Symmetry Breaking and Symmetry-Protected Topology in mixed states

Ruochen Ma
马若辰
University of Chicago

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- Collaborators:

  Chong Wang (Perimeter)

  Meng Cheng (Yale)

  Alex Turzillo (Perimeter)

  Zhen Bi (Penn State)

  Jian-Hao Zhang (Penn State->Boulder)

  Leonardo Lessa (Perimeter)

- References:

  2209.02723

  2305.16399

  2403.13280

  2405.03639

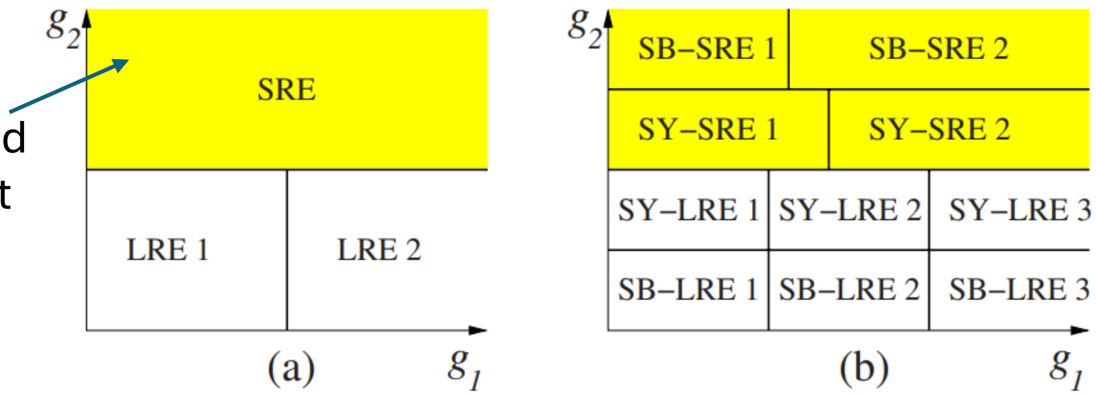





Pure-state Phases of Matter

- Local Hamiltonian with gapped spectrum
- Equivalence relation: A gapped path of local Hamiltonians
- Equivalently: $|\psi_1\rangle \cong |\psi_2\rangle$ in the same phase if $|\psi_1\rangle = U|\psi_2\rangle$
  U: finite-depth local unitary (quasi-adiabatic continuation)
- Symmetry: constraints on local gates, $[gate, G] = 0$

Connected to product states

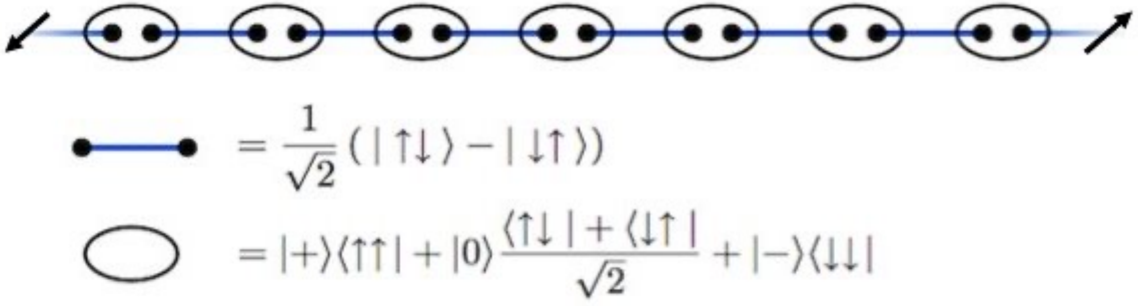


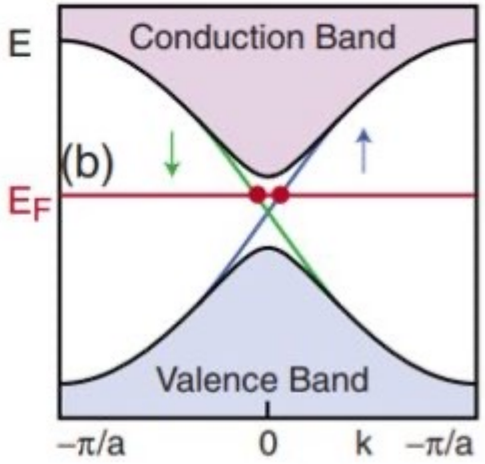
Chen, Gu, Wen 2010

- Spontaneous Symmetry Breaking (SSB)

