Recent Developments and Challenges in Topological Phases
@ Kyoto, Jun. 3, 2024

東京大学
THE UNIVERSITY OF TOKYO

Topology of discrete quantum feedback control

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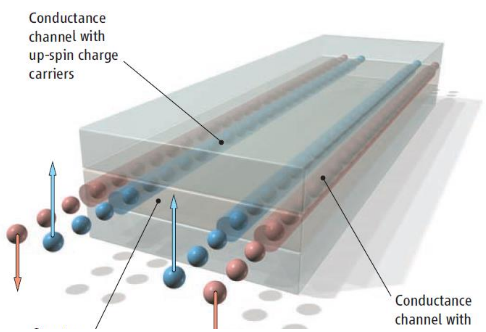
MN and M. Ueda, arXiv:2403.08406

Classes of topological phases

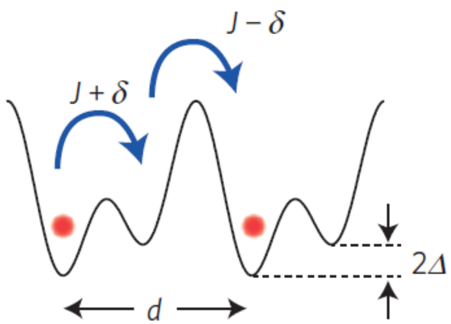
Topological phases in nonequilibrium systems:

Classification of phases in terms of topology of relevant operators

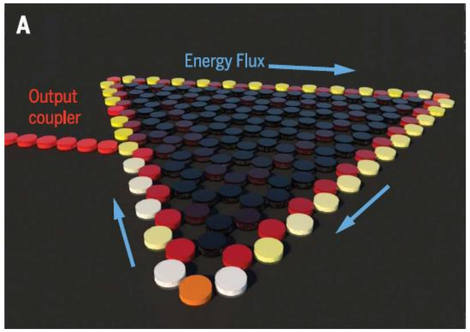
Class	Equilibrium (ground state)	Floquet (periodically driven)	Non-Hermitian
Topology of	Hamiltonian	unitary operator	non-Hermitian Hamiltonian



topological insulator
[König *et al.*, Science (2007)]



topological pump
[Nakajima *et al.*, Nat. Phys. (2016)]



topological laser
[Harari *et al.*, Science (2018)]

Classes of topological phases

Topological phases in nonequilibrium systems:

Classification of phases in terms of topology of relevant operators

Class	Equilibrium (ground state)	Floquet (periodically driven)	Non-Hermitian	Quantum feedback
Topology of	Hamiltonian	unitary operator	non-Hermitian Hamiltonian	quantum channel

This talk:

Topology of quantum feedback control

→ New platform of topological phases

Outline

1. Introduction
2. Topological feedback control
3. Symmetry classification of quantum feedback control
4. Symmetry-protected topological feedback control
5. Summary and future perspective

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1. Introduction

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Topological phases at equilibrium

- **Equilibrium (ground-state) topological phases of free fermions**

  → Topology of a Bloch Hamiltonian

$$H = \sum_{\vec{k}} \boldsymbol{c}^{\dagger}_{\vec{k}} H(\vec{k}) \boldsymbol{c}_{\vec{k}}$$

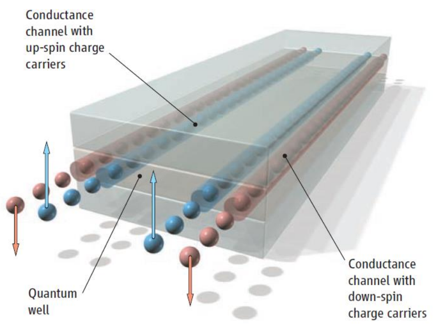
- Symmetry

  Internal: time-reversal, particle-hole, chiral (sublattice)

  Crystalline: inversion, rotation, …

- Example: Z_2 topological insulator

  → protected by time-reversal symmetry



[König *et al.*, Science (2007)]

Topological classification of phases of matter

- Classification of sym.-protected topological phases of free fermions

  [A. Kitaev, AIP Conf. Proc. 1134, 22 (2009)]

  [A. Schnyder, S. Ryu, A. Furusaki, and A. W. W. Ludwig, Phys. Rev. B 78, 195125 (2008)]

Altland-Zirnbauer symmetry class

	TRS	PHS	SLS	$d=1$	$d=2$	$d=3$
A (unitary)	0	0	0	-	$\mathbb{Z}$	-
AI (orthogonal)	+1	0	0	-	-	-
AII (symplectic)	−1	0	0	-	$\mathbb{Z}_2$	$\mathbb{Z}_2$
AIII (chiral unitary)	0	0	1	$\mathbb{Z}$	-	$\mathbb{Z}$
BDI (chiral orthogonal)	+1	+1	1	$\mathbb{Z}$	-	-
CII (chiral symplectic)	−1	−1	1	$\mathbb{Z}$	-	$\mathbb{Z}_2$
D	0	+1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	-
C	0	−1	0	-	$\mathbb{Z}$	-
DIII	−1	+1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
CI	+1	−1	1	-	-	$\mathbb{Z}$

d: Spatial dimension

Z: integer topological invariant

Z_2: Z_2 topological invariant (0 or 1)

(TRS = time-reversal sym., PHS = particle-hole sym., SLS = sublattice sym.)

Topology of quantum dynamics

Topological classification of nonequilibrium dynamics

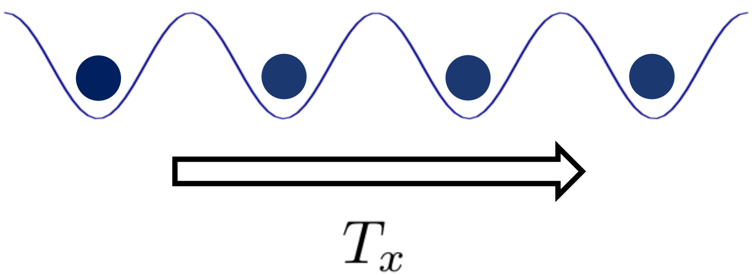
Example: Thouless pumping (1D system)

[D. J. Thouless, PRB 27, 6083 (1983); T. Kitagawa *et al*, PRB 82, 235114 (2010)]

One-site translation (unitary operator)

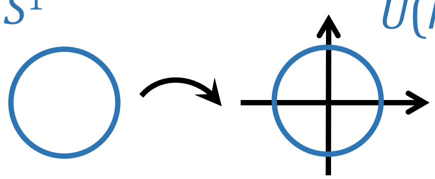
$$U = T_x = \sum_{k_x} e^{-ik_x} \left|k_x\right\rangle \left\langle k_x\right|$$

$=: U(k_x)$



Winding number (topological invariant)

$$w = \int_{-\pi}^{\pi} \frac{dk_x}{2\pi i} U^\dagger(k_x)\partial_{k_x} U(k_x) = -1$$



Interesting consequence: **the translation operator cannot be generated by finite-time evolution under any local Hamiltonian!**

[D. Gross *et al.*, Commun. Math. Phys. 310, 419 (2012)]

∵) $U = e^{-iHt}$ ⇨ $w = 0$ by continuous deformation with $t \to 0$

Topological classification of unitary dynamics

■ Topological classification of unitary operators of free fermions

[S. Higashikawa, MN, and M. Ueda, Phys. Rev. Lett. 123, 066403 (2018)]

d: Spatial dimension

Altland-Zirnbauer symmetry class

Class	$d=0$	$d=1$	$d=2$	$d=3$
A	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	0	0	0	$2\mathbb{Z}$
BDI	$\mathbb{Z}$	0	0	0
D	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
DIII	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
AII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
CII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
C	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
CI	0	0	$2\mathbb{Z}$	0

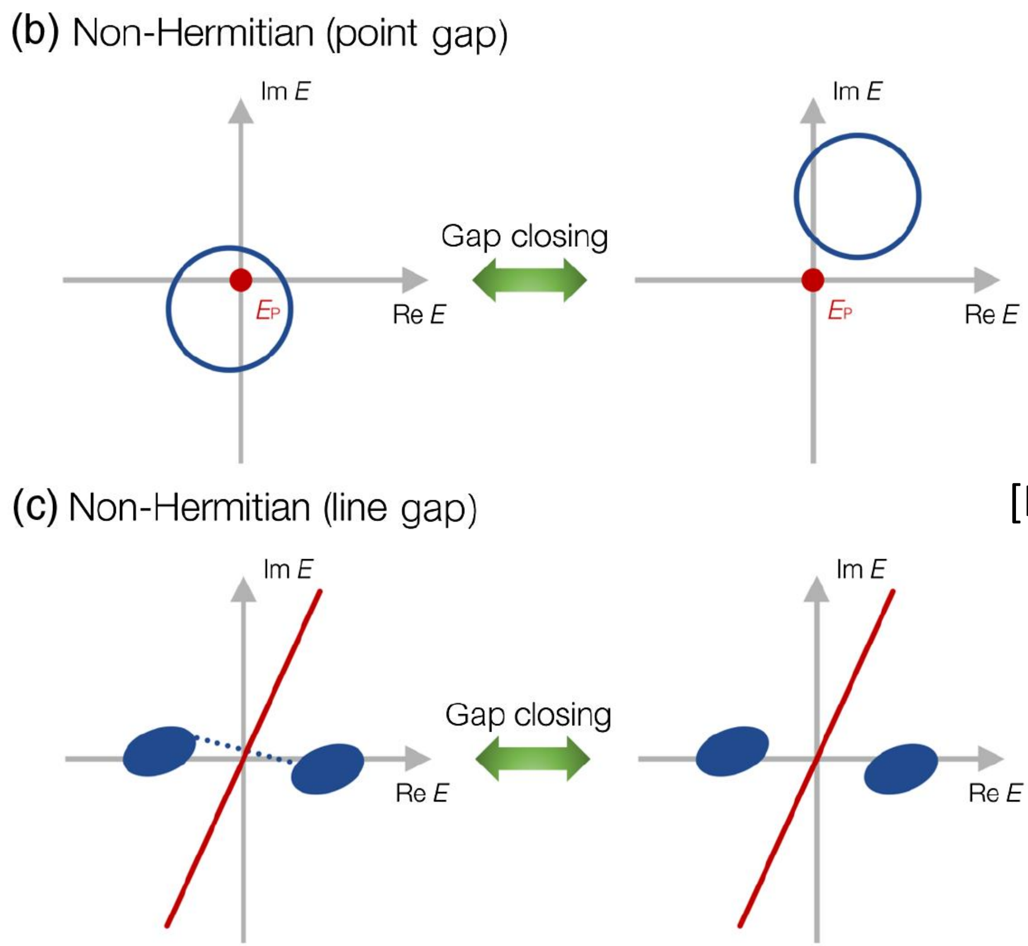
→ equivalent to equivalence classes of unitary operators that cannot be generated by local Hamiltonians with symmetry

[X. Liu, A. B. Culver, F. Harper, and R. Roy, arXiv:2308.02728]

Non-Hermitian topology

Non-Hermitian topological phases

Non-Hermitian Hamiltonian → complex eigenvalues



Two types of energy gaps!

