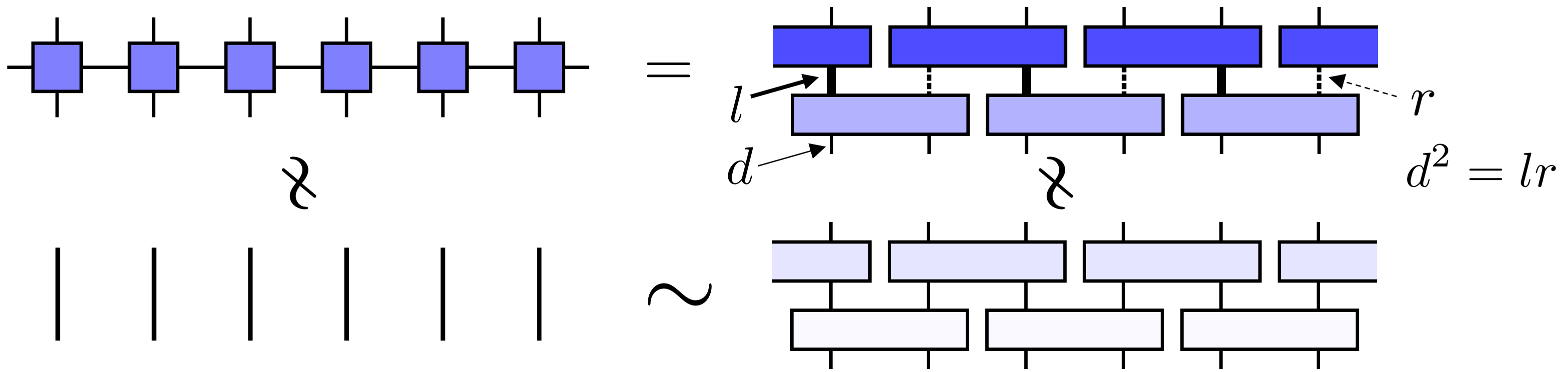
D. Gross *et al*., Commun. Math. Phys. **310**, 419 (2012)

J. I. Cirac *et al*., JSM (2017) 083105

Question: What is the classification (beyond $H^2(G,\mathrm{U}(1))$) of matrix-product unitaries with symmetries?

MPU & MPS w/o symmetry (review)

- MPU is ***not*** always trivial J. I. Cirac *et al.*, JSM (2017) 083105



Chiral index: $\mathrm{ind} = \frac{1}{2}\log\frac{r}{l} = \log\frac{r}{d} \in \log\mathbb{Q}^{+}$

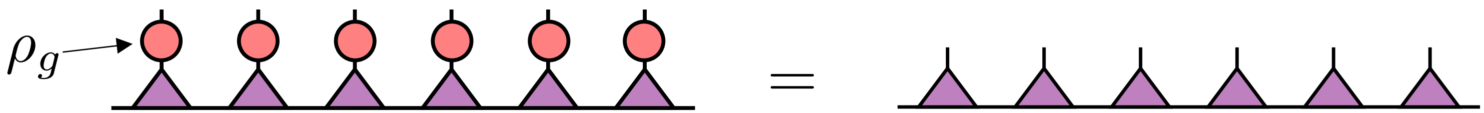
Cf. MPS is always trivial X. Chen, Z.-C. Gu, and X.-G. Wen, PRB **83**, 035107 (2011)

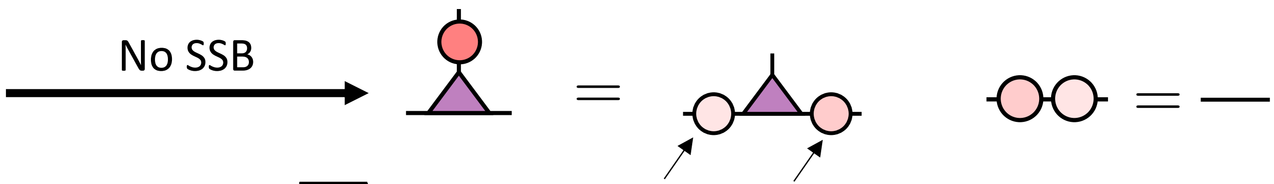
$$|\Psi\rangle \quad \sim \quad |\psi\rangle^{\otimes L}$$

Classification of SPT MPSs (review)

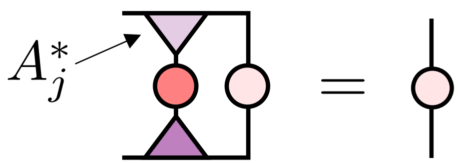
- Symmetry action on the virtual level

D. Pérez-García *et al.*, PRL **100**, 167202 (2008)





$$\sum_i [\rho_g]_{ji} A_i = V_g^\dagger A_j V_g$$



$$V_g \sim \beta_g V_g, \quad \beta_g \in \mathrm{U}(1)$$

- Classification by the 2nd cohomology group

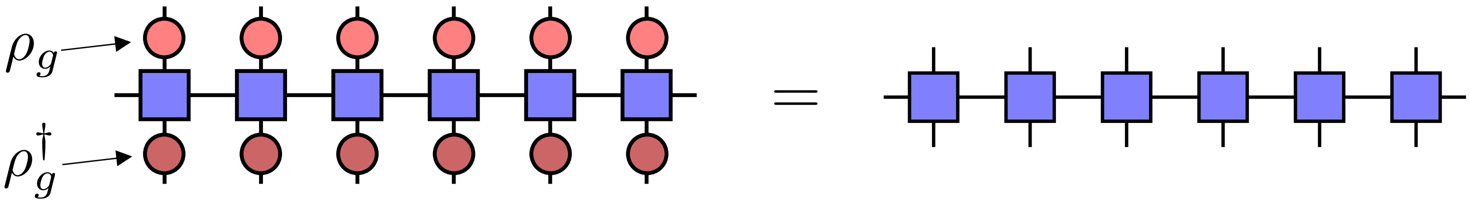
$$\Rightarrow \quad V_g V_h = \omega_{g,h} V_{gh}, \quad \omega_{g,h} \in \mathrm{U}(1), \quad \omega_{g,h} \sim \omega_{g,h} \beta_g \beta_h / \beta_{gh}$$