$$|cat\rangle = \frac{1}{\sqrt{2}}|\uparrow\uparrow\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\downarrow\downarrow\rangle$$

- Symmetry-Protected Topological Phases





Hasan and Kane, 2010

What is a "phase" in mixed states?

- Deformation: finite-depth local channel

$$\varepsilon(\rho) = tr_A[U(\rho \otimes |0_A\rangle\langle 0_A|)U^+]$$

- $A = \otimes_i A_i$: associate an extra ancilla with each vertex
- $|0_A\rangle$ is a product state on A
- U is a finite-depth unitary on $H \otimes A$
- Preserves causality $\varepsilon^+(local) = local$ and locality   Piroli, Cirac 2020
- Phase: ρ_1 and ρ_2 are in the same phase if

$$\rho_2 = \varepsilon_{12}(\rho_1)$$
$$\rho_1 = \varepsilon_{21}(\rho_2)$$

Symmetry in quantum systems

- Pure states

$$U_g|\psi\rangle = e^{i\alpha}|\psi\rangle$$

- Mixed states $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$

Canonical ensemble →

- Strong (exact) symmetry

$$U_k\rho = \rho U_k = e^{i\alpha}\rho$$

Grand canonical ensemble →

- Weak (average) symmetry

$$U_g\rho U_g^{+} = \rho$$

Symmetry in quantum systems

- Pure states

$$[H, U_g] = 0$$

No charge exchange: thermal bath

- Mixed states = system + bath (traced out)
- Strong (exact) symmetry

$$[\text{gate}, U_k \otimes 1_A] = 0$$

Can exchange charge: particle bath

- Weak (average) symmetry

$$[\text{gate}, U_g \otimes U_g^A] = 0$$

Outline

- **Strong-to-weak SSB (SW-SSB)**

1. Fidelity Correlator
2. Renyi-2 Correlator: SSB in the doubled Hilbert space

- Properties of SW-SSB

1. Stability and non-invertibility

- Examples: Thermal States and Ising model

- Average Symmetry-Protected Topological Phases

1. Decorated domain wall construction
2. Example: decohered cluster chain

SSB in pure states

- 1d spin chain with $\mathbb{Z}_2$ spin flip symmetry $X = \prod_i X_i$
- Cat state: $|\psi_\pm\rangle = \frac{1}{\sqrt{2}}|\uparrow\uparrow\uparrow\cdots\rangle \pm \frac{1}{\sqrt{2}}|\downarrow\downarrow\downarrow\cdots\rangle$
- Long-range order (LRO):

$$\langle\psi_\pm|Z_i Z_j|\psi_\pm\rangle \xrightarrow{|i-j|\to\infty} 1$$

- Stability: LRO is a *universal* property of a *phase*

$U|\psi_\pm\rangle$ must have LRO due to the Lieb-Robinson bound

SSB in pure states

- Non-invertibility ("long-range entangled")

$\nexists |\psi_{\pm}^{*}\rangle$ and symmetric finite-depth U, such that

$$U(|\psi_{\pm}\rangle \otimes |\psi_{\pm}^{*}\rangle) = |0\rangle$$

Example: Ising Chain

- Cat state in X basis: $|\psi_\pm\rangle \propto \sum_{\prod_i x_i = \pm 1} |\{x_i\}\rangle$

Coherent superposition of all symmetric product state

Within the charge ± 1 sector

- $\rho \propto I \pm X \propto \sum_{\prod_i x_i = \pm 1} |\{x_i\}\rangle\langle\{x_i\}|$

Incoherent sum of all symmetric product states

Within the charge ± 1 sector

$$tr\left(\rho O_x O_y^+\right) = 0$$

Strong-to-Weak SSB

Order parameter, stability, etc.