Point gap & line gap

[Kawabata *et al*., PRX 9, 041015 (2019)]

Non-Hermitian topology

Non-Hermitian topological phases

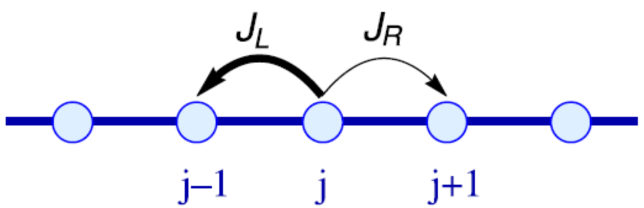
Example: Hatano-Nelson model (1D system)

[Hatano and Nelson, PRL 77, 570 (1996); Gong *et al.*, PRX 8, 031079 (2018)]

Non-Hermitian Hamiltonian with asymmetric hopping

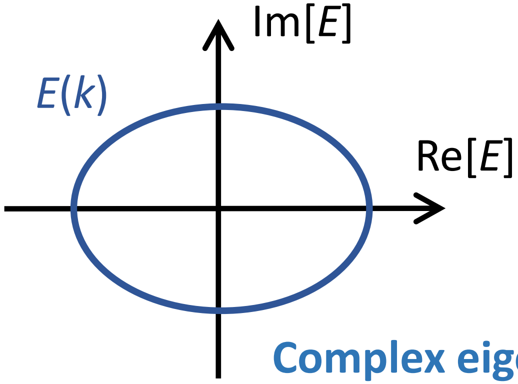
$$H = \sum_j (J_L c_j^\dagger c_{j+1} + J_R c_{j+1}^\dagger c_j)$$

$$= \sum_k H(k) c_k^\dagger c_k$$



Winding number (topological invariant)

$$w = \int_{-\pi}^{\pi} \frac{dk}{2\pi i} H^{-1}(k) \partial_k H(k)$$



Complex eigenvalues
→ winding structure
(point-gap topology)

■ Relevant to open classical systems
& (a limited class of) open quantum systems

Topological classification of non-Hermitian systems

- Topological classification of non-Hermitian systems
  → 38 symmetry classes due to non-Hermiticity
  [Kawabata *et al*., PRX 9, 041015 (2019); Zhou and Lee, PRB 99, 235112 (2019)]

AZ class	Gap	Classifying space	$d=0$	$d=1$	$d=2$	$d=3$
A	P	$\mathcal{C}_1$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
	L	$\mathcal{C}_0$	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	P	$\mathcal{C}_0$	$\mathbb{Z}$	0	$\mathbb{Z}$	0
	L_r	$\mathcal{C}_1$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
	L_i	$\mathcal{C}_0 \times \mathcal{C}_0$	$\mathbb{Z}\oplus\mathbb{Z}$	0	$\mathbb{Z}\oplus\mathbb{Z}$	0

AZ class	Gap	Classifying space	$d=0$	$d=1$	$d=2$	$d=3$
AI	P	$\mathcal{R}_1$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
	L_r	$\mathcal{R}_0$	$\mathbb{Z}$	0	0	0
	L_i	$\mathcal{R}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
BDI	P	$\mathcal{R}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
	L_r	$\mathcal{R}_1$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
	L_i	$\mathcal{R}_2 \times \mathcal{R}_2$	$\mathbb{Z}_2\oplus\mathbb{Z}_2$	$\mathbb{Z}_2\oplus\mathbb{Z}_2$	$\mathbb{Z}\oplus\mathbb{Z}$	0
D	P	$\mathcal{R}_3$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
	L	$\mathcal{R}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
DIII	P	$\mathcal{R}_4$	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
	L_r	$\mathcal{R}_3$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
	L_i	$\mathcal{C}_0$	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AII	P	$\mathcal{R}_5$	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
	L_r	$\mathcal{R}_4$	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
	L_i	$\mathcal{R}_6$	0	0	$2\mathbb{Z}$	0
CII	P	$\mathcal{R}_6$	0	0	$2\mathbb{Z}$	0
	L_r	$\mathcal{R}_5$	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
	L_i	$\mathcal{R}_6 \times \mathcal{R}_6$	0	0	$2\mathbb{Z}\oplus 2\mathbb{Z}$	0
C	P	$\mathcal{R}_7$	0	0	0	$2\mathbb{Z}$
	L	$\mathcal{R}_6$	0	0	$2\mathbb{Z}$	0
CI	P	$\mathcal{R}_0$	$\mathbb{Z}$	0	0	0
	L_r	$\mathcal{R}_7$	0	0	0	$2\mathbb{Z}$
	L_i	$\mathcal{C}_0$	$\mathbb{Z}$	0	$\mathbb{Z}$	0

$\text{AZ}^\dagger$ class	Gap	Classifying space	$d=0$	$d=1$	$d=2$	$d=3$
$\text{AI}^\dagger$	P	$\mathcal{R}_7$	0	0	0	$2\mathbb{Z}$
	L	$\mathcal{R}_0$	$\mathbb{Z}$	0	0	0
$\text{BDI}^\dagger$	P	$\mathcal{R}_0$	$\mathbb{Z}$	0	0	0
	L_r	$\mathcal{R}_1$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
	L_i	$\mathcal{R}_0 \times \mathcal{R}_0$	$\mathbb{Z}\oplus\mathbb{Z}$	0	0	0
$\text{D}^\dagger$	P	$\mathcal{R}_1$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
	L_r	$\mathcal{R}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
	L_i	$\mathcal{R}_0$	$\mathbb{Z}$	0	0	0
$\text{DIII}^\dagger$	P	$\mathcal{R}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
	L_r	$\mathcal{R}_3$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
	L_i	$\mathcal{C}_0$	$\mathbb{Z}$	0	$\mathbb{Z}$	0
$\text{AII}^\dagger$	P	$\mathcal{R}_3$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
	L	$\mathcal{R}_4$	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$\text{CII}^\dagger$	P	$\mathcal{R}_4$	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
	L_r	$\mathcal{R}_5$	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
	L_i	$\mathcal{R}_4 \times \mathcal{R}_4$	$2\mathbb{Z}\oplus 2\mathbb{Z}$	0	$\mathbb{Z}_2\oplus\mathbb{Z}_2$	$\mathbb{Z}_2\oplus\mathbb{Z}_2$
$\text{C}^\dagger$	P	$\mathcal{R}_5$	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
	L_r	$\mathcal{R}_6$	0	0	$2\mathbb{Z}$	0
	L_i	$\mathcal{R}_4$	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$\text{CI}^\dagger$	P	$\mathcal{R}_6$	0	0	$2\mathbb{Z}$	0
	L_r	$\mathcal{R}_7$	0	0	0	$2\mathbb{Z}$
	L_i	$\mathcal{C}_0$	$\mathbb{Z}$	0	$\mathbb{Z}$	0

SLS	AZ class	Gap	Classifying space	$d=0$	$d=1$	$d=2$	$d=3$
$\mathcal{S}_+$	AIII	P	$\mathcal{C}_1$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
		L_r	$\mathcal{C}_1 \times \mathcal{C}_1$	0	$\mathbb{Z}\oplus\mathbb{Z}$	0	$\mathbb{Z}\oplus\mathbb{Z}$
		L_i	$\mathcal{C}_1 \times \mathcal{C}_1$	0	$\mathbb{Z}\oplus\mathbb{Z}$	0	$\mathbb{Z}\oplus\mathbb{Z}$
$\mathcal{S}$	A	P	$\mathcal{C}_1 \times \mathcal{C}_1$	0	$\mathbb{Z}\oplus\mathbb{Z}$	0	$\mathbb{Z}\oplus\mathbb{Z}$
		L	$\mathcal{C}_1$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
$\mathcal{S}_-$	AIII	P	$\mathcal{C}_0 \times \mathcal{C}_0$	$\mathbb{Z}\oplus\mathbb{Z}$	0	$\mathbb{Z}\oplus\mathbb{Z}$	0
		L_r	$\mathcal{C}_0$	$\mathbb{Z}$	0	$\mathbb{Z}$	0
		L_i	$\mathcal{C}_0$	$\mathbb{Z}$	0	$\mathbb{Z}$	0