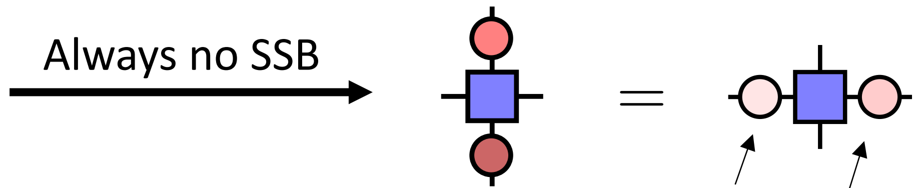
$$\Rightarrow \quad [\omega : G \times G \to \mathrm{U}(1)] \in H^2(G, \mathrm{U}(1))$$

X. Chen, Z.-C. Gu, and X.-G. Wen, PRB **83**, 035107 (2011)
N. Schuch, D. Pérez-García, and I. Cirac, PRB **84,** 165139 (2011)

Cohomology classes for MPUs

- Analogy to MPSs





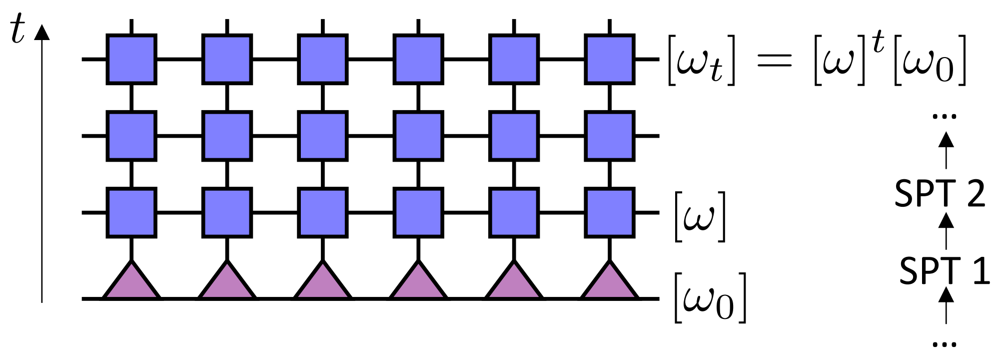
$$\sum_{i',j'}[\rho_g]_{ii'}[\rho_g^*]_{jj'}\mathcal{U}_{i'j'} = z_g^\dagger \mathcal{U}_{ij} z_g$$

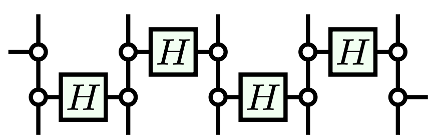
$$z_g z_h = \omega_{g,h} z_{gh}$$

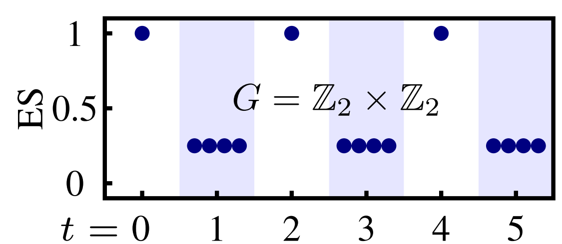
$$[\omega] \in H^2(G, \mathrm{U}(1))$$

- SPT discrete time-crystalline oscillation

A. C. Potter and T. Morimoto, PRB **95**, 155126 (2017)

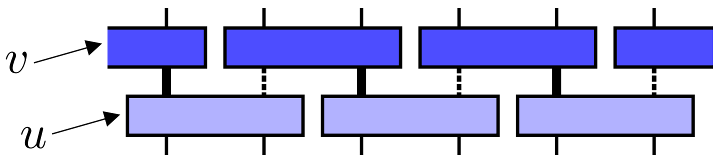


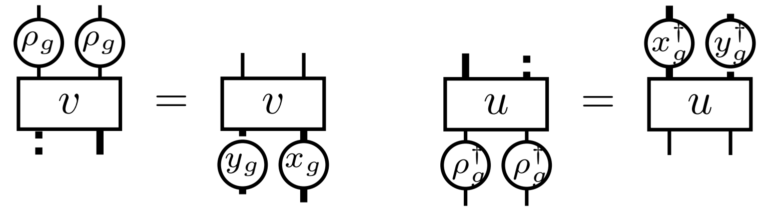




Beyond cohomology

- Symmetry action on the bilayer-unitary form



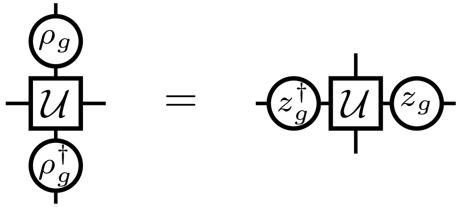


Gauge transformation: J. I. Cirac *et al.*, JSM (2017) 083105

$$v \to v(y \otimes x)$$
$$u \to (x^\dagger \otimes y^\dagger) u$$

$$x_g x_h = \omega_{g,h} x_{gh}$$
$$y_g y_h = \omega_{g,h}^{-1} y_{gh}$$

cf. $z_g z_h = \omega_{g,h} z_{gh}$



- Symmetry-protected index (SPI)

cf. chiral index: $\mathrm{ind} = \frac{1}{2}\log\frac{r}{l} = \log\frac{r}{d}$

$$= \frac{1}{2}\log\frac{\mathrm{Tr}\ y_e}{\mathrm{Tr}\ x_e} = \log\frac{\mathrm{Tr}\ y_e}{\mathrm{Tr}\ \rho_e}$$

Cyclotomic field:
$\mathbb{Q}(\omega_n \equiv e^{\frac{2\pi i}{n}}) \equiv \{\sum_{j=0}^{n-1} q_j \omega_n^j : q_j \in \mathbb{Q}\}$

If $\chi_g \equiv \mathrm{Tr}\ \rho_g \neq 0$, $\mathrm{ind}_g = \frac{1}{2}\log\left|\frac{\mathrm{Tr}\ y_g}{\mathrm{Tr}\ x_g}\right| = \log\left|\frac{\mathrm{Tr}\ y_g}{\mathrm{Tr}\ \rho_g}\right| \in \log|\mathbb{Q}(\omega_{d_g})/\{0\}|$

