Fractionalized non-invertible symmetry

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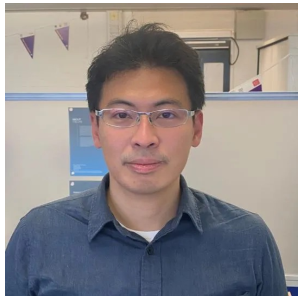
Thanks to



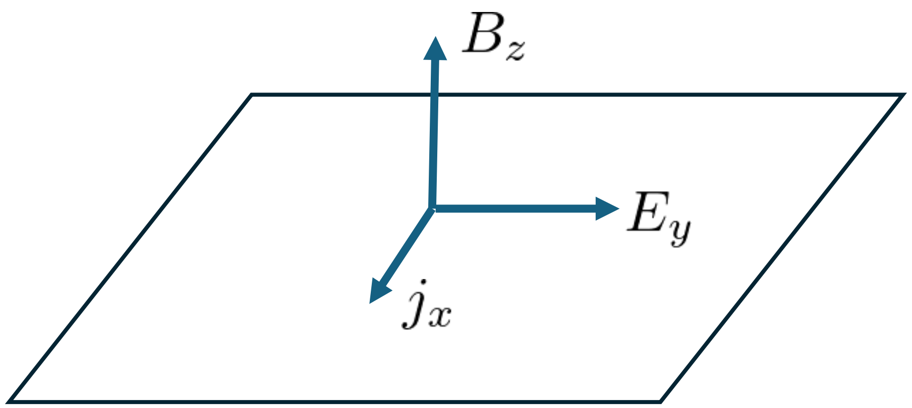
Po-Shen Hsin
(UCLA -> Kings College)

Carolyn Zhang
(Harvard)

Po-Shen Hsin, RK, Carolyn Zhang, arXiv: 2405.20401

“Fractionalization of Coset Non-Invertible Symmetry and Exotic Hall Conductance”

Fractional Quantum Hall effect



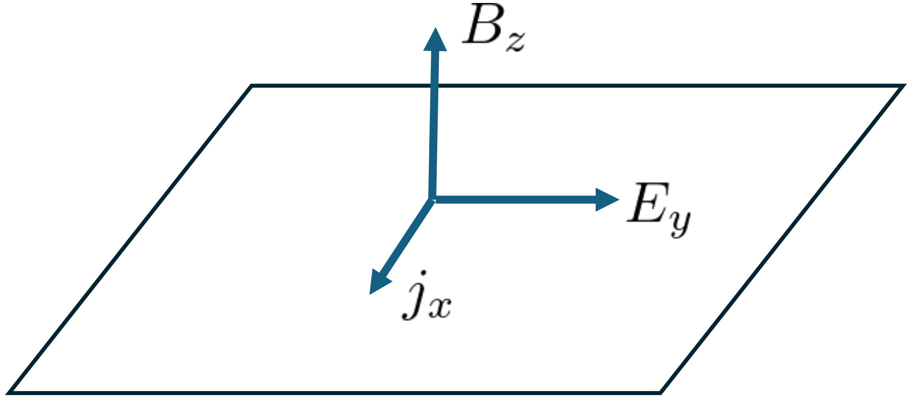
$$j_x = \sigma_{xy} E_y$$

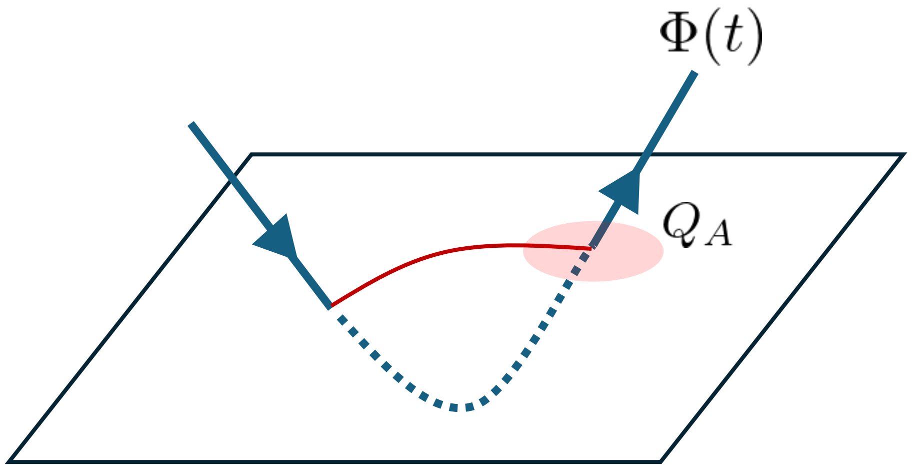
$$\sigma_{xy} = \frac{\nu}{2\pi}$$

Quantized!