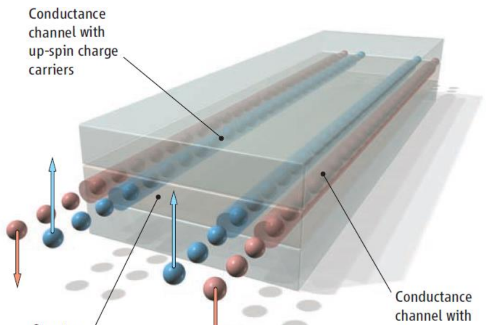
SLS	AZ class	Gap	Classifying space	$d=0$	$d=1$	$d=2$	$d=3$
$\mathcal{S}_{++}$	BDI	P	$\mathcal{R}_1$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
		L_r	$\mathcal{R}_1 \times \mathcal{R}_1$	$\mathbb{Z}_2\oplus\mathbb{Z}_2$	$\mathbb{Z}\oplus\mathbb{Z}$	0	0
		L_i	$\mathcal{R}_1 \times \mathcal{R}_1$	$\mathbb{Z}_2\oplus\mathbb{Z}_2$	$\mathbb{Z}\oplus\mathbb{Z}$	0	0
$\mathcal{S}_{--}$	DIII	P	$\mathcal{R}_3$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
		L_r	$\mathcal{R}_3 \times \mathcal{R}_3$	0	$\mathbb{Z}_2\oplus\mathbb{Z}_2$	$\mathbb{Z}_2\oplus\mathbb{Z}_2$	$\mathbb{Z}\oplus\mathbb{Z}$
		L_i	$\mathcal{C}_1$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
$\mathcal{S}_{++}$	CII	P	$\mathcal{R}_5$	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
		L_r	$\mathcal{R}_5 \times \mathcal{R}_5$	0	$2\mathbb{Z}\oplus 2\mathbb{Z}$	0	$\mathbb{Z}_2\oplus\mathbb{Z}_2$
		L_i	$\mathcal{R}_5 \times \mathcal{R}_5$	0	$2\mathbb{Z}\oplus 2\mathbb{Z}$	0	$\mathbb{Z}_2\oplus\mathbb{Z}_2$
$\mathcal{S}_{--}$	CI	P	$\mathcal{R}_7$	0	0	0	$2\mathbb{Z}$
		L_r	$\mathcal{R}_7 \times \mathcal{R}_7$	0	0	0	$2\mathbb{Z}\oplus 2\mathbb{Z}$
		L_i	$\mathcal{C}_1$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
$\mathcal{S}_-$	AI	P	$\mathcal{C}_1$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
		L_r	$\mathcal{R}_7$	0	0	0	$2\mathbb{Z}$
		L_i	$\mathcal{R}_3$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
$\mathcal{S}_{-+}$	BDI	P	$\mathcal{C}_0$	$\mathbb{Z}$	0	$\mathbb{Z}$	0
		L_r	$\mathcal{R}_0$	$\mathbb{Z}$	0	0	0
		L_i	$\mathcal{R}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
$\mathcal{S}_+$	D	P	$\mathcal{C}_1$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
		L	$\mathcal{R}_1$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
$\mathcal{S}_{-+}$	DIII	P	$\mathcal{C}_0$	$\mathbb{Z}$	0	$\mathbb{Z}$	0
		L_r	$\mathcal{R}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
		L_i	$\mathcal{R}_0$	$\mathbb{Z}$	0	0	0
$\mathcal{S}_-$	AII	P	$\mathcal{C}_1$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
		L_r	$\mathcal{R}_3$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
		L_i	$\mathcal{R}_7$	0	0	0	$2\mathbb{Z}$
$\mathcal{S}_{-+}$	CII	P	$\mathcal{C}_0$	$\mathbb{Z}$	0	$\mathbb{Z}$	0
		L_r	$\mathcal{R}_4$	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
		L_i	$\mathcal{R}_6$	0	0	$2\mathbb{Z}$	0
$\mathcal{S}_+$	C	P	$\mathcal{C}_1$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
		L	$\mathcal{R}_5$	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
$\mathcal{S}_{-+}$	CI	P	$\mathcal{C}_0$	$\mathbb{Z}$	0	$\mathbb{Z}$	0
		L_r	$\mathcal{R}_6$	0	0	$2\mathbb{Z}$	0
		L_i	$\mathcal{R}_4$	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$\mathcal{S}_{--}$	BDI	P	$\mathcal{R}_3$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
		L_r	$\mathcal{C}_1$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
		L_i	$\mathcal{R}_3 \times \mathcal{R}_3$	0	$\mathbb{Z}_2\oplus\mathbb{Z}_2$	$\mathbb{Z}_2\oplus\mathbb{Z}_2$	$\mathbb{Z}\oplus\mathbb{Z}$
$\mathcal{S}_{++}$	DIII	P	$\mathcal{R}_5$	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
		L_r	$\mathcal{C}_1$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
		L_i	$\mathcal{C}_1$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
$\mathcal{S}_{--}$	CII	P	$\mathcal{R}_7$	0	0	0	$2\mathbb{Z}$
		L_r	$\mathcal{C}_1$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
		L_i	$\mathcal{R}_7 \times \mathcal{R}_7$	0	0	0	$2\mathbb{Z}\oplus 2\mathbb{Z}$

Motivation

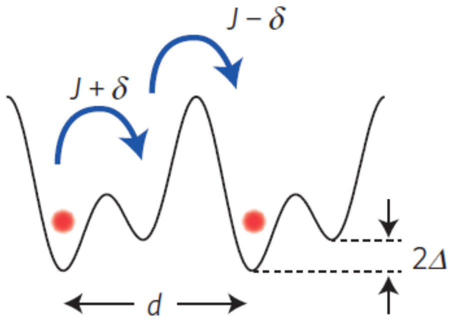
Topological phases:

Classification of phases in terms of topology of relevant operators

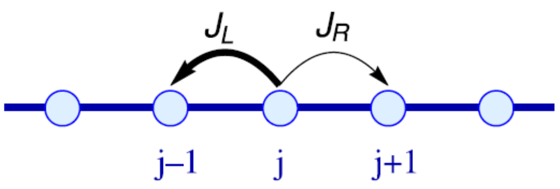
Class	Equilibrium (ground state)	Floquet (periodically driven)	Non-Hermitian
Topology of	Hamiltonian	unitary operator	non-Hermitian Hamiltonian



topological insulator
[König *et al*., Science (2007)]



Thouless pump
[Nakajima *et al*., Nat. Phys. (2016)]



Hatano-Nelson model
[Gong *et al*., PRX (2018)]

Motivation

Topological phases:

Classification of phases in terms of topology of relevant operators

Class	Equilibrium (ground state)	Floquet (periodically driven)	Non-Hermitian	Quantum feedback
Topology of	Hamiltonian	unitary operator	non-Hermitian Hamiltonian	quantum channel

Topology of quantum feedback control

→ New platform of topological phases

[MN and M. Ueda, arXiv:2403.08406]

[For topology of quantum channels in 0-dim. systems, see Gong *et al.*, PRX 8, 031079 (2018)]

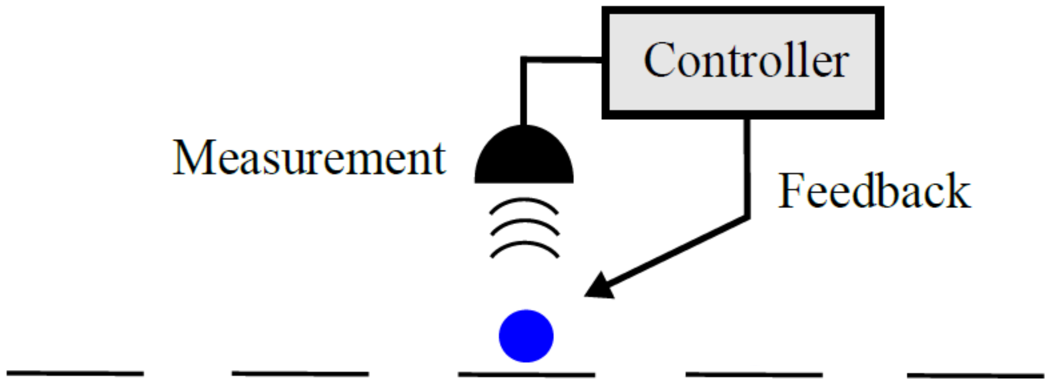
Outline

1. Introduction
2. Topological feedback control
3. Symmetry classification of quantum feedback control
4. Symmetry-protected topological feedback control
5. Summary and future perspective

Feedback control

Feedback control:

Operation conditioned on measurement outcomes

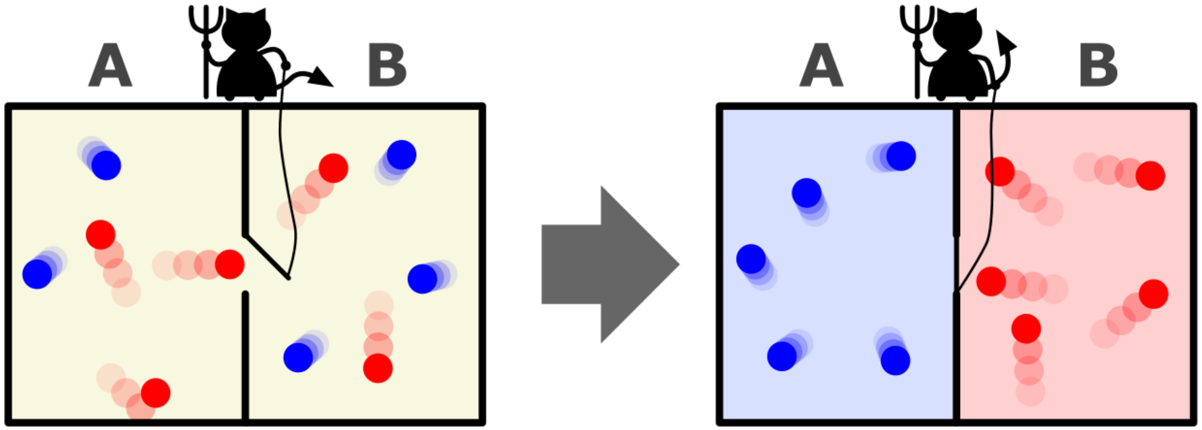


Versatile applications to quantum technology

- Suppression of quantum noise [Inoue *et al*., PRL 110, 163602 (2013)]
- State preparation [Sayrin *et al*., Nature 477, 73 (2011)]
- Quantum error correction [Cramer *et al*., Nat. Commun. 7, 11526 (2016)]

Maxwell's demon

Maxwell's demon



[Figure from Wikipedia]