SW-SSB: Fidelity Correlator

- Conventional order parameter fails to detect SW-SSB:

$$tr\left(\rho O_x O_y^{+}\right) = 0$$

- SSB in terms of similarity (overlap, distance, etc.)

$\langle\psi|Z_x Z_y|\psi\rangle$: similarity between $|\psi\rangle$ and $\mathrm{Z}_x Z_y|\psi\rangle$

- Intuition: similarity between

ρ and $\sigma = O_x O_y^{+} \rho O_x^{+} O_y$

SW-SSB: Fidelity Correlator

- Intuition: similarity (distance) between

$$\rho \text{ and } \sigma = O_x O_y^{+} \rho O_x^{+} O_y$$

- Definition: A strongly symmetric ρ has SW-SSB if:

$$F(\rho, \sigma) = tr\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}} = tr\sqrt{\sqrt{\rho} O_x O_y^{+} \rho O_x^{+} O_y \sqrt{\rho}} \xrightarrow{|x-y|\to\infty} O(1)$$

$$\text{And } tr(\rho O_x O_y^{+}) \xrightarrow{|x-y|\to\infty} 0$$

Basic Properties of Fidelity

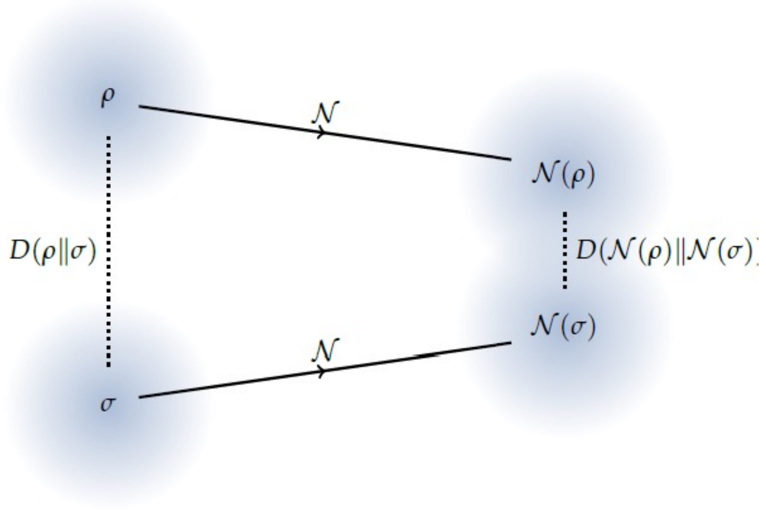
- For two states ρ and σ,

$$0 \le F(\rho,\sigma) \le 1$$

$$F(\rho,\sigma) = 0 \Leftrightarrow \rho \perp \sigma, F(\rho,\sigma) = 1 \Leftrightarrow \rho = \sigma$$

- Data-processing inequality

$$F(\rho,\sigma) \le F(\varepsilon(\rho),\varepsilon(\sigma))$$



Khatri, Wilde 2021

Example: Ising Chain

- Product state after maximal dephasing

$$\rho = \varepsilon(|+\rangle\langle+|)$$

$$\varepsilon = \prod_{<ij>} \varepsilon_{ij}\ ,\ \varepsilon_{ij}(\rho) = \frac{1}{2}(\rho + Z_i Z_j \rho Z_i Z_j)$$

- $\rho \propto I + X \propto \sum_{\prod_i x_i = 1} |\{x_i\}\rangle\langle\{x_i\}|$