- Topological order w/ U(1) global symmetry
- Quasiparticles = Anyons

Flux insertion



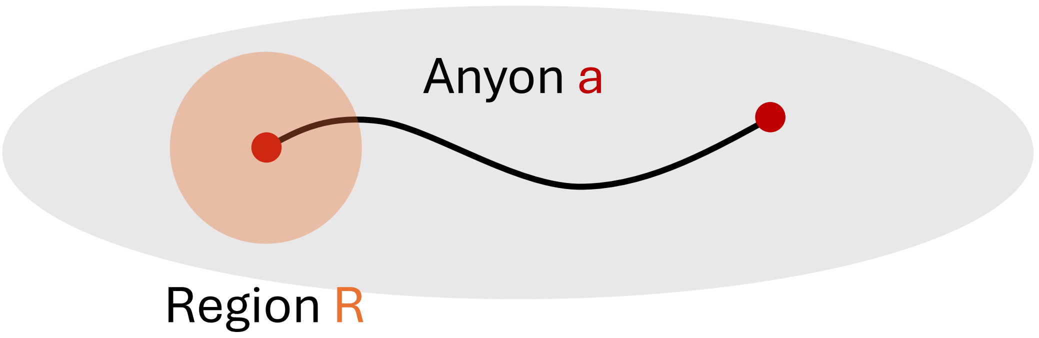


$$\Delta Q_A = (\Delta\Phi) \cdot \sigma_{xy}$$
$$= 2\pi\sigma_{xy} = \nu$$

2pi vortex carries electric charge

= Abelian anyon ”vison”

Symmetry fractionalization: Anyons carry fractional charge under U(1) symmetry



The symmetry acts on anyons in a projective fashion:

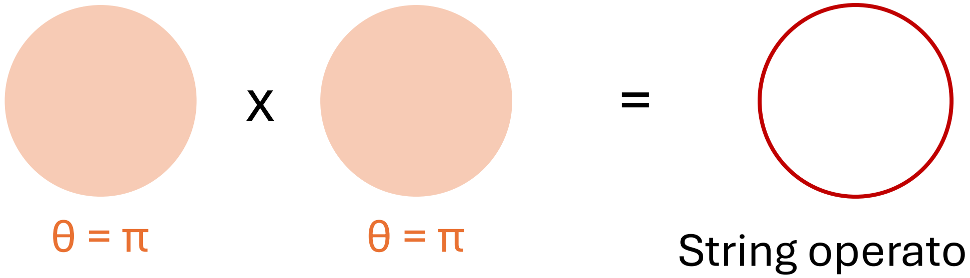
$$U_R(\pi) \times U_R(\pi) = \exp(2\pi i q_a) U_R(0)$$

U(1) symmetry on R

$\theta = \pi$

Projective phase = fractional charge

$$U_R(\pi) \times U_R(\pi) = \exp(2\pi i q_a) U_R(0)$$



String operator of Abelian anyon:

2pi flux = Vison

Fractional charge = Mutual braiding between anyon and a vison

Hall conductance = Fractional charge of a vison = Mutual braiding of vison and vison = (Spin of a vison) x 2

Formula: $\sigma_H = 2h_v$ (mod 2)

Spin of vison v

Non-invertible symmetry

Symmetry operator doesn't have inverse. Examples include:

- String operator of non-Abelian anyon (Ising anyon… $\sigma \times \sigma = 1 + \psi$) [Kitaev, Levin=Wen…]
- Kramers-Wannier duality (In 1d, $D \times D = 1 + \eta$)
- Rep(D8) symmetry in 1d Cluster state ($D \times D = 1 + U_1 + U_2 + U_1U_2$)
- Anyon chains

  …

[Aasen=Fendley=Mong, Koide=Nagoya=Yamaguchi, Kaidi=Ohmori=Zheng, Choi=Cordova=Hsin=Lam=Shao…]

[Thorngren=Wang, Seifnashri=Shao…]

In this talk, I want to discuss fractionalization of non-invertible symmetry in topological order.

Projective symmetry group

In spin liquids, it typically happens that...

Microscopic model has a global symmetry G. (projective symmetry group)

Subgroup K can act trivially at low energy, and it is gauged. (invariant gauge group)

[Wen]

The global symmetry is a coset: G/K

(e.g. parton wave function of spin liquids)

When K is a normal subgroup, G/K is a group

When K is a non-normal subgroup, G/K is not a group, but a non-invertible symmetry

[Heidenreich=McNamara=Montero=Reece=Rudelius=Valenzuela, Arias-Tamargo=Rodriguez-Gomez, Nguyen=Unsal=Tanizaki, Schafer-Nameki...]

Thought experiment: FQH with Alice electrodynamics

Fractional Quantum Hall has (emergent) charge conjugation symmetry.

Permute the anyons of Abelian FQH as: $a \to a^{-1}$

Let's suppose that the charge conjugation has been gauged,

i.e., coupled to dynamical Z2 gauge field.

U(1) Chern-Simons theory becomes O(2) Chern-Simons after gauging Z2.

Non-Abelian anyon: (a, a^{-1})

U(1) symmetry is no longer a symmetry, since it's not gauge invariant: $\exp(i\theta Q) \to \exp(-i\theta Q)$

Cosine symmetry

But this is not an end of the story...