Measurement of speed of particles

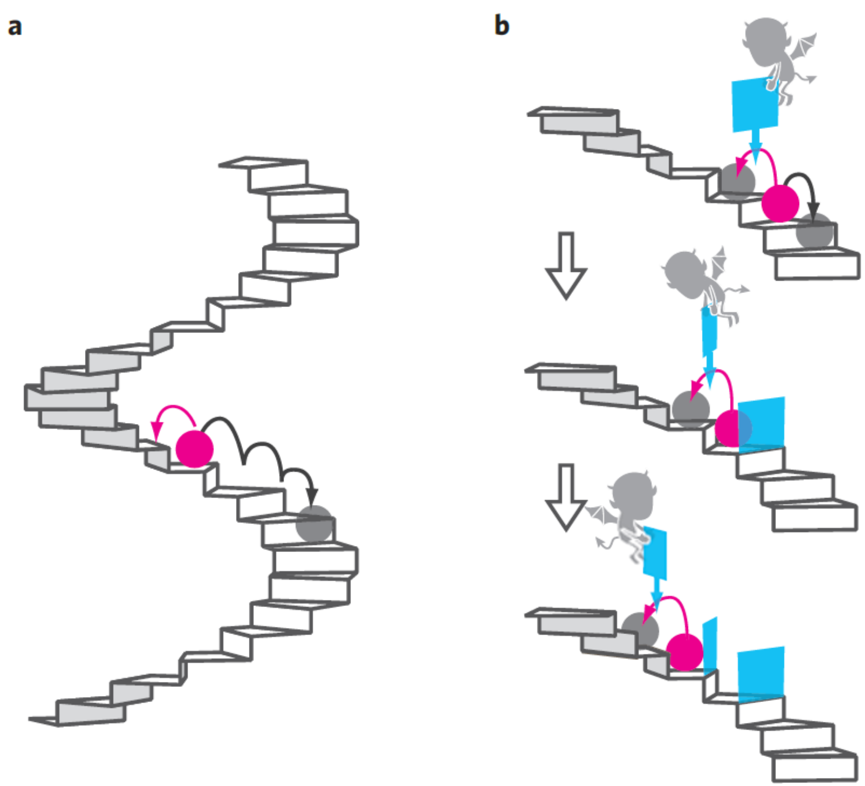
& **feedback control** of a window

→ decrease the entropy of a gas against the 2nd law of thermodynamics

Maxwell's demon

Experimental realization of Maxwell's demon

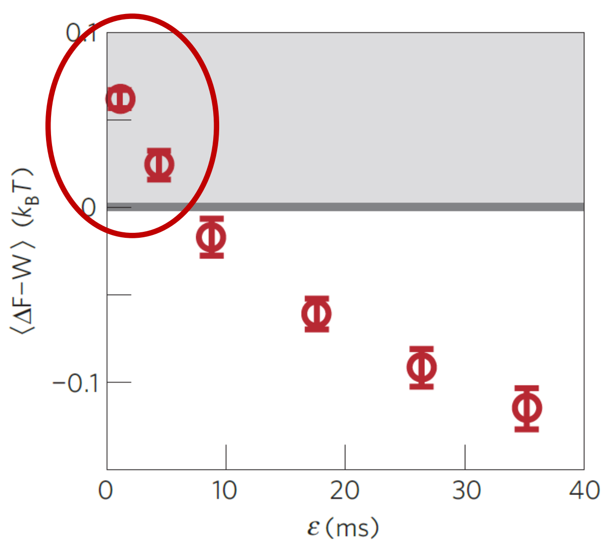
[Toyabe *et al.*, Nat. Phys. 6, 988 (2010)]



Surpassing the conventional 2nd law by using feedback!

(information thermodynamics)

$$\Delta F - W \leq k_B T I$$



classical colloidal particle on a staircase potential

feedback controller = "demon"

Quantum feedback control

(Discrete) quantum feedback control

ρ: density matrix of a quantum system

Step 1: measurement

M_m measurement operator (*m*: measurement outcome), $\sum_m M_m^\dagger M_m = I$

$p_m = \mathrm{Tr}[M_m^\dagger M_m \rho]$ probability of outcome *m*

$\rho \to \rho_m = \dfrac{M_m \rho M_m^\dagger}{p_m}$ measurement backaction

Step 2: feedback operation $\rho_m \to U_m \rho_m U_m^\dagger$

U_m unitary operator conditioned on measurement outcome *m*

$$\mathcal{E}(\rho) := \sum_m p_m U_m \rho_m U_m^\dagger = \sum_m U_m M_m \rho M_m^\dagger U_m^\dagger$$

averaged over outcomes

quantum channel (CPTP map)

Formalism

■ Single-particle quantum system on a *d*-dimensional lattice

$|\vec{i}, a\rangle$ quantum state at site $\vec{i}$ with internal state *a*

$$K_m = \sum_{\vec{i},\vec{j}} \sum_{a,b} (K_m)_{\vec{i},a;\vec{j},b} \, |\vec{i}, a\rangle \langle \vec{j}, b| \,,$$ Kraus operator

$$(K_m = U_m M_m)$$

■ Vectorization of the density matrix

$$\rho = \sum_{\vec{i},\vec{j}} \sum_{a,b} \rho_{\vec{i},a;\vec{j},b} \, |\vec{i}, a\rangle \langle \vec{j}, b| \Rightarrow |\rho\rangle := \sum_{\vec{i},\vec{j}} \sum_{a,b} \rho_{\vec{i},a;\vec{j},b} \, |\vec{i}, a\rangle \otimes |\vec{j}, b\rangle$$

■ Matrix representation of a quantum channel

$$\mathcal{E}(\rho) = \sum_m K_m \rho K_m^\dagger \Rightarrow \tilde{\mathcal{E}} = \sum_m K_m \otimes K_m^*$$

Non-Hermitian, non-unitary operator on the doubled Hilbert space!

Formalism (cont'd)

- Translational symmetry (T_λ: translation operator)

$$\sum_m (T_\lambda K_m T_\lambda^\dagger)\rho(T_\lambda K_m T_\lambda^\dagger)^\dagger = \sum_m K_m \rho K_m^\dagger \quad (\lambda = 1, \cdots, d)$$

$$\Rightarrow \quad \tilde{\mathcal{E}} = \sum_{\vec{k}} \tilde{\boldsymbol{c}}_{\vec{k}}^\dagger X(\vec{k}) \tilde{\boldsymbol{c}}_{\vec{k}}$$

momentum-space representation

"Bloch matrix"
→ characterized by a topological invariant

$$\tilde{c}^\dagger_{\vec{i},a,c,\vec{\mu}}(|\mathrm{vac}\rangle \otimes |\mathrm{vac}\rangle) = |\vec{i}, a\rangle \otimes |\vec{i} + \vec{\mu}, c\rangle \,,$$

auxiliary creation operator in the doubled Hilbert space

$$\tilde{c}_{\vec{k},a,c,\vec{\mu}} := \frac{1}{\sqrt{N_{\mathrm{cell}}}} \sum_{\vec{j}} \tilde{c}_{\vec{j},a,c,\vec{\mu}} e^{-i\vec{k}\cdot\vec{R}_{\vec{j}}}$$

$$X_{a,c,\vec{\mu};b,d,\vec{\nu}}(\vec{k}) = \frac{1}{N_{\mathrm{cell}}} \sum_{\vec{j},\vec{j}'} \sum_m (K_m)_{\vec{j},a;\vec{j}',b} (K_m)^*_{\vec{j}+\vec{\mu},c;\vec{j}'+\vec{\nu},d} e^{-i\vec{k}\cdot(\vec{R}_{\vec{j}} - \vec{R}_{\vec{j}'})} .$$

Locality of measurement and feedback

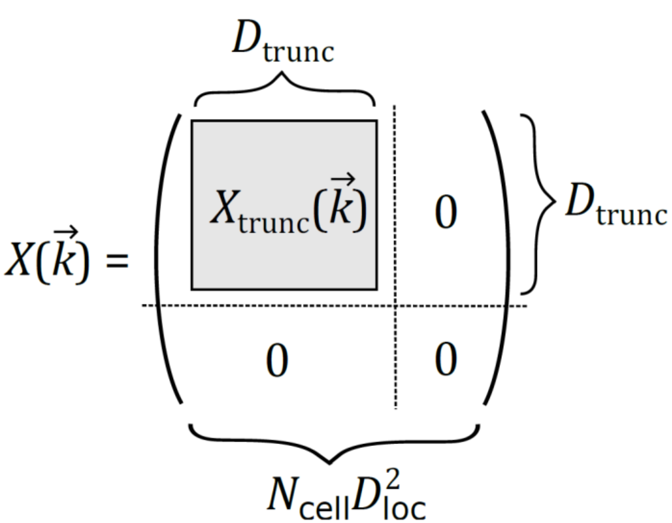
- Dimension of the Hilbert space: $D = N_{\mathrm{cell}} D_{\mathrm{loc}}$ # of internal DOF

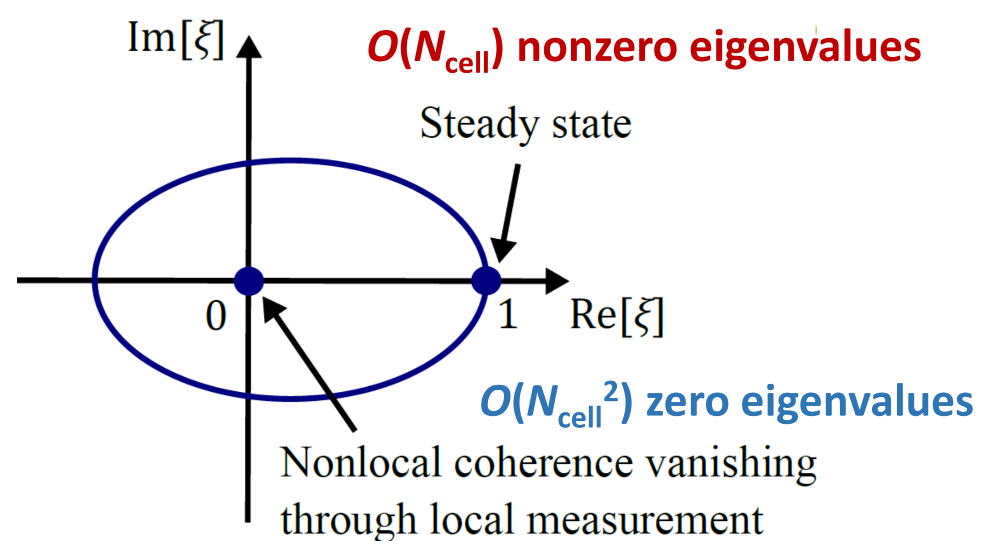
  # of unit cells

$$\tilde{\mathcal{E}} = \sum_{\vec{k}} \tilde{\boldsymbol{c}}_{\vec{k}}^{\dagger} X(\vec{k}) \tilde{\boldsymbol{c}}_{\vec{k}}$$

A quantum channel is a $D^2 \times D^2$ matrix!
[Dim. of $X(k)$] = $N_{\mathrm{cell}} D_{\mathrm{loc}}^2 \to \infty$ ($N_{\mathrm{cell}} \to \infty$)

- Assuming the locality of measurement and feedback processes
  → Many zero eigenvalues! $\hat{K}_m |\vec{i}, a\rangle \langle \vec{j}, b| \hat{K}_m^{\dagger} = 0$ for $|\vec{i} - \vec{j}| > \ell,$
  → Truncation to a finite-dimensional matrix





Topological feedback control

- Model: chiral Maxwell's demon (1D, spinless)