Incoherent sum of all symmetric product states

Within the charge $+1$ sector

$$\Rightarrow tr\left(\rho Z_x Z_y\right) = 0$$

$$F\left(\rho, Z_x Z_y \rho Z_x Z_y\right) = 1$$

Example: Ising Chain

- If we decompose $\rho \propto I + X$ in local Z basis:

$$\rho = \sum_s (|s\rangle + X|s\rangle)(\langle s| + \langle s|X)$$

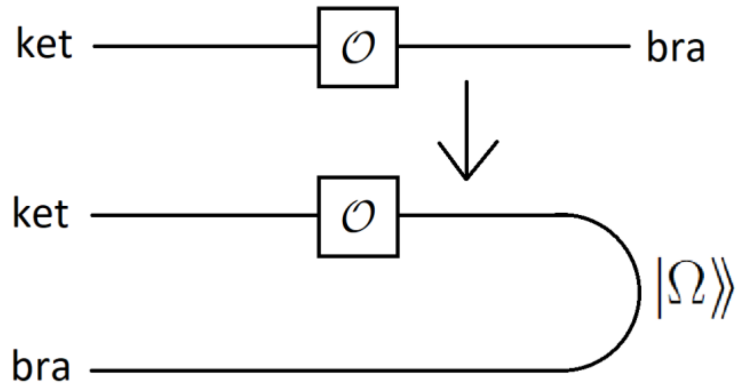
$\{s\}$: all possible bit strings in Z basis

$$F\left(\rho, Z_x Z_y \rho Z_x Z_y\right) = \overline{|\langle Z_x Z_y \rangle|}$$

Mixed-state “Edward-Anderson” order parameter

Renyi-2 Correlator: SSB in the doubled space

- Operator-State map:



- Renyi-2 Correlator:

$$\frac{\langle\langle\rho|O_x O_y^+ \otimes O_x^+ O_y|\rho\rangle\rangle}{\langle\langle\rho|\rho\rangle\rangle} = \frac{tr\left(\rho O_x O_y^+ \rho O_x^+ O_y\right)}{tr(\rho^2)} = O(1)$$

- Example: $\rho \propto I + X$

Lee, Jian, Xu 2023

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- Strong-to-weak SSB (SW-SSB)

1. Fidelity Correlator
2. Renyi-2 Correlator: SSB in the doubled Hilbert space

- **Properties of SW-SSB**

1. Stability and non-invertibility

- Examples: Thermal States and Ising model

- Average Symmetry-Protected Topological Phases

1. Decorated domain wall construction
2. Example: decohered cluster chain

Properties of SW-SSB

- Stability: If ρ has SW-SSB, ε is a symmetric finite-depth channel,

$$\Rightarrow \varepsilon(\rho) \text{ also has SW-SSB}$$

- Symmetric finite-depth channel: can be purified to a symmetric finite-depth local unitary

$$\varepsilon(\rho) = tr_A[U(\rho \otimes |0_A\rangle\langle 0_A|)U^+]$$
$$[gate\ of\ U, U_k \otimes 1_A] = 0$$

- Proof:
- Uhlmann Theorem: $F(\rho,\sigma) = \max_{|\psi_\rho\rangle,|\phi_\sigma\rangle} |\langle\psi_\rho|\phi_\sigma\rangle|$
- For $\sigma = O_x O_y^+ \rho O_x^+ O_y$

$$F(\rho,\sigma) = \max_{|\psi_\rho\rangle,|\phi_\rho\rangle} \left|\left\langle\psi_\rho\middle|O_x O_y^+ \middle|\, \phi_\rho\right\rangle\right| = O(1)$$

$$\Rightarrow \left|\left\langle\psi_\rho \otimes 0_A\middle|U^+ U O_x O_y^+ U^+ U\middle|\, \phi_\rho \otimes 0_A\right\rangle\right| = O(1)$$

$$\Rightarrow \left|\left\langle\psi_{\varepsilon(\rho)}\middle|O_x{}' O_y^+{}'\middle|\, \phi_{\varepsilon(\rho)}\right\rangle\right| = O(1)$$