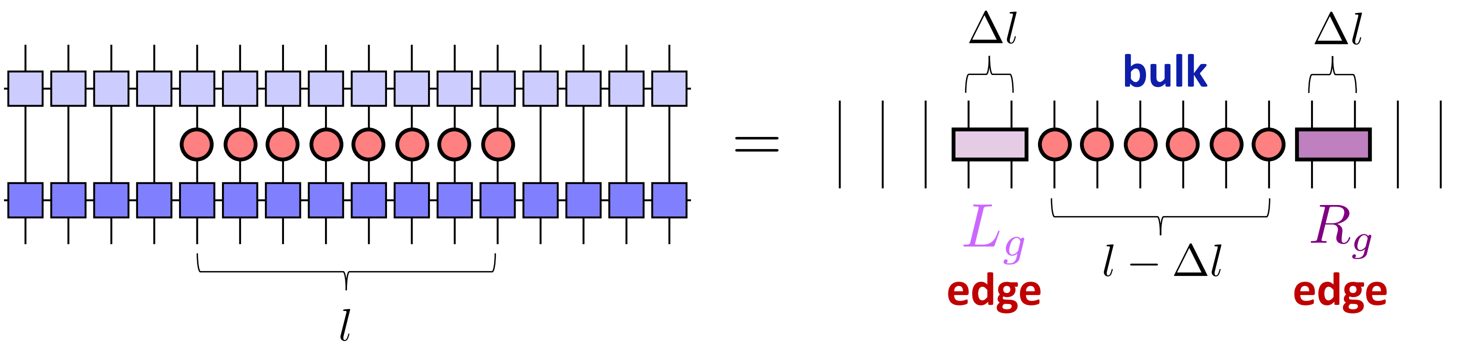
$d_g \equiv \min\{d \in \mathbb{Z}^+ : g^d = e\}$

Interpreting the symmetry-protected index

- Evolved symmetry-string operator

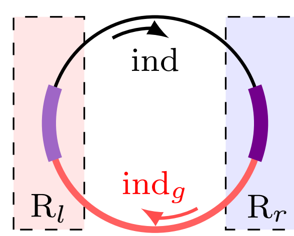
cf. edge modes in equilibrium SPT phases

$$U^\dagger \rho_g^{\otimes l} U = L_g \otimes \rho_g^{\otimes (l-\Delta l)} \otimes R_g$$



$$\mathrm{ind}_g - \mathrm{ind} = \frac{1}{2} \log \left| \frac{\mathrm{Tr}\ L_g}{\mathrm{Tr}\ R_g} \right|$$ ***– Measure of asymmetry***