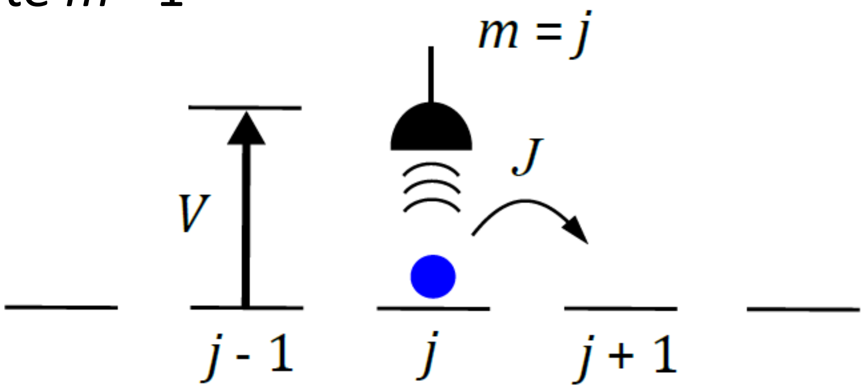
  [MN and M. Ueda, arXiv:2403.08406; cf. K. Liu, MN, and M. Ueda, arXiv:2303.08326]

  [Quantum version of Toyabe *et al*., Nat. Phys. 6, 988 (2010)]

- Projective position measurement $M_m = |m\rangle\langle m|$

- Feedback: raising the potential @ site *m* - 1

$$H_m = -J\sum_{i=1}^{L}(|i+1\rangle\langle i| + \mathrm{H.c.}) + V|m-1\rangle\langle m-1|$$



- Quantum channel

$$\mathcal{E}(\rho) = \sum_{m=1}^{L} U_m M_m \rho M_m^\dagger U_m^\dagger, \quad U_m = \exp[-iH_m\tau]$$

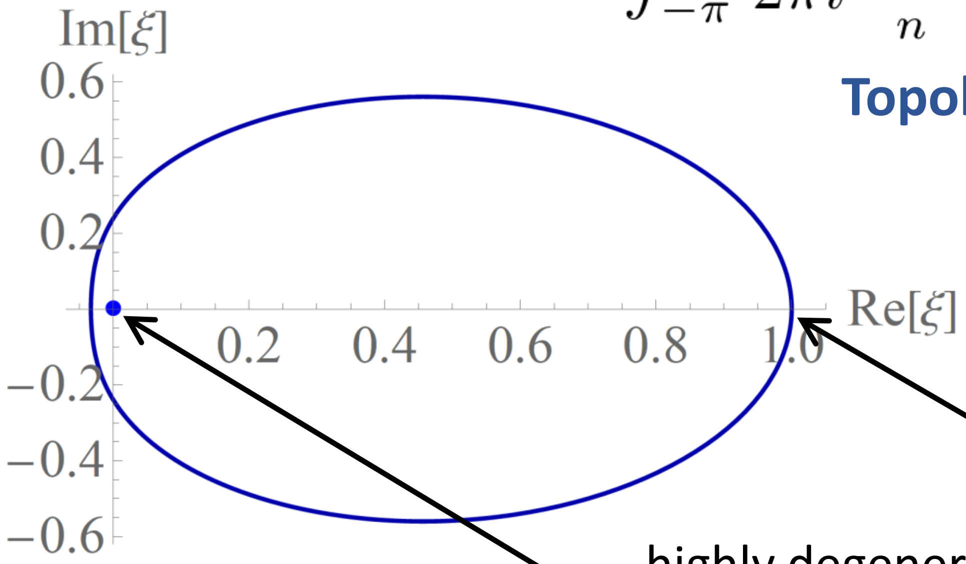
Chiral transport induced by feedback control!

Eigenspectrum under PBC

- Eigenspectrum of the quantum channel
  [periodic boundary condition (PBC)]

$$X(k)v_{k,n} = \xi_n(k)v_{k,n}$$

Winding number $w = \int_{-\pi}^{\pi} \frac{dk}{2\pi i} \sum_n \partial_k \ln[\xi_n(k) - \xi_{\mathrm{PG}}] = -1$



Topology of the quantum channel!

(ξ_{PG}: location of a point gap)

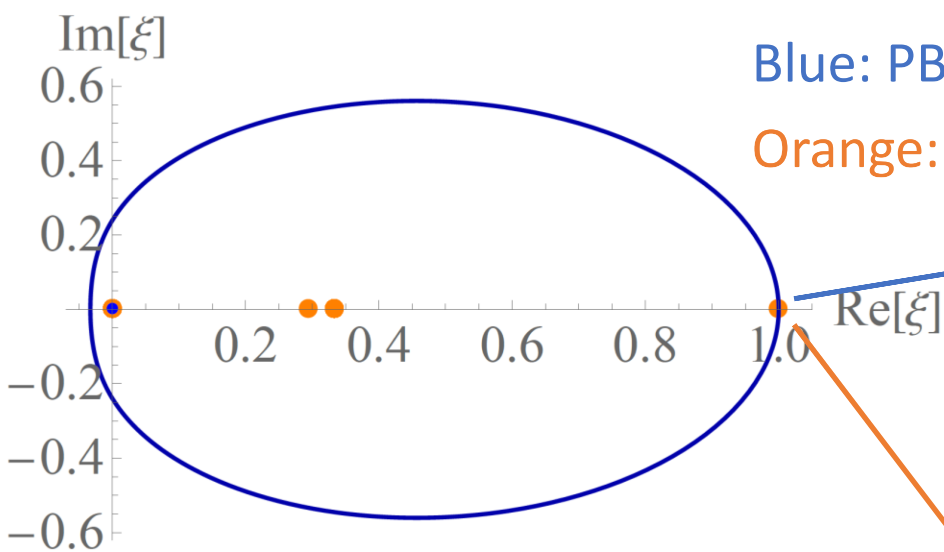
eigenvalue = 1 → steady state

highly degenerate zero eigenvalue
(projective measurement
→ off-diagonal elements vanish)

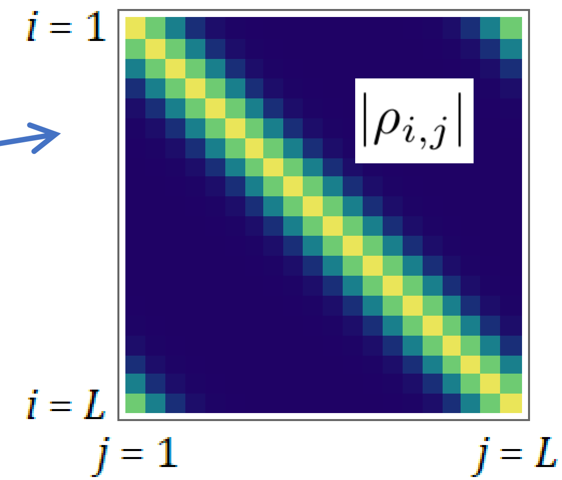
(J = 1, V = +∞, τ = 1)

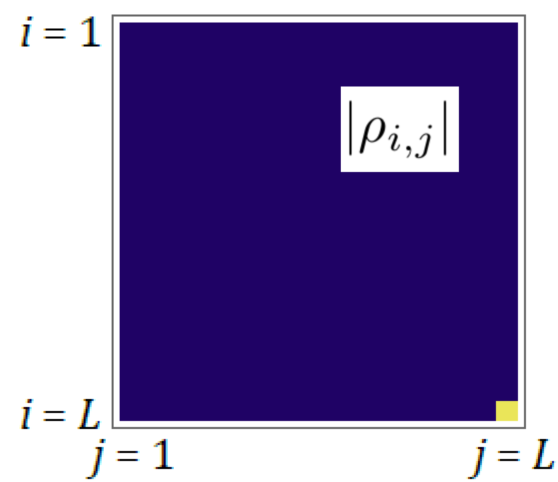
Non-Hermitian skin effect

■ Signature of topology under the open boundary condition (OBC)



Steady state under PBC





Steady state under OBC

✓ **Drastic change of eigenspectrum**

✓ **Localization of eigenmodes**

→ **<u>Non-Hermitian skin effect:</u>**

Hallmark of non-Hermitian topology!

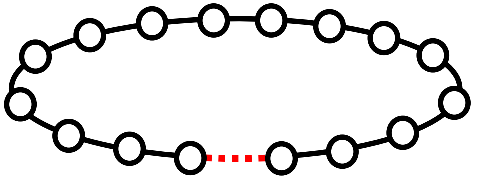
[Okuma *et al*, PRL (2020); Zhang *et al*., PRL (2020)]

Boundary condition

Remark on boundary conditions

Open boundary condition for a Hamiltonian

$$H_{\mathrm{OBC}} = H_{\mathrm{PBC}} - H_{\mathrm{boundary}}$$



Open boundary condition for a quantum channel

$$\mathcal{E}_{\mathrm{OBC}} \neq \mathcal{E}_{\mathrm{PBC}} - \mathcal{E}_{\mathrm{boundary}}$$

$$\mathcal{E}_{\mathrm{OBC}}(\rho) = \sum_m e^{-i\tau H_{m,\mathrm{OBC}}} M_m \rho M_m^\dagger e^{i\tau H_{m,\mathrm{OBC}}}$$

Time evolution under the OBC
→ Reconstruction of matrix elements near the boundary!

Quantum channel vs Lindblad

■ Remark on the difference from Lindblad dynamics

$$\frac{d\rho}{dt} = \mathcal{L}(\rho) = -i[H,\rho] + \sum_{\alpha}(L_\alpha \rho L_\alpha^\dagger - \frac{1}{2}\{L_\alpha^\dagger L_\alpha, \rho\})$$ Lindblad eq.

Generator (Lindbladian)

⇨ $$\rho(t) = e^{\mathcal{L}t}\rho(0)$$

■ **A quantum channel with zero eigenvalues cannot have a generator!**

$$\mathcal{E} \neq e^{\mathcal{L}t}$$

Outline

1. Introduction
2. Topological feedback control
3. Symmetry classification of quantum feedback control
4. Symmetry-protected topological feedback control
5. Summary and future perspective

Symmetry classification of topological phases

Classification of topological feedback control?

Symmetry classification of topological phases in **static** systems

[Kitaev, AIP Conf. Proc. 1134, 22 (2009)]
[Schnyder, Ryu, Furusaki, and Ludwig, Phys. Rev. B 78, 195125 (2008)]

Altland-Zirnbauer symmetry class

	TRS	PHS	SLS	$d=1$	$d=2$	$d=3$
A (unitary)	0	0	0	-	$\mathbb{Z}$	-
AI (orthogonal)	+1	0	0	-	-	-
AII (symplectic)	−1	0	0	-	$\mathbb{Z}_2$	$\mathbb{Z}_2$
AIII (chiral unitary)	0	0	1	$\mathbb{Z}$	-	$\mathbb{Z}$
BDI (chiral orthogonal)	+1	+1	1	$\mathbb{Z}$	-	-
CII (chiral symplectic)	−1	−1	1	$\mathbb{Z}$	-	$\mathbb{Z}_2$
D	0	+1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	-
C	0	−1	0	-	$\mathbb{Z}$	-
DIII	−1	+1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
CI	+1	−1	1	-	-	$\mathbb{Z}$

d: Spatial dimension

Z: integer topological invariant

Z_2: Z_2 topological invariant (0 or 1)

(TRS = time-reversal sym., PHS = particle-hole sym., SLS = sublattice sym.)