There is a continuous non-invertible symmetry, generated by $\tilde{U}_\theta = \text{“}\cos(\theta Q)\text{”}$

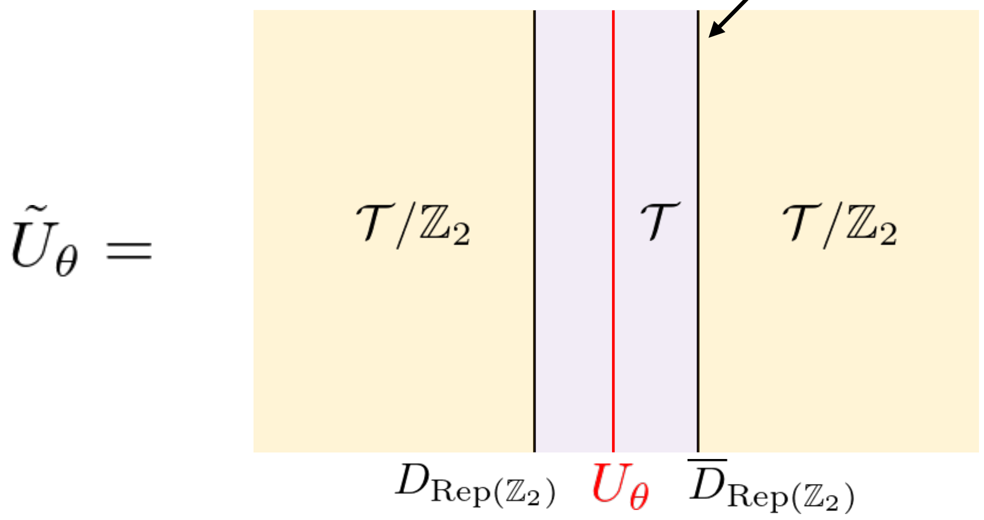
“Cosine symmetry”

Cosine symmetry has a non-invertible fusion rule: $\tilde{U}_\theta \times \tilde{U}_{\theta'} = \tilde{U}_{\theta+\theta'} + \tilde{U}_{\theta-\theta'}$

”product to sum formula” of cosθ. $2\cos\theta \times 2\cos\theta' = 2\cos(\theta+\theta') + 2\cos(\theta-\theta')$

Cosine symmetry defect from "sandwich"

Condensation of Z2 electric particle



T: theory with O(2) symmetry

T/Z2: theory obtained by gauging Z2 of T. Has a cosine symmetry.

[2405.20401]

Cosine symmetry defect

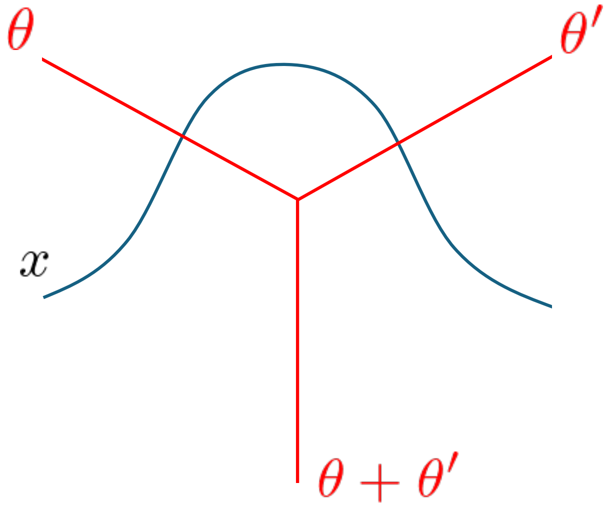
= U(1) symmetry defect (U) sandwiched by a gapped interface (D)

Gapped interface connects the Z2 gauge theory T/Z2 and the original theory T (half gauging)

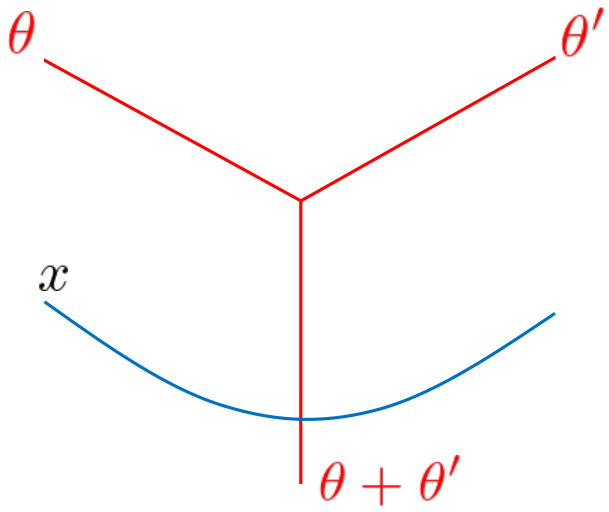
“Fractional charge” under cosine symmetry