- Symmetry-charge-pump picture



State-like vs. Genuinely dynamical

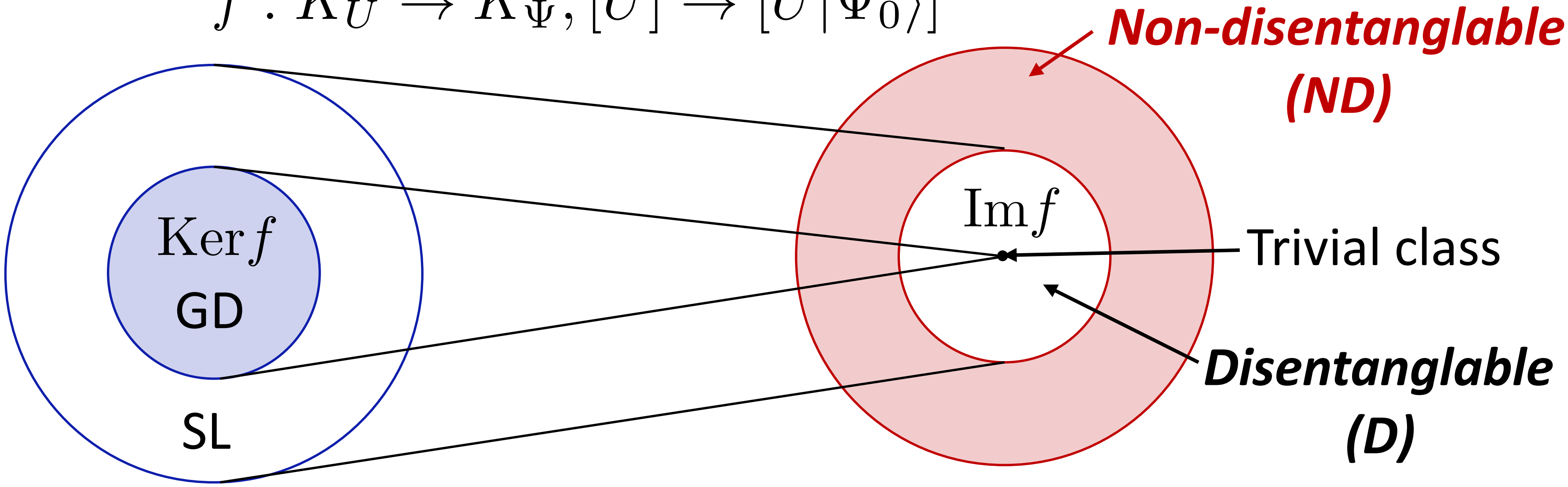
- Two types of topological invariants

Cohomology class: trivial -> SPT – ***state-like (SL)***

SPI / index: trivial -> trivial – ***genuinely dynamical (GD)***

- Homomorphism from QCA to short-range states

$$f : K_U \to K_\Psi, [U] \to [U|\Psi_0\rangle]$$

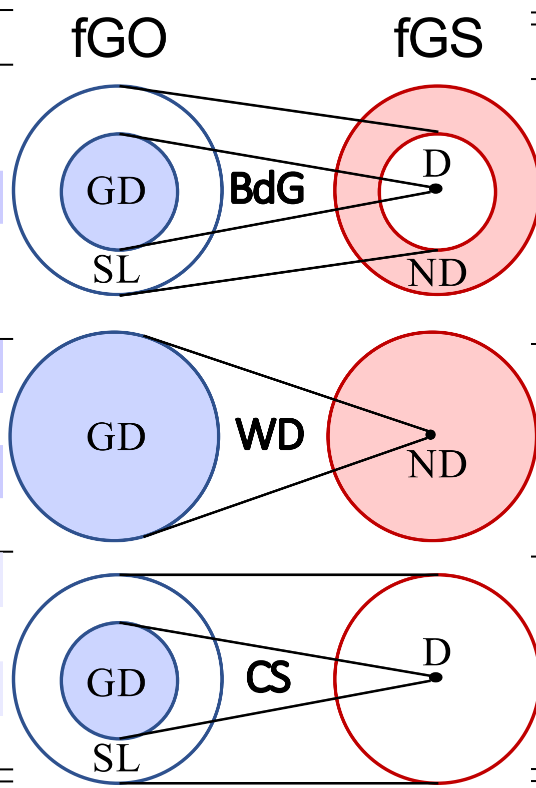


Refining the periodic table

- Map from fermionic Gaussian QCA to fermionic Gaussian states with fundamental symmetries

Cf. A. Schnyder *et al.*, PRB **78**, 195125 (2008)

d	0	1	2	3
D	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
DIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$
C	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
CI	0	$\mathbb{Z}$	0	$\mathbb{Z}$
A	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
AII	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
AIII	0	$\mathbb{Z}^2$	0	$\mathbb{Z}^2$
BDI	$\mathbb{Z}_2^2$	$\mathbb{Z}^2$	0	0
CII	0	$2\mathbb{Z}^2$	0	$\mathbb{Z}_2^2$



0	1	2	3
$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
0	0	$2\mathbb{Z}$	0
0	0	0	$2\mathbb{Z}$
$\mathbb{Z}$	0	$\mathbb{Z}$	0
$\mathbb{Z}$	0	0	0
$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
0	$\mathbb{Z}$	0	$\mathbb{Z}$
$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$

ZG and T. Guaita, arXiv: 2106.05044

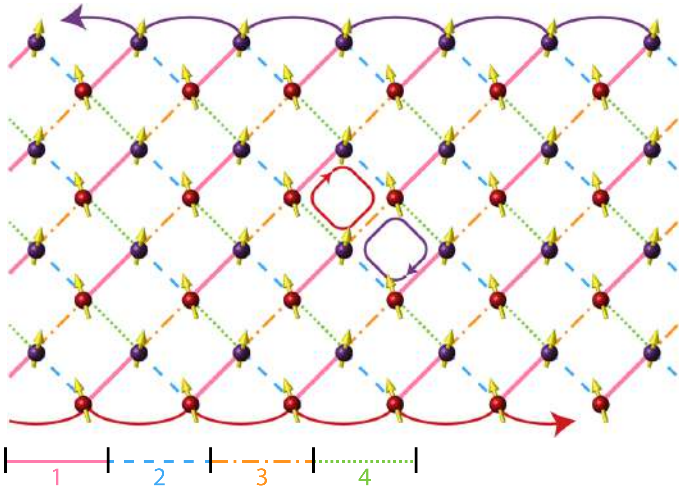
Part II

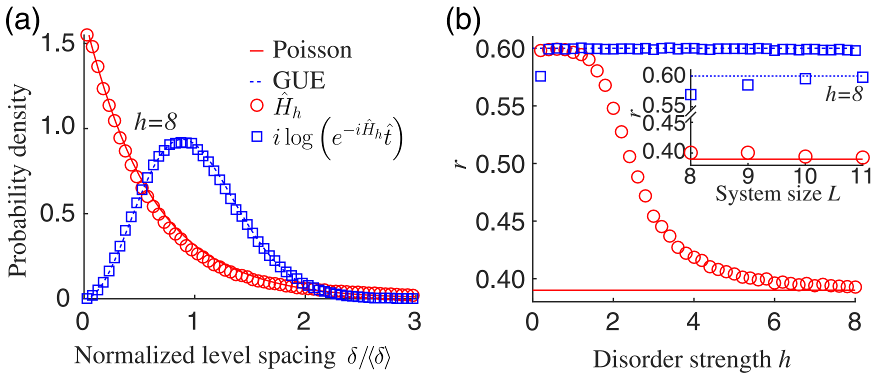
Impact of nontrivial dynamical topology on entanglement growth

Collaborators: Lorenzo Piroli, J. Ignacio Cirac

Phys. Rev. Lett. **126**, 160601 (2021)

Motivation





H. C. Po *et al.*, PRX **6**, 041070 (2016)

2D Chiral Floquet MBL Phases

1D Edge cannot be MBL

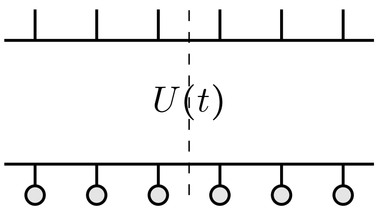
Locality + MBL ⟶ QCA

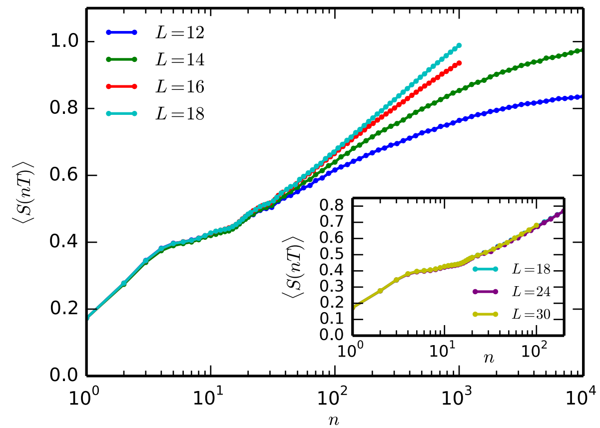
Question: Can we rigorously rule out MBL?