Symmetry classification of quantum feedback control

10-fold symmetry classification of quantum feedback control with projective measurement

[MN and M. Ueda, arXiv:2403.08406]

Class	C	K	Q	Example of conditions
A	0	0	0	US
psH (AIII)	0	0	1	US, 2PM, and modified reciprocity
$\mathrm{AI}^\dagger$	$+1$	0	0	US, 2PM, and reciprocity
$\mathrm{AI}+\mathrm{psH}_+$ ($\mathrm{BDI}^\dagger$)	$+1$	$+1$	1	2PM and reciprocity
AI ($\mathrm{D}^\dagger$)	0	$+1$	0	No symmetry or particle-hole symmetry
$\mathrm{AI}+\mathrm{psH}_-$ ($\mathrm{DIII}^\dagger$)	-1	$+1$	1	2PM, reciprocity, and additional FB
$\mathrm{AII}^\dagger$	-1	0	0	US, 2PM, reciprocity, and additional FB
$\mathrm{AII}+\mathrm{psH}_+$ ($\mathrm{CII}^\dagger$)	-1	-1	1	US, 2PM, AUS_-, reciprocity, and additional FB
AII ($\mathrm{C}^\dagger$)	0	-1	0	US and AUS_-
$\mathrm{AII}+\mathrm{psH}_-$ ($\mathrm{CI}^\dagger$)	$+1$	-1	1	US, 2PM, AUS_-, and reciprocity

Bernard-LeClair symmetry classes

■ Bernard-LeClair symmetry classes of non-Hermitian matrices

[Bernard and LeClair (2001); arXiv:0110649]

$$\tilde{\mathcal{E}} = \sum_m K_m \otimes K_m^*$$ quantum channel → non-Hermitian operator

P sym. $\tilde{\mathcal{V}}_P \tilde{\mathcal{E}} \tilde{\mathcal{V}}_P^{-1} = - \tilde{\mathcal{E}},$ anticommutation

C sym. $\tilde{\mathcal{V}}_C \tilde{\mathcal{E}}^T \tilde{\mathcal{V}}_C^{-1} = \epsilon_C \tilde{\mathcal{E}},$ transpose

K sym. $\tilde{\mathcal{V}}_K \tilde{\mathcal{E}}^* \tilde{\mathcal{V}}_K^{-1} = \epsilon_K \tilde{\mathcal{E}},$ complex conjugation

Q sym. $\tilde{\mathcal{V}}_Q \tilde{\mathcal{E}}^\dagger \tilde{\mathcal{V}}_Q^{-1} = \epsilon_Q \tilde{\mathcal{E}},$ Hermitian conjugation

$\tilde{\mathcal{V}}_X$: unitary

$\epsilon_X = \pm 1$

$(X = C, K, Q)$

$\tilde{\mathcal{V}}_X^* \tilde{\mathcal{V}}_X = \pm 1$

$(X = C, K)$

■ Four types of symmetry and the combination thereof

→ 38-fold symmetry classification of non-Hermitian systems

[Kawabata *et al*., PRX 9, 041015 (2019); Zhou and Lee, PRB 99, 235112 (2019)]

Symmetry of projective measurement channels

- Symmetry with ϵ_X = -1 leads to pairs of eigenvalues

$$\tilde{\mathcal{E}}\,|R_n\rangle = \xi_n\,|R_n\rangle\,,\;\; \tilde{\mathcal{E}}^\dagger\,|L_n\rangle = \xi_n^*\,|L_n\rangle$$ right/left eigenvectors

e.g.) $\tilde{\mathcal{V}}_P\tilde{\mathcal{E}}\tilde{\mathcal{V}}_P^{-1} = -\tilde{\mathcal{E}} \Rightarrow \tilde{\mathcal{E}}\tilde{\mathcal{V}}_P\,|R_n\rangle = -\xi_n\tilde{\mathcal{V}}_P\,|R_n\rangle$ → (ξ_n, -ξ_n) pair

$\tilde{\mathcal{V}}_Q\tilde{\mathcal{E}}^\dagger\tilde{\mathcal{V}}_Q^{-1} = -\tilde{\mathcal{E}} \Rightarrow \tilde{\mathcal{E}}\tilde{\mathcal{V}}_Q\,|L_n\rangle = -\xi_n^*\tilde{\mathcal{V}}_Q\,|L_n\rangle$ → (ξ_n, -ξ_n^*) pair

- Symmetry with ϵ_X = -1 is **NOT** consistent with projective measurement

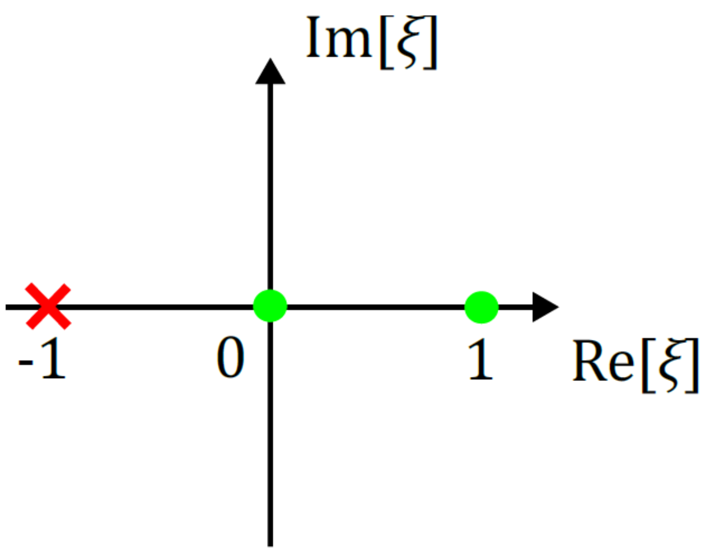
Projective measurement channel

$$\mathcal{E}_{\mathrm{proj}}(\rho) = \sum_m P_m \rho P_m,$$

✓ **Eigenvalues are either zero or one**

✓ **ϵ_X = -1 implies pair eigenvalues ±1**

→ no symmetry with ϵ_X = -1



Symmetry classification of quantum feedback control

■ Symmetry of feedback control with projective measurement

Assumption: Symmetry should not depend on the operation time!

$$\mathcal{E}_\tau(\rho) = \sum_m U_m(\tau) P_m \rho P_m U_m^\dagger(\tau) \xrightarrow{\tau \to 0} \mathcal{E}_{\mathrm{proj}}(\rho) = \sum_m P_m \rho P_m$$

$$(U_m(\tau) = e^{-iH_m \tau})$$

Operation time

Projective measurement channel!

■ Allowed symmetries for quantum channels

C sym. $\tilde{\mathcal{V}}_C \tilde{\mathcal{E}}^T \tilde{\mathcal{V}}_C^{-1} = \tilde{\mathcal{E}}, \quad \tilde{\mathcal{V}}_C^* \tilde{\mathcal{V}}_C = \pm 1$

K sym. $\tilde{\mathcal{V}}_K \tilde{\mathcal{E}}^* \tilde{\mathcal{V}}_K^{-1} = \tilde{\mathcal{E}}, \quad \tilde{\mathcal{V}}_K^* \tilde{\mathcal{V}}_K = \pm 1$

Q sym. $\tilde{\mathcal{V}}_Q \tilde{\mathcal{E}}^\dagger \tilde{\mathcal{V}}_Q^{-1} = \tilde{\mathcal{E}}, \quad \tilde{\mathcal{V}}_Q^2 = 1$

Class	C	K	Q
A	0	0	0
psH (AIII)	0	0	1
$\mathrm{AI}^\dagger$	$+1$	0	0
$\mathrm{AI+psH}_+$ ($\mathrm{BDI}^\dagger$)	$+1$	$+1$	1
AI ($\mathrm{D}^\dagger$)	0	$+1$	0
$\mathrm{AI+psH}_-$ ($\mathrm{DIII}^\dagger$)	-1	$+1$	1
$\mathrm{AII}^\dagger$	-1	0	0
$\mathrm{AII+psH}_+$ ($\mathrm{CII}^\dagger$)	-1	-1	1
AII ($\mathrm{C}^\dagger$)	0	-1	0
$\mathrm{AII+psH}_-$ ($\mathrm{CI}^\dagger$)	$+1$	-1	1

→ 10-fold symmetry classification of quantum feedback control with projective measurement (equivalent to AZ† classes of $i\tilde{\mathcal{E}}$)

Outline

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Antiunitary symmetry of quantum channels

■ K symmetry

$$\tilde{\mathcal{V}}_K \tilde{\mathcal{E}}^* \tilde{\mathcal{V}}_K^{-1} = \tilde{\mathcal{E}}$$

$$\Longleftrightarrow \ (\tilde{\mathcal{V}}_K \mathcal{K}) \tilde{\mathcal{E}} (\tilde{\mathcal{V}}_K \mathcal{K})^{-1} = \tilde{\mathcal{E}}$$

antiunitary symmetry
($\mathcal{K}$: complex conjugation)

■ Every quantum channel has an antiunitary symmetry (modular conjugation symmetry)

[Gong *et al*., PRX (2018); Kawabata *et al*., PRX Quantum (2023); Sa *et al*., PRX (2023)]

$$[\mathcal{E}(\rho)]^\dagger = \mathcal{E}(\rho^\dagger)$$

A quantum channel must preserve Hermiticity of the density matrix

$$\Longleftrightarrow \ \mathcal{J} \mathcal{E} \mathcal{J}^{-1} = \mathcal{E}, \ \mathcal{J}(\rho) = \rho^\dagger$$

antiunitary symmetry!

$$(\mathcal{J}^2 = 1)$$

Unitary symmetry of quantum channels

- Two types of unitary symmetry of quantum channels

[cf. Buca and Prosen, New J. Phys. 14, 073007 (2012); Albert and Jiang, PRA 89, 022118 (2014)]

$$\mathcal{E}(\rho) = \sum_m K_m \rho K_m^\dagger \Rightarrow \tilde{\mathcal{E}} = \sum_m K_m \otimes K_m^*$$

quantum channel

matrix rep.

$$(K_m = U_m M_m)$$

- Weak symmetry: a conserved quantity is not necessary

$$\mathcal{U}\mathcal{E}\mathcal{U}^{-1} = \mathcal{E}, \quad \mathcal{U}$$: unitary superoperator

→ **The quantum channel can be block diagonalized by symmetry eigenvalues**

- Strong symmetry: unitary symmetry with a conserved quantity

$$[K_m, A] = 0 \ (\forall m) \Rightarrow (e^{-i\theta A} \otimes I)\tilde{\mathcal{E}}(e^{i\theta A} \otimes I) = \tilde{\mathcal{E}}$$

$$\Rightarrow \mathrm{Tr}[A\mathcal{E}(\rho)] = \sum_m \mathrm{Tr}[K_m^\dagger A K_m \rho] = \mathrm{Tr}[A\rho]$$

Feedback with unitary symmetry

Feedback control with unitary symmetry

Example: projective position measurement that does not disturb spin

$$M_m = |m,\uparrow\rangle\langle m,\uparrow| + |m,\downarrow\rangle\langle m,\downarrow| \quad \Rightarrow \quad [M_m, S^z] = 0$$

$[U_m, S^z] = 0$ Feedback operation also conserves the magnetization

$$\Rightarrow \quad [K_m, S^z] = 0 \ (K_m = U_m M_m)$$

strong unitary symmetry!