Warm-up: U(1) symmetry

anyon

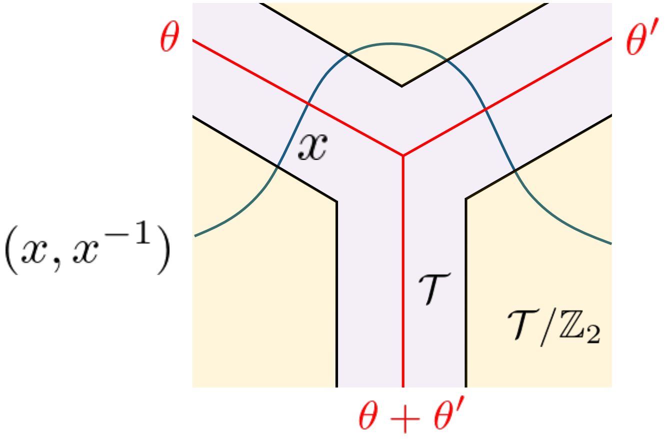


$$= \exp\left(iq_x(\theta + \theta' - [\theta + \theta']_{2\pi})\right)$$



Fractional charge = Phase under crossing anyon through the junction

"Fractional charge" under cosine symmetry



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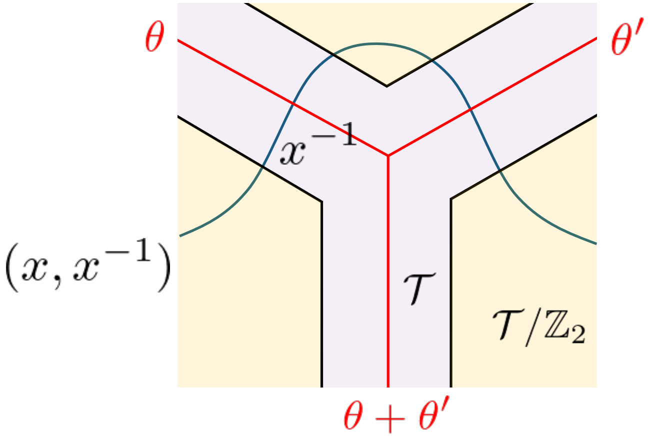

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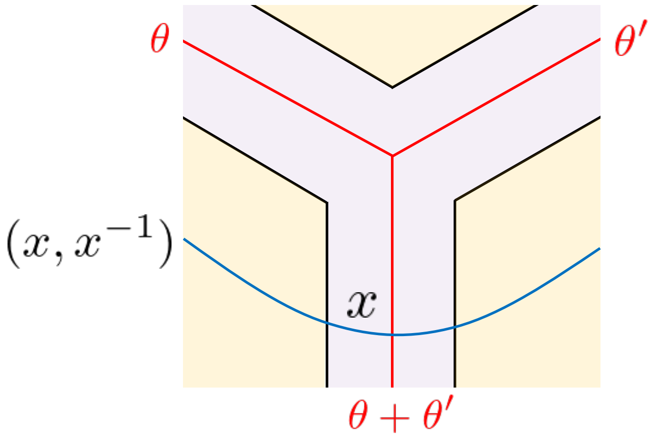
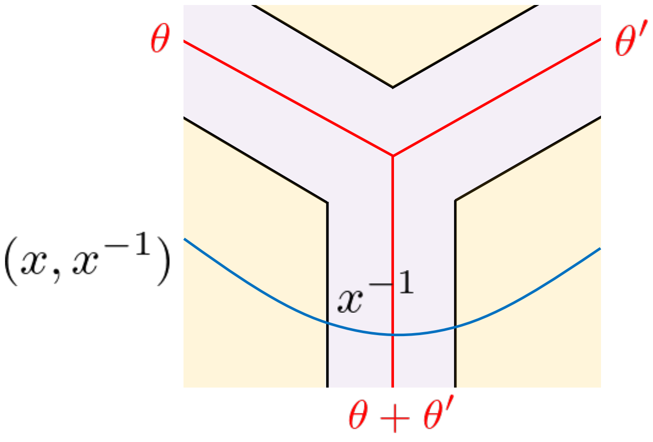


x $\exp\left(iq_x(\theta + \theta' - [\theta + \theta']_{2\pi})\right)$

$\oplus$



x $\exp(-iq_x(\theta + \theta' - [\theta + \theta']_{2\pi}))$

Fractional cosine charge:

Superposition of
opposite fractional U(1) charge

[2405.20401]

Hall conductance for cosine symmetry

Even after charge conjugation is gauged, one can still define electric Hall conductance,

which is now associated with non-invertible cosine symmetry.

Hall conductance is contact term of current two-point function

$$\langle j_\mu(x,y,t) j_\nu(0) \rangle \supset \sigma_H \epsilon_{\mu\nu\lambda} \partial^\lambda \delta^3(x,y,t)$$

Contact term is not modified by gauging charge conjugation, so Hall conductance stays well-defined.

[2405.20401]

Cosine Hall conductance $\neq$ U(1) Hall conductance

In ordinary U(1) FQHE, Hall conductance is associated with the spin of the Abelian anyon (vison) :

Formula: $$\sigma_H = 2h_v \quad \text{(mod 2)}$$

↑ Spin of vison v

After gauging charge conjugation symmetry, the vison is generally a non-Abelian anyon $v \oplus v^{-1}$

Cosine Hall conductance is generally associated with the non-Abelian anyon

⇒ the Hall conductance for cosine symmetry is generally not realized by the value of U(1) conductance

[2405.20401]

Gapless edge mode protected by cosine symmetry

Boundary of gauged FQH has a gapless edge mode protected by cosine symmetry.