Entanglement growth

	MBL	Translation

State entanglement



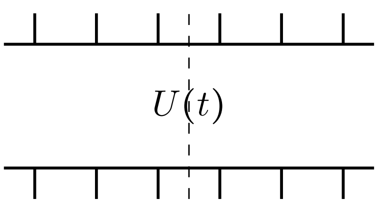


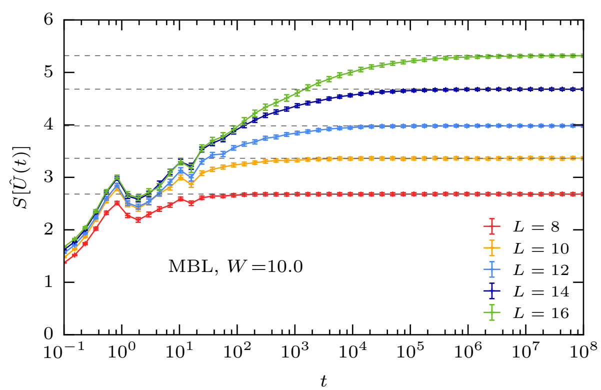
$$|\Psi_{\text{prod}}\rangle \to |\Psi'_{\text{prod}}\rangle$$

$$S(t) = 0$$

J. H. Bardarson, F. Pollmann, and J. E. Moore, PRL **109**, 017202 (2012)
P. Ponte *et al.*, PRL **114**, 140401 (2015)

Operator entanglement





?

P. Zanardi, PRA **63**, 040304(R) (2001)

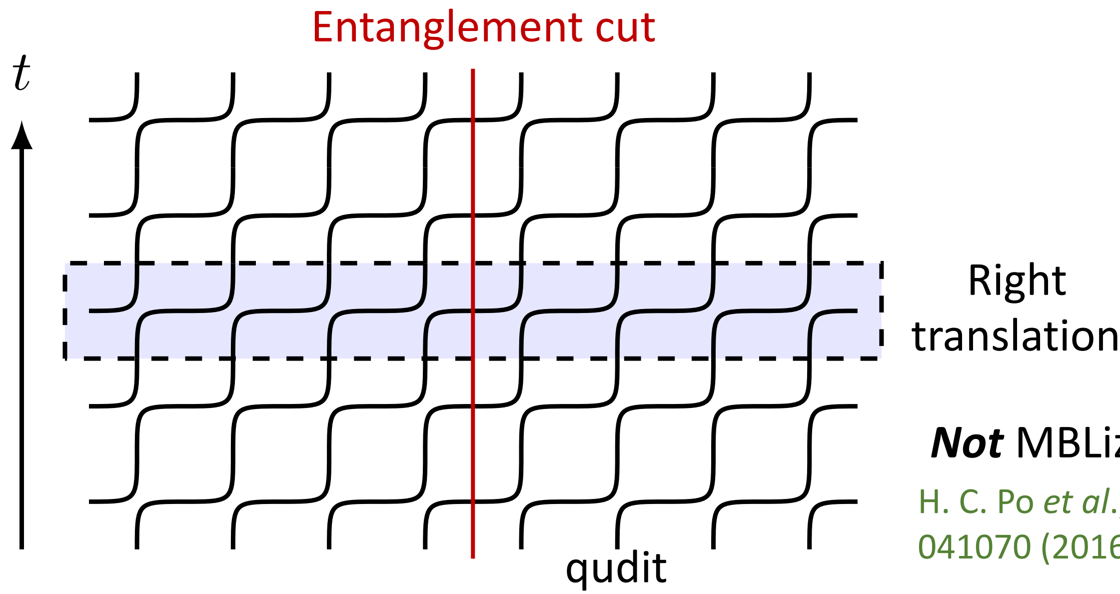
T. Zhou and D. J. Luitz, PRB **95**, 094206 (2017)

Conjecture (@2020 MPHQ interview)

- Idea:

$$\mathrm{ind} = \log d$$

$$S_{\mathrm{OEE}} = t \log d$$



Not MBLizable

H. C. Po *et al*., PRX **6**, 041070 (2016)

Operator entanglement bounded by the chiral index?

- Impact: Rigorous result on topology & thermalization

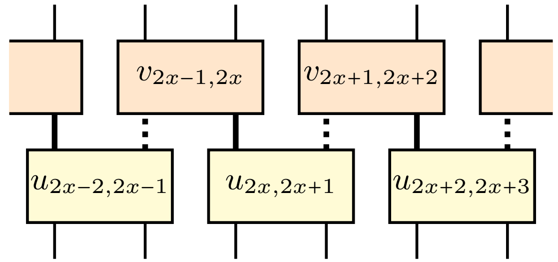
  Lower bound on chaos (cf. MSS bound)

J. Maldacena, S. H. Shenker, and D. Stanford, JHEP 2016, 106

Setup and the main result

- Index of a QCA

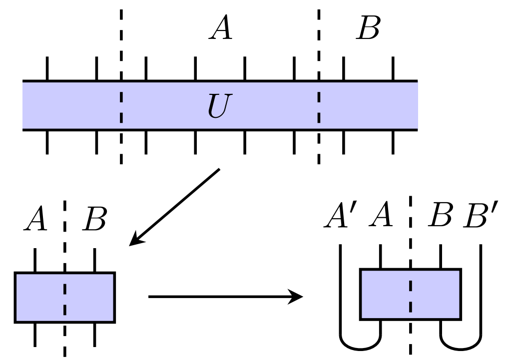
$$\mathrm{ind} = \log \frac{d'_{2x}}{d_{2x}} = \log \frac{d_{2x+1}}{d'_{2x+1}} \in \log \mathbb{Q}^+$$



- Operator entanglement

$$|U\rangle \equiv (U \otimes \mathbb{I})|I\rangle, \quad |I\rangle \equiv d^{-N/2}(\textstyle\sum_{j=1}^{d} |jj\rangle)^{\otimes N}$$

$$S^{(\alpha)}_{AA'} \equiv \tfrac{1}{1-\alpha} \log \mathrm{Tr}\rho^{\alpha}_{AA'}$$