Block diagonalization of a quantum channel by unitary symmetry

$$\mathcal{E} = \mathcal{E}_{\uparrow\uparrow} + \mathcal{E}_{\downarrow\uparrow} + \mathcal{E}_{\uparrow\downarrow} + \mathcal{E}_{\downarrow\downarrow},$$

$$\mathcal{E}_{\sigma\sigma'}(\rho) = \sum_m U_m M_{m,\sigma} \rho M_{m,\sigma'}^\dagger U_m^\dagger, \quad M_{m,\sigma} = |m,\sigma\rangle\langle m,\sigma|$$

$$\mathcal{J}\mathcal{E}_{\sigma\sigma'}\mathcal{J}^{-1} = \mathcal{E}_{\sigma'\sigma}$$

Each block does not necessarily have the modular conjugation symmetry! (if $\sigma \neq \sigma'$)

Three groups of symmetry classes

- Ten symmetry classes can be divided into three groups

Class	C	K	Q
AI+psH$_+$	$+1$	$+1$	1
AI	0	$+1$	0
AI+psH$_-$	-1	$+1$	1

Symmetry class with K symmetry

$$(\tilde{\mathcal{V}}_K^* \tilde{\mathcal{V}}_K = +1)$$

→ Basic symmetry class of quantum channels
(Every quantum channel has antiunitary sym. because of Hermiticity preservation)

Class	C	K	Q
A	0	0	0
psH	0	0	1
AI†	$+1$	0	0
AII†	-1	0	0

Symmetry class without K symmetry

→ Quantum channels with unitary symmetry (block diagonalization)

Class	C	K	Q
AII+psH$_+$	-1	-1	1
AII	0	-1	0
AII+psH$_-$	$+1$	-1	1

Symmetry class with K symmetry

$$(\tilde{\mathcal{V}}_K^* \tilde{\mathcal{V}}_K = -1)$$

→ Quantum channels with unitary symmetry and additional K symmetry

Order-reversing symmetry

Class	C	K	Q
AI+psH$_+$	$+1$	$+1$	1
AI	0	$+1$	0
AI+psH$_-$	-1	$+1$	1

Class AI (only the modular conjugation sym.)
→ Quantum channel without additional sym.
(example: chiral Maxwell's demon)

■ Class AI+psH$_\pm$: C symmetry, Q symmetry

$$\tilde{\mathcal{V}}_C \tilde{\mathcal{E}}^T \tilde{\mathcal{V}}_C^{-1} = \tilde{\mathcal{E}}$$ **Transpose**

$$\tilde{\mathcal{V}}_Q \tilde{\mathcal{E}}^\dagger \tilde{\mathcal{V}}_Q^{-1} = \tilde{\mathcal{E}}$$ **Hermitian conjugation**

■ Order-reversing nature of transpose and Hermitian conjugation

$$\tilde{\mathcal{E}}^T = \sum_m K_m^T \otimes K_m^\dagger, \quad K_m^T = (U_m M_m)^T = M_m^T U_m^T$$

The order of measurement and feedback is reversed!

How to realize feedback control with C or Q symmetry?

Feedback with two-point measurement

■ Feedback control with a two-point projective measurement

feedback unitary

$$\mathcal{E}(\rho) = \sum_{m_1,m_2} M_{m_2} U_{m_1} M_{m_1} \rho M^\dagger_{m_1} U^\dagger_{m_1} M^\dagger_{m_2} = \sum_{m_1,m_2} K_{m_1,m_2} \rho K^\dagger_{m_1,m_2}$$

2nd measurement 1st measurement

$$M_m = M_{\vec{j},\sigma} = |\vec{j},\sigma\rangle\langle\vec{j},\sigma|$$ projective measurement of position and spin

$$\Rightarrow K_{m_1,m_2} \otimes K^*_{m_1,m_2} = p(\vec{j}_2,\sigma_2|\vec{j}_1,\sigma_1)\,|\vec{j}_2,\sigma_2\rangle\langle\vec{j}_1,\sigma_1| \otimes |\vec{j}_2,\sigma_2\rangle\langle\vec{j}_1,\sigma_1|$$

$$:= \left|(U_{\vec{j}_1,\sigma_1})_{\vec{j}_2,\sigma_2;\vec{j}_1,\sigma_1}\right|^2$$ transition probability

$$K^T_{m_1,m_2} \otimes K^\dagger_{m_1,m_2} = p(\vec{j}_2,\sigma_2|\vec{j}_1,\sigma_1)\,|\vec{j}_1,\sigma_1\rangle\langle\vec{j}_2,\sigma_2| \otimes |\vec{j}_1,\sigma_1\rangle\langle\vec{j}_2,\sigma_2|$$

C symmetry and Q symmetry

→ Symmetry between transition probabilities (e.g., reciprocity)

Feedback with two-point measurement

■ Feedback control with a two-point projective measurement

feedback unitary

$$\mathcal{E}(\rho) = \sum_{m_1,m_2} M_{m_2} U_{m_1} M_{m_1} \rho M_{m_1}^\dagger U_{m_1}^\dagger M_{m_2}^\dagger = \sum_{m_1,m_2} K_{m_1,m_2} \rho K_{m_1,m_2}^\dagger$$

2nd measurement **1st measurement**

$$M_m = M_{\vec{j},\sigma} = |\vec{j},\sigma\rangle\langle\vec{j},\sigma|$$

projective measurement of position and spin

$$\Rightarrow \quad \tilde{\mathcal{E}} = \sum_{\vec{k}} \tilde{\boldsymbol{c}}_{\vec{k}}^\dagger X(\vec{k}) \tilde{\boldsymbol{c}}_{\vec{k}}$$

momentum-space representation

$$X(\vec{k}) = \begin{pmatrix} \tilde{X}(\vec{k}) & O \\ O & O \end{pmatrix}, \quad \tilde{X}(\vec{k}) = \begin{pmatrix} \xi_{\uparrow\uparrow}(\vec{k}) & \xi_{\uparrow\downarrow}(\vec{k}) \\ \xi_{\downarrow\uparrow}(\vec{k}) & \xi_{\downarrow\downarrow}(\vec{k}) \end{pmatrix},$$

$$\xi_{\sigma\sigma'}(\vec{k}) = \frac{1}{N_{\mathrm{cell}}} \sum_{\vec{j}_1,\vec{j}_2} p(\vec{j}_2,\sigma|\vec{j}_1,\sigma') e^{-i\vec{k}\cdot(\vec{R}_{\vec{j}_2} - \vec{R}_{\vec{j}_1})}.$$

transition probability

Spin winding number

- C symmetry with $\tilde{\mathcal{V}}_C^* \tilde{\mathcal{V}}_C = -1$ (class AI + psH$_-$)

$$(-i\sigma_y)\tilde{X}^T(-\vec{k})(-i\sigma_y)^{-1} = \tilde{X}(\vec{k}),$$

$$\Longleftrightarrow \xi_{\uparrow\uparrow}(-\vec{k}) = \xi_{\downarrow\downarrow}(\vec{k}),\ \xi_{\uparrow\downarrow}(-\vec{k}) = -\,\xi_{\uparrow\downarrow}(\vec{k}),\ \xi_{\downarrow\uparrow}(-\vec{k}) = -\,\xi_{\downarrow\uparrow}(\vec{k}).$$

$$\Longleftrightarrow p(\vec{j}_2, \uparrow | \vec{j}_1, \uparrow) = p(\vec{j}_1, \downarrow | \vec{j}_2, \downarrow),$$ **reciprocity**

$$p(\vec{j}_2, \uparrow | \vec{j}_1, \downarrow) = p(\vec{j}_2, \downarrow | \vec{j}_1, \uparrow) = 0$$ since $p(\vec{j}_2, \sigma | \vec{j}_1, \sigma') \geq 0$

Positivity of probability imposes nontrivial constraints on symmetry!

$$\tilde{X}(\vec{k}) = \begin{pmatrix} \xi_{\uparrow\uparrow}(\vec{k}) & 0 \\ 0 & \xi_{\downarrow\downarrow}(\vec{k}) \end{pmatrix}$$