That is, one can say:

If the edge preserves cosine symmetry, the edge theory must be gapless.

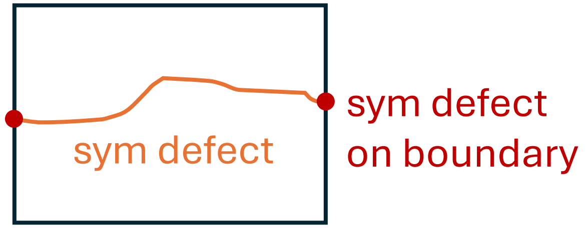
[2405.20401]

But what do we mean by "boundary preserves non-invertible symmetry" ?

At the level of effective field theory, we require two conditions
(generally bifurcate for non-invertible symmetry):

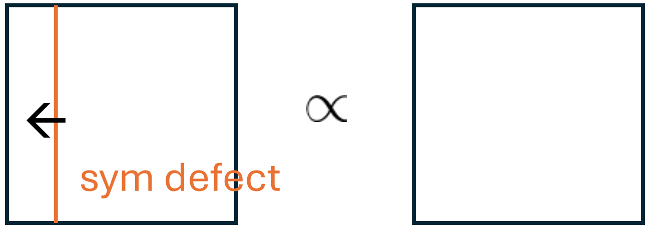
[Choi=Rayhaun=Sanghavi=Shao]

- Cosine symmetry defect can terminate at boundary



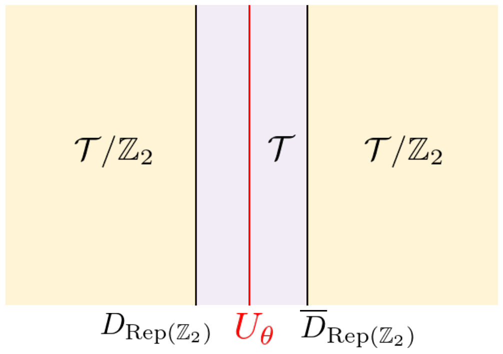
- The boundary is invariant under "pushing" the defect to boundary

(= boundary state is an eigenstate of symmetry action)



Gapless edge mode protected by cosine symmetry

$\tilde{U}_\theta =$



Condensation of electric particle for θ = 0

Cosine symmetry enforces gapless edge modes:

- The boundary is invariant under “pushing” the defect to boundary

  Invariance under pushing $\tilde{U}_{\theta=0}$ ⟹ Z2 electric particle must be condensed at boundary

- Cosine symmetry defect can terminate at boundary

  When Z2 electric particle is condensed, cosine $\tilde{U}_\theta$ is transformed into invertible U_θ

  ⟹ U(1) defect can end at the boundary, which requires $\sigma_H = 0$ for gapped boundary

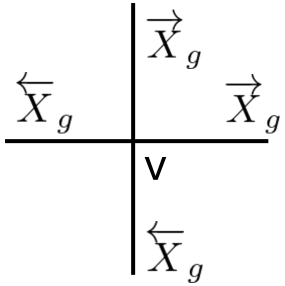
S3 quantum double model

Arguably the simplest exactly solvable model for non-Abelian topological order

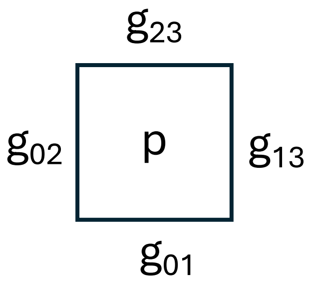
$$H = -\sum_v A_v - \sum_p B_p$$

S3 group element per edge

$$A_v = \sum_{g \in S_3}$$



$$B_p = \delta(g_{01} g_{13} g_{23}^{-1} g_{02}^{-1})$$



S3 gauge theory = smallest non-Abelian topological order

[Kitaev, Bombin=Martin-Delgado...]

This model has a cosine symmetry that exhibits fractionalization.

[2405.20401]

Cosine symmetry in S3 quantum double model

It's convenient to describe S3 element by a Z3 qudit and Z2 qubit; $(a, b) \in S_3$

Operator for cosine symmetry is described by

$$\tilde{U}_\theta = 2^{|v|} \cdot \Pi^b \mathcal{D}^b U_\theta \mathcal{D}^b \Pi^b$$

Π^b : Projection onto low energy space of S3 quantum double (product of Z2 Gauss law op)

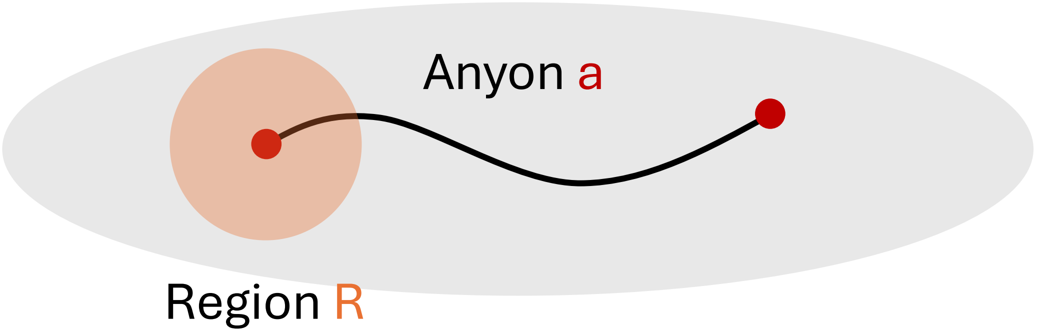
$\mathcal{D}^b$: Projection onto Hilbert space of Z3 toric code; b = 0 $\mathcal{D}^b = \prod_e \frac{1+Z_e^b}{2}$