- Lower bound on chaos

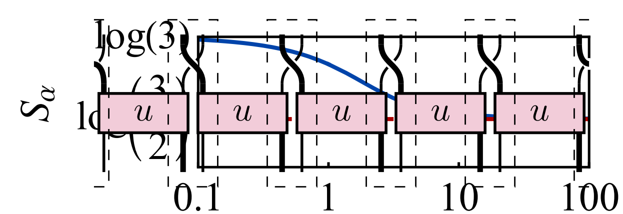
$$S^{(\alpha)}_{AA'} \geq 2|\mathrm{ind}|$$

If $\min\{|A|, N - |A|\} \geq 2r$

Tight for

(i) $|\mathrm{ind}| \in \log \mathbb{Z}^+$

(ii) $\alpha = \infty$



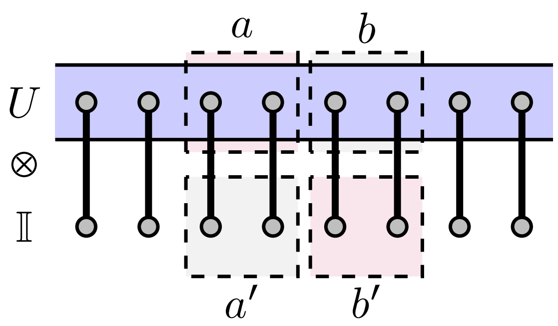
Proof of the main result

- Entropy formula of the index

$$\mathrm{ind} = \frac{1}{2}(S^{(\alpha)}_{ab'} - S^{(\alpha)}_{a'b})$$

$$S_{aba'b'} \geq |S_{ab'} - S_{a'b}| = 2|\mathrm{ind}|$$

$$\Rightarrow S^{(\alpha\leq 1)}_{aba'b'} \geq 2|\mathrm{ind}|$$



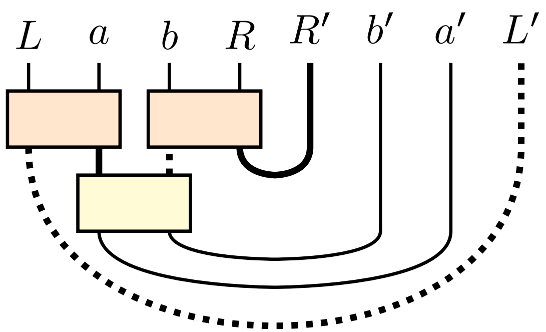
- Weak subadditivity

W. van Dam and P. Hayden, arXiv:quant-ph/0204093

$$S^{(\alpha)}_{aba'b'} = S^{(\alpha)}_{LL'} + S^{(\alpha)}_{RR'}$$

$$S^{(\alpha)}_{LL'} \geq \max\{S^{(\alpha)}_{L} - S^{(0)}_{L'}, S^{(\alpha)}_{L'} - S^{(0)}_{L}\} = |\mathrm{ind}|$$

$$S^{(\alpha)}_{RR'} \geq \max\{S^{(\alpha)}_{R} - S^{(0)}_{R'}, S^{(\alpha)}_{R'} - S^{(0)}_{R}\} = |\mathrm{ind}|$$



Stability against exponential tails

$$U = \hat{\mathrm{T}} e^{-i\int_0^T dt \sum_j h_j(t)} U_{\mathrm{QCA}}$$

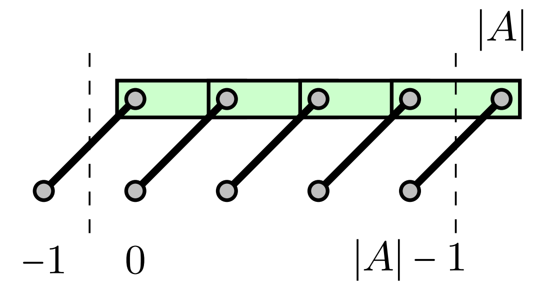
Local and bounded $h \equiv \max_{j,t} \|h_j(t)\|$

- Example of violating $S^{(\alpha)} \geq 2\,\mathrm{ind}$

$$H(t) = h\mathbb{S}^{[j_t, j_t+1]}, \quad j_t = \lfloor t|A|/T \rfloor$$

$$S^{(\infty)} = 2\log d - \log[1 + (d^2 - 1)\epsilon]$$

$$\epsilon = \sin^{2|A|}(hT/|A|) \sim e^{-\mathcal{O}(|A|\log|A|)}$$



- General proof for $S^{(\alpha)} > 2\,\mathrm{ind} - e^{-\mathcal{O}(|A|\log|A|)}$

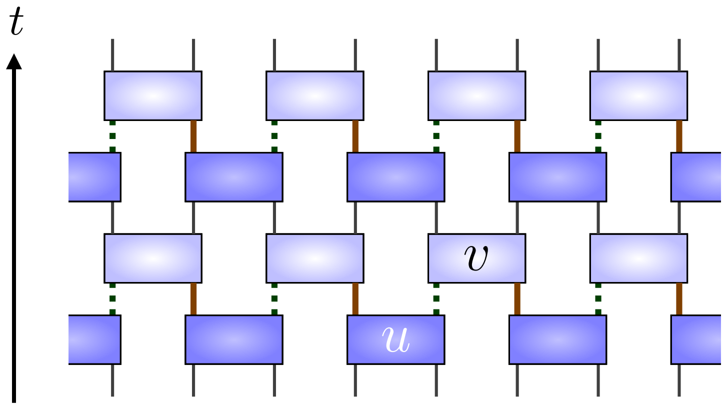
Step 1 – Approximate Hamiltonian evolution by quantum circuit

T. J. Osborne, PRL **97**, 157202 (2006)

Step 2 – Optimizing the (time-dependent) Lieb-Robinson bound

M. B. Hastings, arXiv:1008.5137; ZG et al., PRA **101**, 052122 (2020)

Exactly solvable chaotic anomalous dynamics: Random QCA



i.i.d Haar random

$$u : \mathbb{C}^d \otimes \mathbb{C}^d \to \mathbb{C}^p \otimes \mathbb{C}^q$$
$$v : \mathbb{C}^q \otimes \mathbb{C}^p \to \mathbb{C}^d \otimes \mathbb{C}^d$$
$$\Rightarrow \text{ind} = \frac{1}{2} \log \frac{q}{p}$$