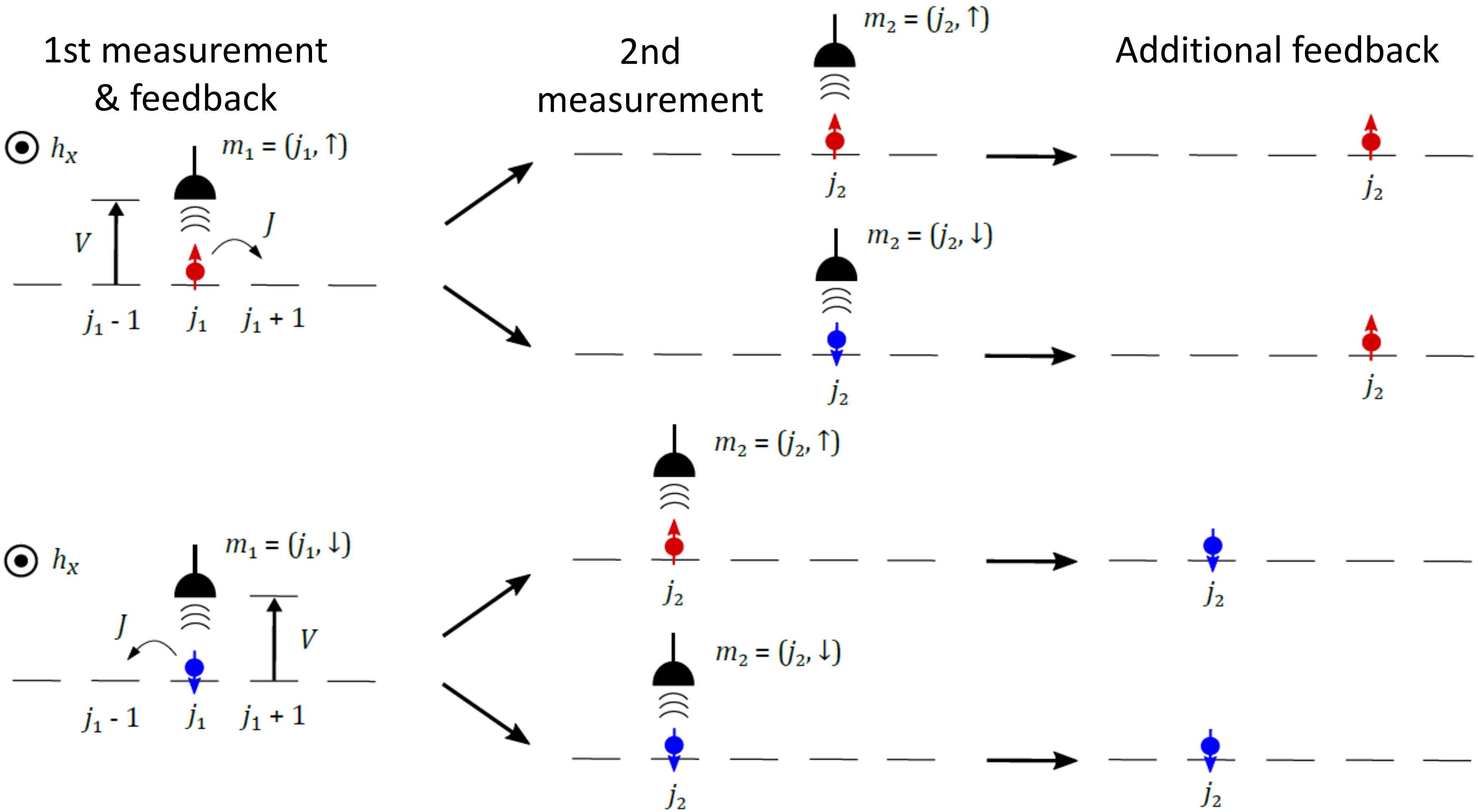
$$\Rightarrow\ w_{\mathrm{spin}} = \frac{1}{2}(w_\uparrow - w_\downarrow)$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \frac{dk}{2\pi i}\left(\partial_k \ln[\xi_{\uparrow\uparrow}(k) - \xi_{\mathrm{PG}}] - \partial_k \ln[\xi_{\downarrow\downarrow}(k) - \xi_{\mathrm{PG}}]\right)$$

Spin winding number:
topological invariant (in 1D)

Symmetry-protected feedback control

■ Model: helical Maxwell's demon (1D, spin 1/2, class AI + psH$_-$)



Helical spin transport by feedback control!

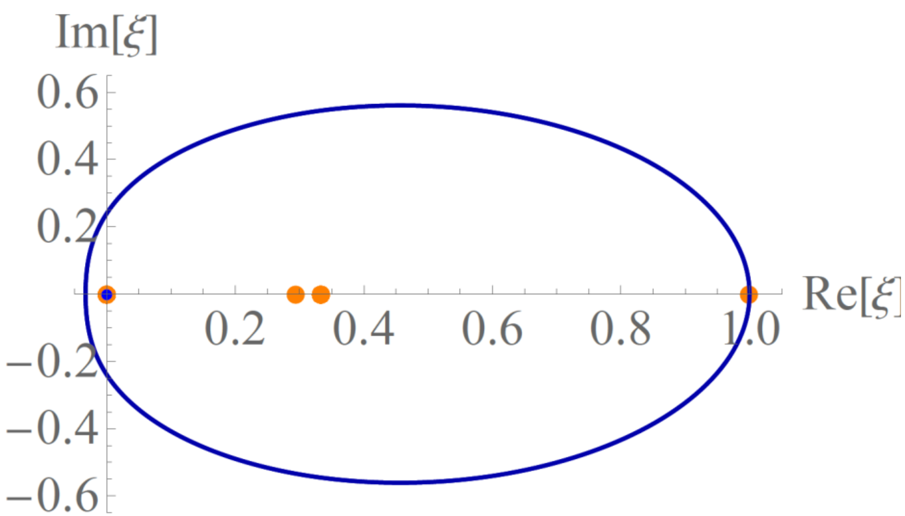
Symmetry-protected topological feedback control

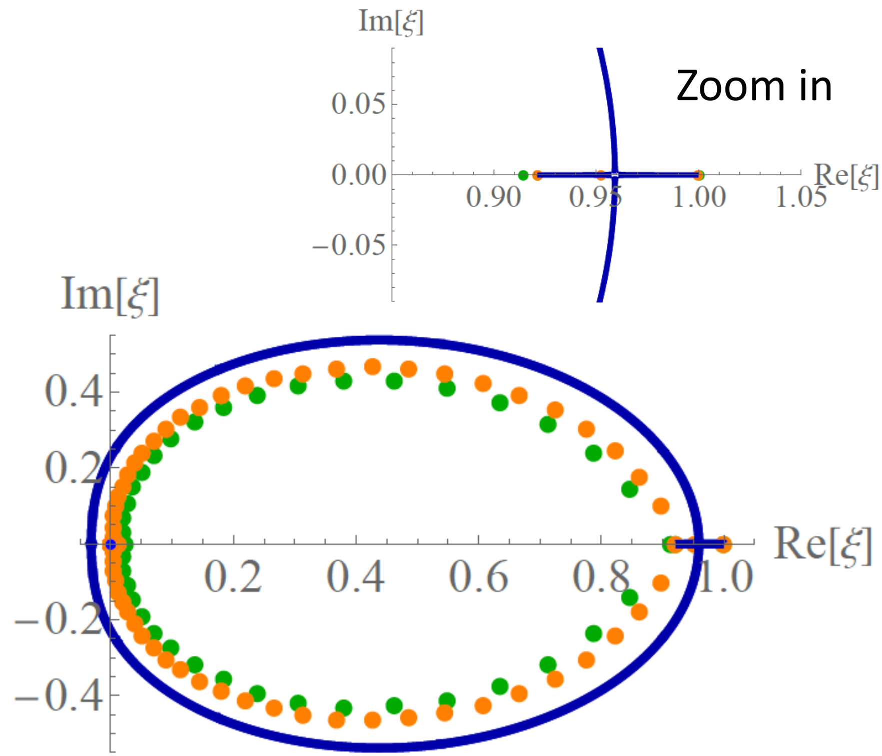
■ Eigenspectrum of the CPTP map for helical Maxwell's demon

Blue: PBC

Orange: OBC with 30 sites

Green: OBC with 20 sites





Symmetry is unbroken
→ non-Hermitian skin effect

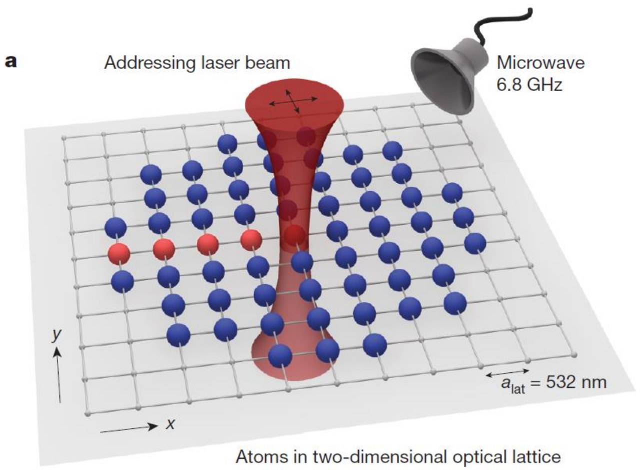
Symmetry is broken
→ no skin effect

Symmetry-protected topological feedback control!

Experimental platforms

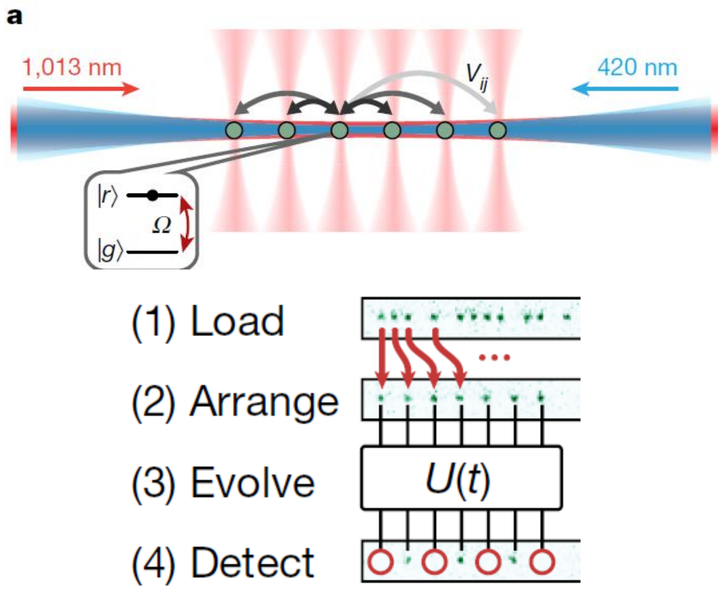
- Experimental platforms?

Cold atoms: quantum-gas microscopy & single-site addressing



[Weitenberg *et al.*, Nature 471, 319 (2011)]

Optical tweezer array



[Bernien *et al.*, Nature 551, 582 (2017)]

High-precision quantum measurement & control

→ Platform of topological Maxwell's demon

Outline

1. Introduction

2. Topological feedback control

3. Symmetry classification of quantum feedback control

4. Symmetry-protected topological feedback control

5. Summary and future perspective

Summary

- Topology of quantum feedback control
  - **Non-Hermitian topology of quantum channels!**
  - **10-fold symmetry classification**
- Topological Maxwell's demon
  - **Chiral/helical transport by feedback control**
  - **Non-Hermitian skin effect induced by feedback control**
- Outlook:
  - Feedback-controlled line-gap topology
  - Topology and information thermodynamics
  - Classification of time evolution in open quantum systems (cf. Floquet → classification of unitary time evolution)

MN and M. Ueda, arXiv:2403.08406