- For symmetric finite-depth U,

$$O_x' = \sum_i O_x^i \otimes W_A^i$$

such that (1) finite sum; (2) O_i carries K charge; (3) W_A^i unitary

$$\Rightarrow F\left(\varepsilon(\rho), O_x^i O_y^{j+} \varepsilon(\rho) O_x^{i+} O_y^j\right) = O(1)$$

Intuition

- Data-processing inequality

$$\sigma = O_x O_y^+ \rho O_x^+ O_y$$
$$O(1) \sim F(\rho, \sigma) \leq F(\varepsilon(\rho), \varepsilon(\sigma))$$
$$= F\left(\varepsilon(\rho), \varepsilon\left(O_x O_y^+ \rho O_x^+ O_y\right)\right) \sim O(1)$$

- When O commutes with ε, stability is straightforward.

- When ε is symmetric finite-depth, pushing O through the channel still gives a local operator with the same charge.

$$F\left(\varepsilon(\rho), O'_x O'^+_y \varepsilon(\rho) O'^+_x O'_y\right) \sim O(1)$$

Properties of SW-SSB

- Non-invertibility:

  $\nexists \tilde{\rho}$ and symmetric finite-depth channel ε,

  such that $\varepsilon(\rho \otimes \tilde{\rho}) = |0\rangle\langle 0|$

  (Directly from stability)

- Fidelity: stable $\rightarrow$ universal

- Renyi-2: not universal

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- **Examples: Thermal States and Ising model**

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Example: Thermal States

- Stat. Mech.:Local equivalence between canonical and grand-canonical ensembles
- **Conjecture**: If a Gibbs state within a fixed charge sector has no SSB, it must have SW-SSB

$$\rho \propto P_\lambda e^{-\beta H}$$

- Supporting evidence:

H is a sum of complete set of $\mathbb{Z}_2$ symmetric commuting projector

e.g. $H = -\sum_i X_i$

$$F\left(\rho, Z_i Z_j \rho Z_i Z_j\right) \xrightarrow{L\to\infty} 1/\cosh^2 \beta$$

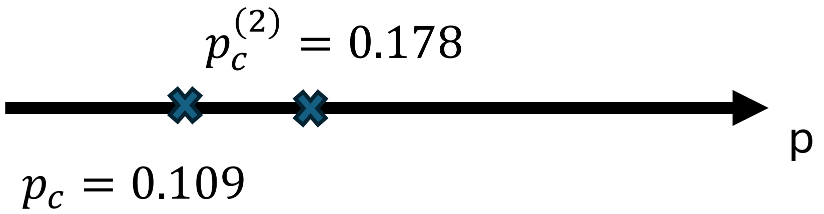
Nonzero for any finite T

Example: decohered Ising model

- 2d Ising symmetric product state $|0\rangle = \otimes_i |+\rangle_i$
- Apply ZZ dephasing: $\varepsilon = \prod_{<ij>} \varepsilon_{ij}$, $\varepsilon_{ij}(\rho) = (1-p)\rho + pZ_iZ_j\rho Z_iZ_j$
- "Ungauged" toric code under bit-flip noise
- Inequivalence between fidelity and Renyi-2

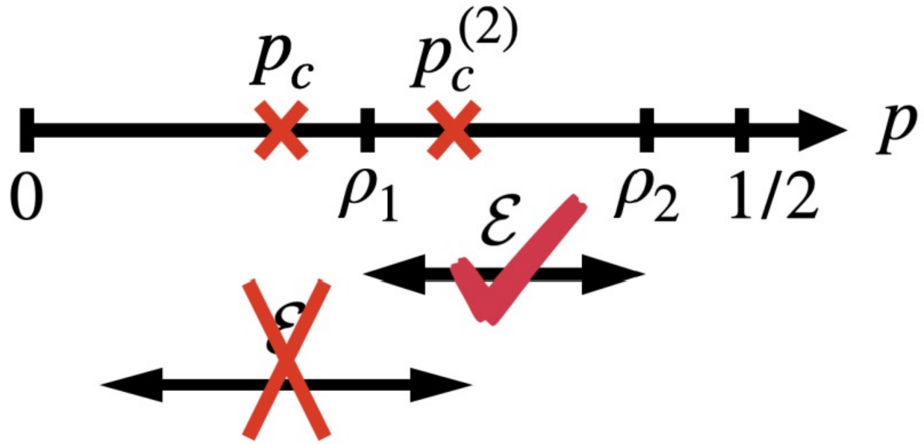
Fidelity correlator→correlator in random bond Ising model