Cf. Random quantum circuits ($p = q = d$)

$$v_B = \frac{d^2-1}{d^2+1}$$

$$v_E = \log_d\left(\frac{d^2+1}{2d}\right)$$

Butterfly velocity:

$$v_L = \frac{p^2q^2 - q(p+q)+1}{p^2q^2-1}, v_R = \frac{p^2q^2-p(p+q)+1}{p^2q^2-1}$$

State- & Operator-entanglement velcotiy:

$$v_E = \log_{d^2}\left[\frac{(pq+1)^2}{2d(p+q)}\right], v_E^{(\mathrm{o})} = \log_{d^2}\left[\frac{(pq+1)^2(pq-1)}{d(\sqrt{p(q^2-1)}+\sqrt{q(p^2-1)})^2}\right]$$

Tripartite-information velocity: $v_{\mathrm{tri}} = v_E^{(\mathrm{o})} - \frac{|\mathrm{ind}|}{\log(pq)}$

C. W. von Keyserlingk *et al.*, PRX **8**, 021013 (2018); A. Nahum, S. Vijay, and J. Haah, PRX **8**, 021014 (2018)

ZG, A. Nahum, and L. Piroli, **128**, 080602 (2022)

Entanglement-membrane theory

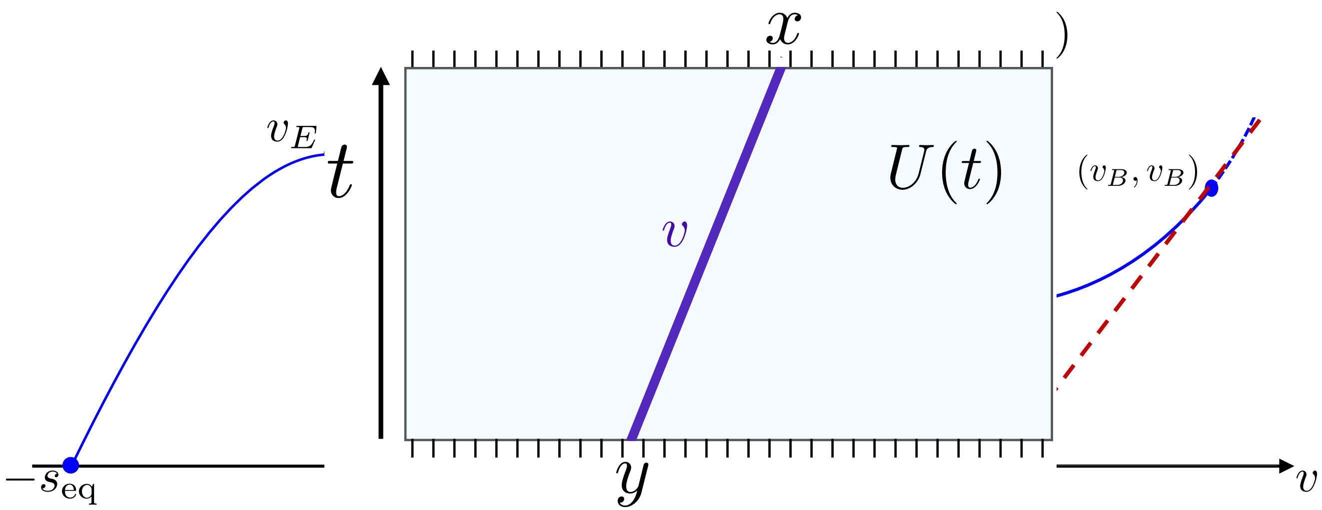
- Hydrodynamic equation (zero index)

$$\frac{\partial S}{\partial t} = s_{\text{eq}} \Gamma \left(\frac{\partial S}{\partial x} \right)$$

$S(x,t)$: Entanglement entropy of subsystem $(-\infty, x]$ at time t

Formal solution:

$$S(x,t) = \min_y \left(t s_{\text{eq}} \mathcal{E} \left(\frac{x-y}{t} \right) + S(y,0) \right) \qquad \mathcal{E}(v) = \max_s \left(\Gamma(s) + \frac{vs}{s_{\text{eq}}} \right)$$



C. Jonay, D. A. Huse, and A. Nahum, arXiv:1803.00089

Entanglement-membrane theory

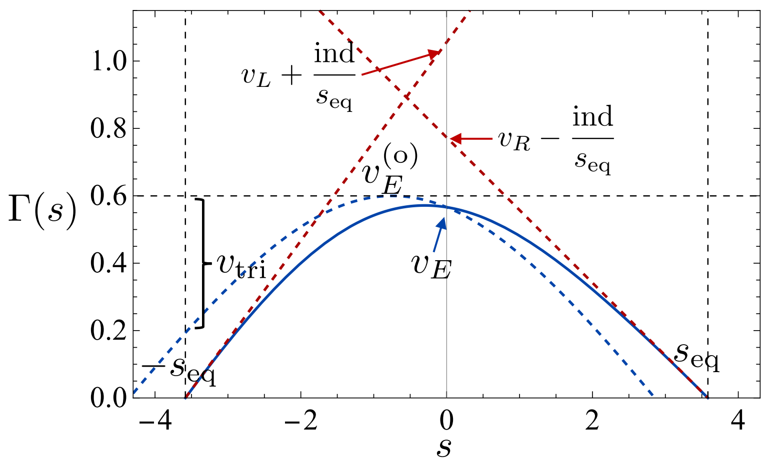
- Hydrodynamic equation (nonzero index)