U_θ : U(1) symmetry of Z3 toric code w/ fractionalization (m carries frac charge 1/3)

- ✓ Exact symmetry of the S3 quantum double model
- ✓ Cosine fusion rule: $\tilde{U}_\theta \times \tilde{U}_{\theta'} = \tilde{U}_{\theta+\theta'} + \tilde{U}_{\theta-\theta'}$
- Fractionalizes non-Abelian anyon (m, m^{-1})

We want to see how the cosine symmetry fractionalizes.

In U(1) case, the fractionalization is observed from:



$$U_R(\pi) \times U_R(\pi) = \exp(2\pi i q_a) U_R(0)$$

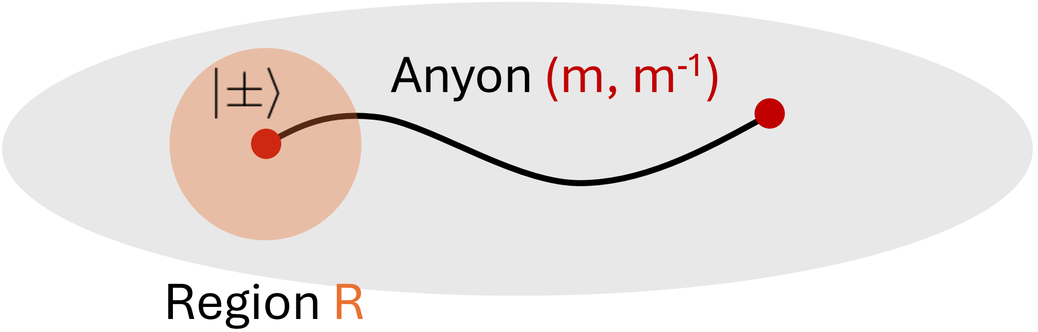
U(1) symmetry on R

θ = π

fractional charge

Fractionalization of cosine symmetry

Cosine symmetry defect supported at region R:



Let's consider its action on a non-Abelian anyon (m, m^{-1})

Anyon carries quantum dimension 2:
2d internal Hilbert space at anyon excitation $|+\rangle\,,|-\rangle$

$$\tilde{U}_\theta(R)\times\tilde{U}_{\theta'}(R)\,|\pm\rangle$$
$$=\left[\cos\left(\frac{2\pi}{3}\eta(\theta,\theta')+\frac{1}{3}[\theta+\theta']_{2\pi}\right)+\cos\left(\frac{2\pi}{3}\eta(\theta,-\theta')+\frac{1}{3}[\theta-\theta']_{2\pi}\right)\right](|+\rangle+|-\rangle)$$

Fractional charge 1/3

$$\left[\tilde{U}_{\theta+\theta'}(R)+\tilde{U}_{\theta-\theta'}(R)\right]|\pm\rangle$$
$$=\left[\cos\left(\frac{1}{3}[\theta+\theta']_{2\pi}\right)+\cos\left(\frac{1}{3}[\theta-\theta']_{2\pi}\right)\right](|+\rangle+|-\rangle)$$

$$\eta(\theta,\theta')=\frac{[\theta]_{2\pi}+[\theta']_{2\pi}-[\theta+\theta']_{2\pi}}{2\pi}$$

Projective symmetry action = Cosine of fractional charge

[2405.20401]

Fractionalization of general coset symmetry G/K

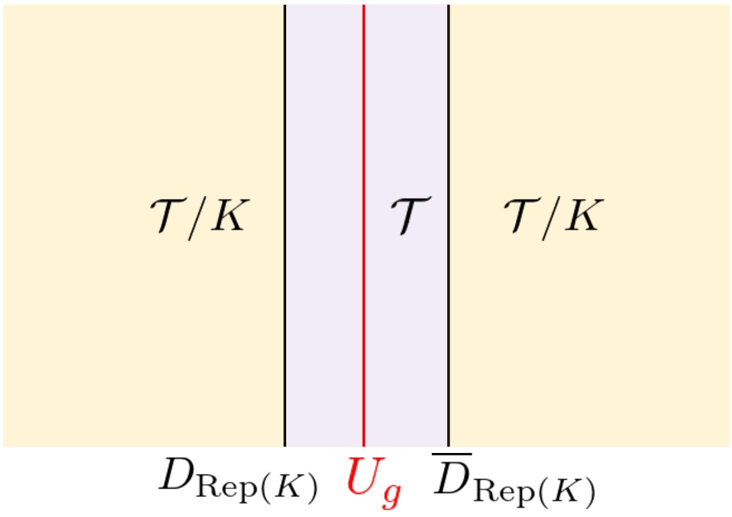
One can generally consider:

Start with topological order with G symmetry, and gauge (finite) non-normal subgroup K

One can obtain general algebraic picture for G/K symmetry fractionalization:

- sandwich expression of the coset symmetry defect

$\tilde{U}_g =$



Fusion rules: $\overline{D}_{\mathrm{Rep}(K)} \times D_{\mathrm{Rep}(K)} = \sum_{k\in K} U_k$

which leads to $\tilde{U}_g \times \tilde{U}_{g'} = \sum_{k\in K} \tilde{U}_{gkg'k^{-1}}$

(generalization of cosine symmetry)

Fractionalization of general coset symmetry G/K

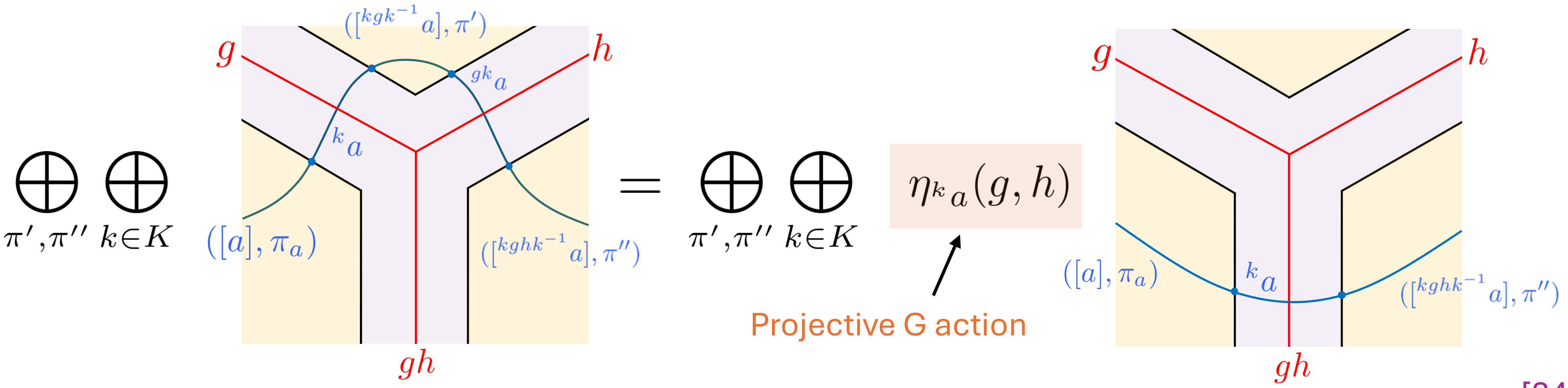
One can generally consider:

Start with topological order with G symmetry, and gauge (finite) non-normal subgroup K

One can obtain general algebraic picture for G/K symmetry fractionalization:

[Barkeshli=Bonderson=Cheng=Wang]

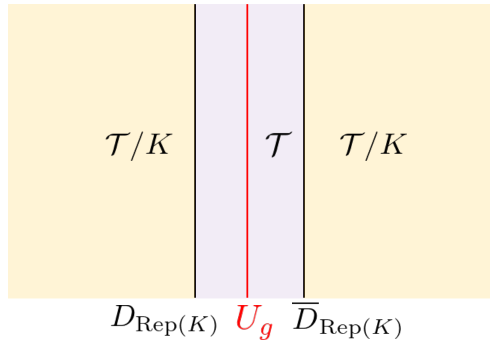
- Symmetry fractionalization of G symmetry, combined with gapped interface



[2405.20401]

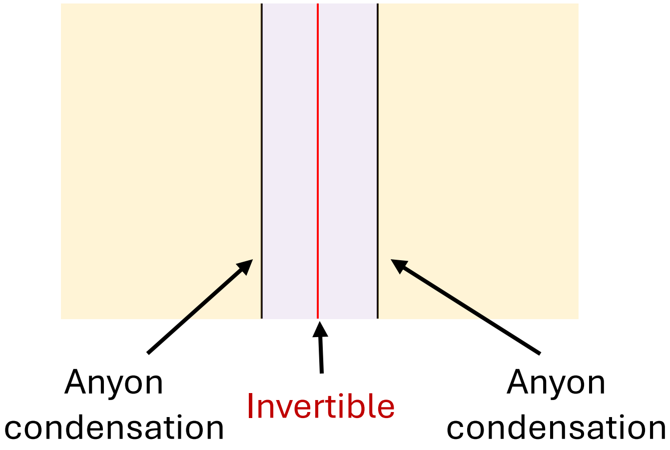
Discussions: Sandwich and SVD

We've considered symmetry defects in a "sandwich" form: $\tilde{U}_g =$



The "sandwich" of invertible symmetry is very general in (2+1)D topological order:

Gapped interface = Invertible symmetry sandwiched by anyon condensation



Realization in microscopic models $\overset{?}{=}$ Singular value decomposition

[Davydov=Nikshych=Ostrik, Huston=Burnell=Jones=Penneys]

This is the case for cosine symmetry in our lattice model:

$$\tilde{U}_\theta = 2^{|v|} \cdot \Pi^b \mathcal{D}^b U_\theta \mathcal{D}^b \Pi^b$$