Renyi-2→correlator in Ising model



SSB *phase*

- Question: whether ρ_1 and ρ_2 can be two way connected?

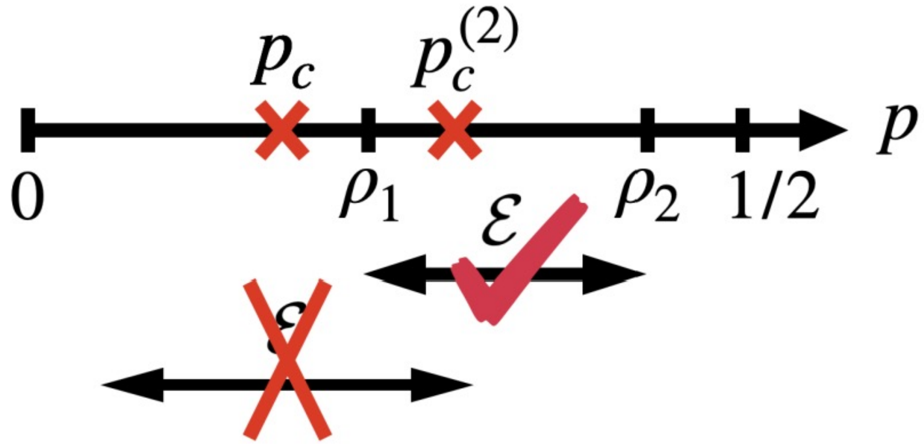


- Question: whether local decoherence is locally recoverable?

SSB *phase*

- Answer: Fidelity correlator is universal

  Renyi-2 correlator is not



- Recovery channel:

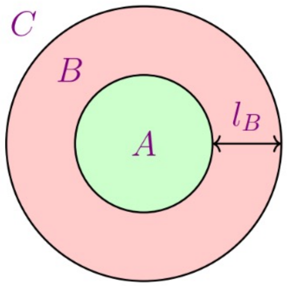
  Petz Recovery Map

  Sang, Hsieh 2024

- Condition:

  $\partial_p I(A{:}C|B)_{\rho_p} \sim O(e^{-l_B/\xi})$

- ξ ~ correlation length of the RBIM



Take home

- The behavior of SSB diagnostics in a mixed state with a strong symmetry

Symmetry	$F_O(x,y)$	$\mathrm{Tr}(\rho O(x) O^\dagger(y))$
Unbroken	0	0
SW-SSB	$O(1)$	0
Completely Broken	$O(1)$	$O(1)$

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2. Decorated domain wall construction

Cluster Chain in 1d

- 1d Cluster state as a $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPT   Chen, Lu, Vishwanath 2013

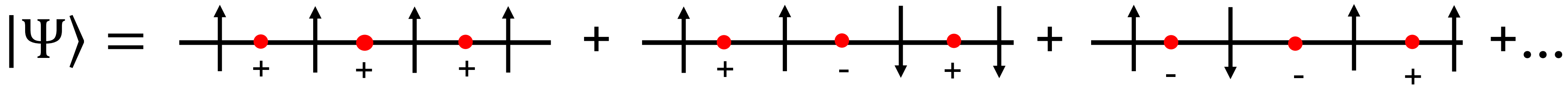
$$H = -\sum_i Z_{2i-1} X_{2i} Z_{2i+1} - \sum_i Z_{2i} X_{2i+1} Z_{2i+2}$$