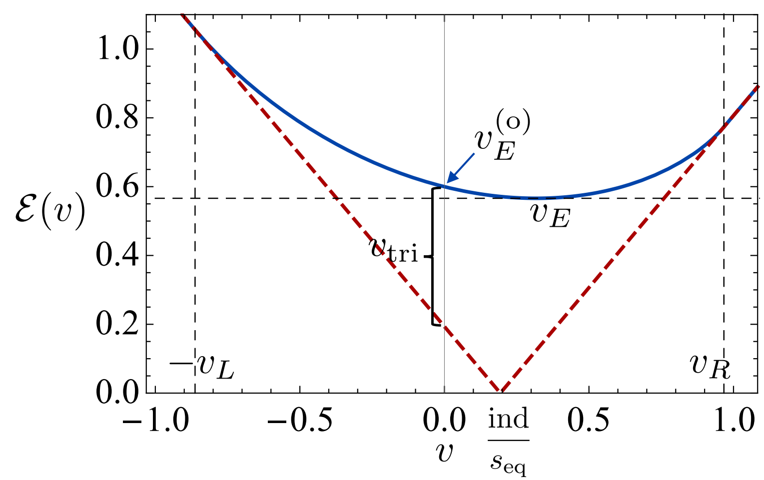
$$\frac{\partial S}{\partial t} + \boxed{\frac{\text{ind}}{s_{\text{eq}}}\frac{\partial S}{\partial x}} = s_{\text{eq}}\Gamma\left(\frac{\partial S}{\partial x}\right)$$

Index appears as a background velocity

Formal solution:

$$S(x,t) = \min_y \left(t s_{\text{eq}} \mathcal{E}\left(\frac{x-y}{t}\right) + S(y,0) \right) \quad \mathcal{E}(v) = \max_s \left(\Gamma(s) + \frac{vs}{s_{\text{eq}}} - \boxed{\frac{\text{ind}}{s_{\text{eq}}}\frac{s}{s_{\text{eq}}}} \right)$$





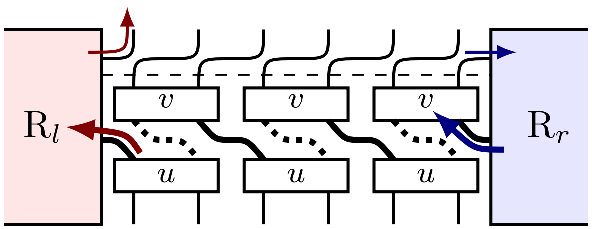
ZG, A. Nahum, and L. Piroli, PRL **128**, 080602 (2022)

Summary

- Classification of symmetric 1D QCA

Part I

Cohomology + SPI



State-like vs. genuinely dynamical

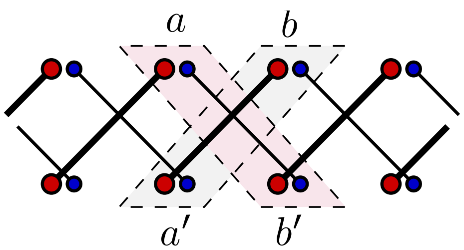
arXiv:2106.05044

PRL **124**, 100402 (2020)

- Impact on entanglement dynamics

Part II

$$S^{(\alpha)} \geq 2|\mathrm{ind}|$$



Generalized entanglement-membrane theory

PRL **128**, 080602 (2022)

PRL **126**, 160601 (2021)

Outlook

Part I

- Complete classification in 1D Cf. C. Zhang, arXiv:2306.03171
- Nontrivial QCA in higher dimensions Cf. J. Haah, L. Fidkowski, and M. B. Hastings, Commun. Math. Phys. **398**, 469 (2023)
- Homomorphism for interacting systems Cf. T. D. Ellison and L. Fidkowski, PRX **9**, 011016 (2019)

Part II

- Tighter bound for finite α
- Impact of SPI
- Modular commutator, generalized Kitaev sum

Cf. R. Fan, P. Zhang, and Y. Gu, SciPost Phys. **15**, 249 (2023)

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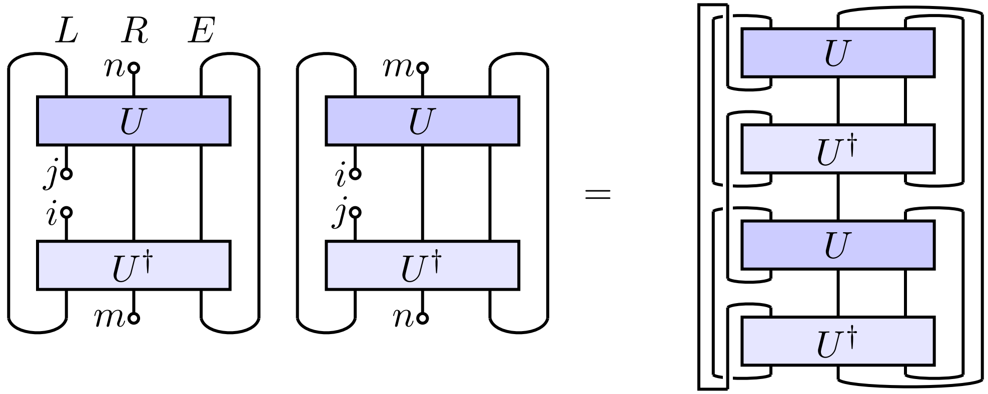
Appendix: Equivalence between algebra overlap and Rényi-2 entropy

$$\text{ind} = \ln \frac{\eta(\mathcal{A}_L^U, \mathcal{A}_R)}{\eta(\mathcal{A}_L, \mathcal{A}_R^U)} = \frac{1}{2}(S^{(2)}_{LR'} - S^{(2)}_{L'R})$$

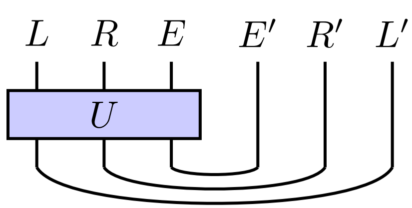
$$\eta(\mathcal{A}_L^U, \mathcal{A}_R) = \frac{\sqrt{d_L d_R}}{d_\Lambda} \sqrt{\sum_{i,j=1}^{d_L} \sum_{m,n=1}^{d_R} |\text{Tr}_\Lambda [U e_{ij}^{L\dagger} U^\dagger e^R_{mn}]|^2}$$

$\sum_{i,j,m,n}$



$$\eta(\mathcal{A}_L^U, \mathcal{A}_R)^2 = d_L d_R \text{Tr} \rho^2_{L'R}$$

$|U\rangle = \frac{1}{\sqrt{d_\Lambda}}$



$\rho_{L'R} = \text{Tr}_{LR'EE'}[|U\rangle\langle U|] = \frac{1}{d_\Lambda}$