" M = U Σ V "   MH = HM, HU = UH', H'Σ = ΣH', H'V = VH

Summary

- Fractionalization of coset non-invertible symmetry
- Hall conductance of cosine symmetry
- S3 quantum double model has a fractionalized cosine symmetry

To think:

- Explore realization of coset symmetry in FQH or spin liquids:

  E.g Coset symmetry in "orbifold FQH states" with $[U(1) \times U(1)] \rtimes Z2$ gauge group [Barkeshli=Wen]

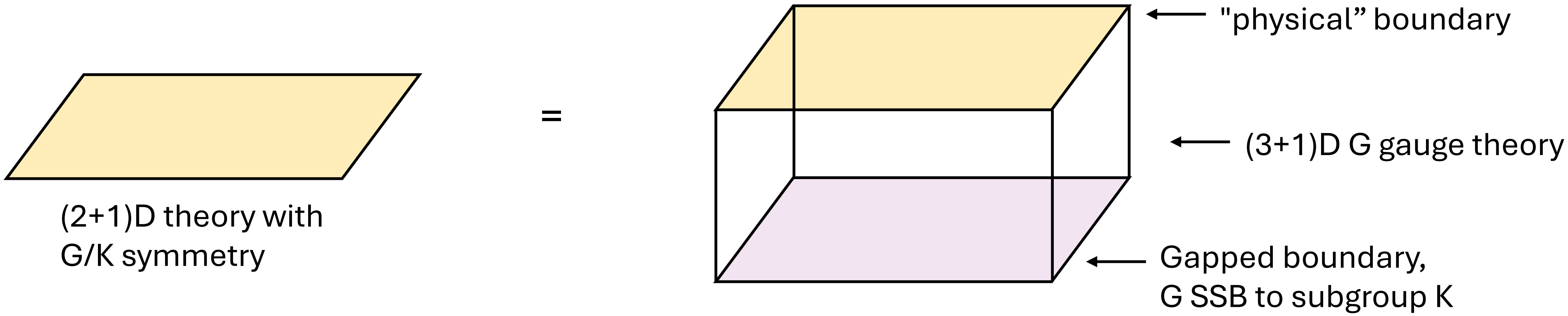
- Fractionalization of larger class of non-invertible symmetries beyond coset?
- Microscopic definition of cosine Hall conductance?
- Gauging continuous non-normal subgroup?
- Higgs phase with coset symmetry?

What I didn't talk about

- Anomalies of coset symmetry (anomaly free condition derived through symmetry TFT)

Anomalies of coset symmetry G/K

A generic (2+1)D QFT with (finite) G/K symmetry can be obtained from (3+1)D bulk topological order:

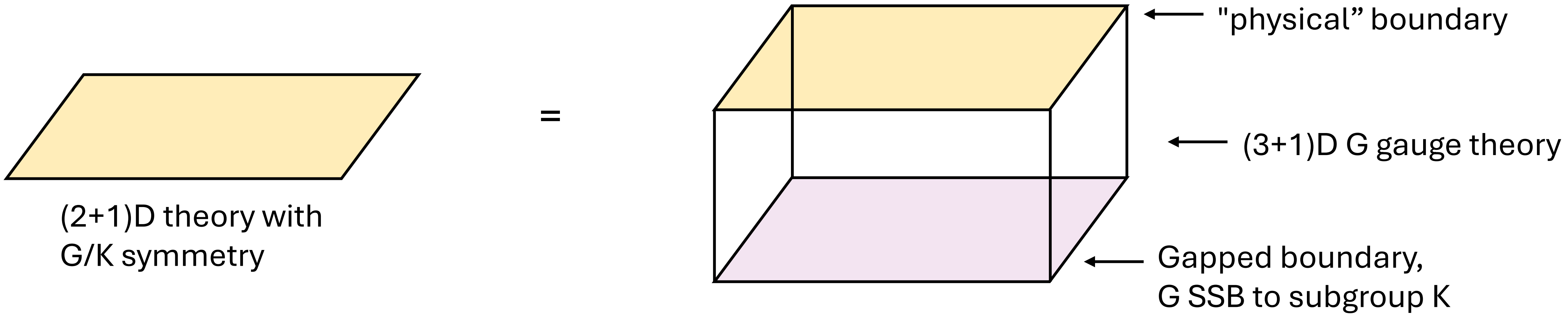


G/K symmetry is free of ‘t Hooft anomaly → $\exists$ SPT phase with G/K symmetry

↔ $\exists$ physical boundary that realizes SPT with G/K

Anomalies of coset symmetry G/K

A generic (2+1)D QFT with (finite) G/K symmetry can be obtained from (3+1)D bulk topological order:



Necessary conditions for G/K symmetry being anomaly free:

There exists subgroup K' of G s.t.

- K and K' doesn't have an overlap: $K \cap K' = \{id\}$
- For any irrep of G, its decomposition into irreps of K, K' doesn't contain trivial rep simultaneously

Example: S3/Z2 symmetry is always anomalous. There is no SPT with S3/Z2 symmetry.