$\mathbb{Z}_2^e \times \mathbb{Z}_2^o$ symmetry: $X = \otimes_i X_{2i}$, $\mathrm{X} = \otimes_i X_{2i+1}$

Ground State $|\Psi\rangle$: $Z_i X_{i+1} Z_{i+2} = 1$

Decorated domain wall construction

- $|\Psi\rangle$ in the basis of X on even sites and Z on odd sites



$$Z_i X_{i+1} Z_{i+2} = 1$$

- A domain wall of $\mathbb{Z}_2^o$ traps a charge of $\mathbb{Z}_2^e$

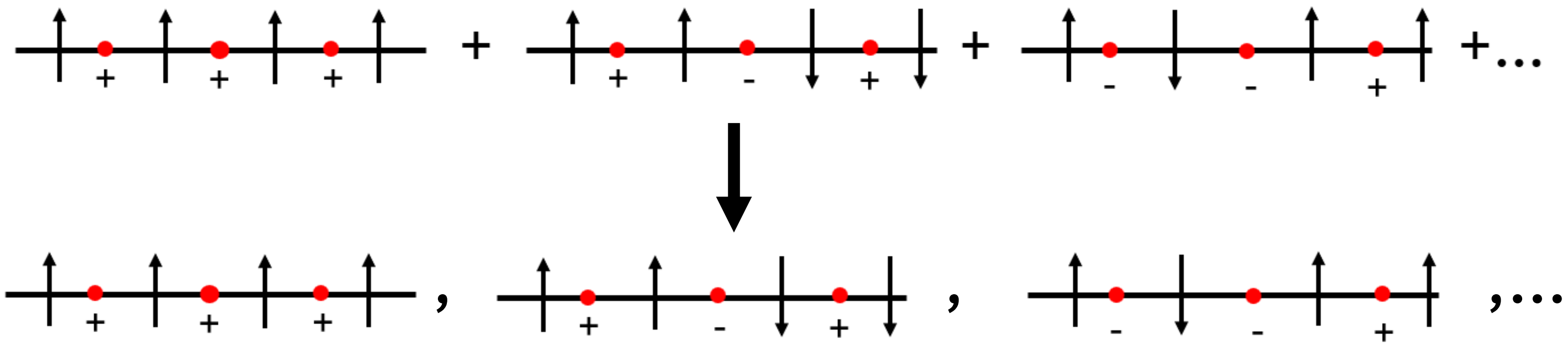
Decohering the cluster state

- $\varepsilon = \prod_i \varepsilon_{2i}\,, \varepsilon_{2i}(\rho) = (1-p)\rho + p Z_{2i} \rho Z_{2i}$

- For $p > 0$, $\mathbb{Z}_2^o$ is strong and $\mathbb{Z}_2^e$ becomes weak

- $\rho_{\frac{1}{2}} = \prod_i \frac{1 + Z_{2i} X_{2i+1} Z_{2i+2}}{2}$

$|\Psi\rangle =$



Boundary Correlation

- $\rho_{\frac{1}{2}} = \prod_i \frac{1+Z_{2i}X_{2i+1}Z_{2i+2}}{2}$
- $\langle Z_{2n}X_{2n+1}X_{2n+3} \dots X_{2m-1}Z_{2m}\rangle = 1$
- Global strong symmetry: $\prod_i X_{2i+1}\rho = \rho$
- $\langle X_1X_3 \dots X_{2n-1}Z_{2n} \cdot Z_{2m}\, X_{2m+1}X_{2m+3} \dots X_{2k+1}\rangle = 1$
- $\langle X_1X_3 \dots X_{2n-1}Z_{2n}\rangle$ and $\langle Z_{2m}X_{2m+1}X_{2m+3} \dots X_{2k+1}\rangle = 0$ due to the weak $\mathbb{Z}_2^e$ symmetry

SRE mixed states

- SRE mixed states: ρ can be purified to an SRE state

  $\Leftrightarrow$ ρ can be prepared from $|0\rangle\langle 0|$ by a finite depth channel

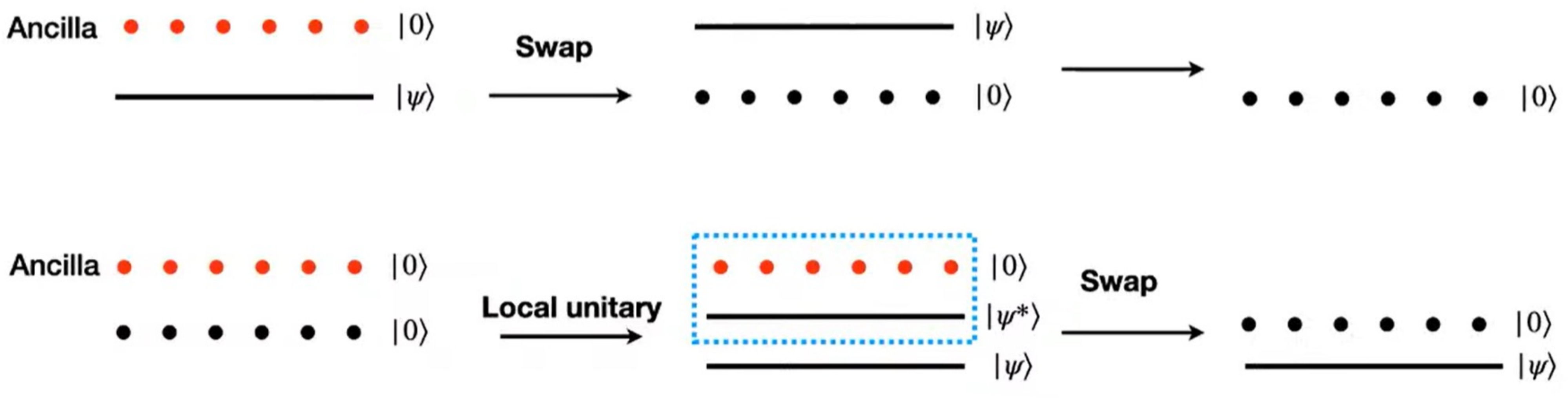
- Equivalence: symmetric finite-depth channel

$$\rho_2 = \varepsilon_{12}(\rho_1)$$
$$\rho_1 = \varepsilon_{21}(\rho_1)$$

- Trivial: In the same equivalence class as symmetric product states

Classification

- All SPT states with only weak symmetry become trivial
- Invertible states (e.g. E_8) also become trivial



- No phase factor in incoherent sum

Non-trivial ones

- Strong symmetry K and weak symmetry G
- Any SPT protected jointly by $K \times G$ or K alone are non-trivial
- $|\Psi\rangle$ can not be connected to product state

  Proof: If $\exists \varepsilon$:symmetric finite depth channel, such that

$$\varepsilon(|\Psi\rangle\langle\Psi|) = |0\rangle\langle 0|$$

$$\Rightarrow U|\Psi\rangle \otimes |0_A\rangle = |0\rangle \otimes |\Psi'_A\rangle$$

$$\Rightarrow |\Psi'_A\rangle \text{ and } |\Psi\rangle \text{ belong to the same SPT}$$

- Since K acts non-trivially on A, this is impossible when $|\Psi\rangle$ requires K symmetry

Classification

- Classification:

$$H^{d+1}(K\times G, U(1))/H^{d+1}(G, U(1))$$

- Decorated domain wall:

$$|\Psi\rangle = \sum_D \sqrt{p_D}\, e^{i\theta_D} |\psi_D\rangle |a_D\rangle$$

Decoherence: G becomes weak

$$\rho = \sum_D p_D |\psi_D\rangle\langle\psi_D| \otimes |a_D\rangle\langle a_D|$$

Summary

- Strong-to-Weak Spontaneous Symmetry Breaking

  Similarity measure

  Fidelity correlator: stability, non-invertibility

  Renyi-2: not "universal"

- Average symmetry-protected topological phases

  decorated domain wall construction