Higher bracket structure of density operators for topological insulators

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Outline

- Review of 2d case
  - Noncommutative coordinates (single-particle)
  - GMP algebra (many-body)
  - Bulk-boundary correspondence
- Higher dimensional generalization (mainly 4d)
  - Higher dimensional “currrent algebra”

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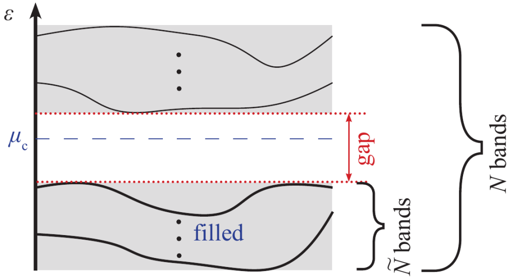
Based on: arXiv:2401.09683

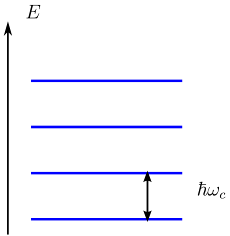
Topological band insulators



- Occupied bands are characterized by the Berry connection $A_\mu^{ab}(\boldsymbol{k})$ (curvature $F_{\mu\nu}^{ab}(\boldsymbol{k})$) and topological invariants, e.g., Chern number
- Non-trivial boundary states; bulk-boundary correspondence
- Density operator $\hat{\rho}(\boldsymbol{q})$ projected to the occupied bands below the band gap.

Coordinate algebra in 2d

- Physics confined within the lowest Landau level (LLL):



- Position operators *(in the first quantization)* projected on to LLL $(= X_i)$ do not commute

$$[X_1, X_2] = -i\ell_0^2,$$

where $\ell_0 = \sqrt{1/eB}$ is the magnetic length.

Coordinate algebra in 2d

- For functions $f(x_1, x_2)$ and their projected counterparts, $f(X_1, X_2)$,

$$[f(X_1, X_2), g(X_1, X_2)] = i\ell_0^2 \{\{f(X_1, X_2), g(X_1, X_2)\}\},$$

  where $\{\{\cdots\}\}$ is the Moyal bracket

$$\{\{f_1, f_2\}\} := i\ell_0^{-2} \sum_{n=1}^{\infty} \frac{(-)^n}{n!} \left(\partial_z^n f_1 \partial_{\bar{z}}^n f_2 - \partial_{\bar{z}}^n f_1 \partial_z^n f_2\right).$$

- To lowest order in ℓ_0, Moyal $\Rightarrow$ Poisson bracket,

$$[f(X_1, X_2), g(X_1, X_2)] \sim -i\ell_0^2 \, \epsilon_{ij} \, \partial_i f \, \partial_j g \equiv -i\ell_0^2 \{f, g\}$$

Bulk density operator algebra in 2d

- Projected density operator $\hat{\rho}(x,y)$ (*many-body!*) weighted by an envelop function $f(x,y)$:

$$\hat{\rho}(f) := \int d^2x\, f(x,y)\hat{\rho}(x,y)$$

- $\hat{\rho}(f)$ satisfies the Girvin-MacDonald-Platzman algebra

$$[\hat{\rho}(f_1), \hat{\rho}(f_2)] = i\,\ell_0^2\,\hat{\rho}(\{\{f_1, f_2\}\}) \sim i\,\ell_0^2\,\hat{\rho}(\{f_1, f_2\}\})$$

[Girvin-MacDonald-Platzman(85), Iso-Karabali-Sakita(92), Cappelli-Trugenberger-Zemba(93), Fairlie-Fletcher-Zachos(89)]

- Note: $f(x,y) = e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$ as an example.

$$[\hat{\rho}(\boldsymbol{k}), \hat{\rho}(\boldsymbol{q})] = 2i\ e^{\boldsymbol{k}\cdot\boldsymbol{q}\ell_0^2}\ \sin\left(\frac{\boldsymbol{k}\times\boldsymbol{q}\ell_0^2}{2}\right)\hat{\rho}(\boldsymbol{k}+\boldsymbol{q})$$

Boundary $U(1)$ current algebra

- Boundary of (F)QH states support chiral edge states
- Density operator at the boundary of QH droplet obeys the $U(1)$ current algebra

$$[\hat{\rho}(x), \hat{\rho}(x')] = \underbrace{-i\frac{\nu}{2\pi}\partial_x \delta(x - x')}_{\text{Schwinger term}}.$$

- For integer filling ν, can be derived from the non-interacting chiral edge mode described by $\hat{H} = \int dx \sum_{a=1}^{\nu} \hat{\psi}_a^\dagger i\partial_x \hat{\psi}_a$.
- Can also be written as (Note: $\hat{\rho}[f_i(x) = \delta(x - x_i)] = \hat{\rho}(x_i)$)

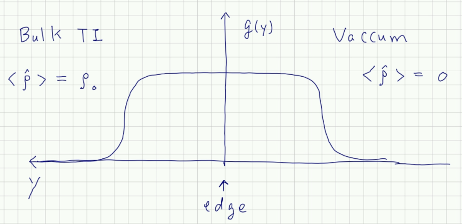
$$[\hat{\rho}(f_1), \hat{\rho}(f_2)] = \frac{-i\nu}{2\pi} \int_{\partial M_2} f_1 df_2$$

Bulk-boundary correspondence (i)

- [Martinez-Stone (93)]: set $f_i(x,y) = f_i(x)g(y)$,

$$[\hat{\rho}(f_1), \hat{\rho}(f_2)] \sim \frac{i\ell_0^2}{2} \int dx \int dy \, [(\partial_x f_1) f_2 - (\partial_x f_2) f_1] \, \partial_y g^2 \, \hat{\rho}(x,y).$$

- $g(y)$ is non-zero (constant) around the edge, and changes only deep inside/outside of the droplet (where $\partial_y g^2$ is finite);



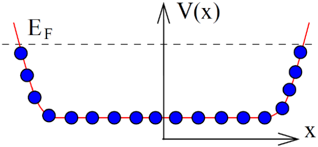
- Replacing $\hat{\rho} \to \langle \hat{\rho} \rangle =: \rho_0$ in the bulk,

$$[\hat{\varrho}(f_1), \hat{\varrho}(f_2)] \sim -i\rho_0 \ell_0^2 \int_{\partial M_2} f_1 df_2,$$

where $\hat{\varrho}$ denotes boundary density operator.

Bulk-boundary correspondence (ii)

- The many-body ground state in the presence of the boundary:

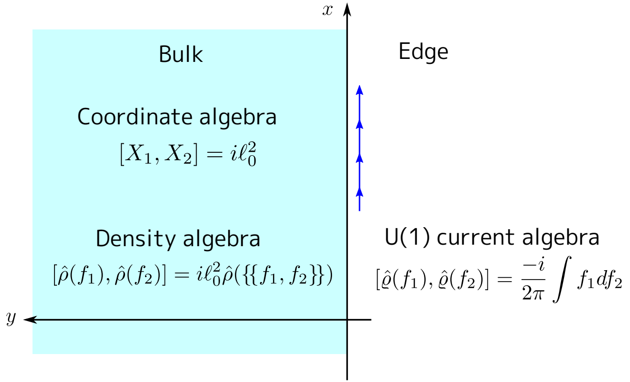


- Operators have to be *normal ordered* with respect to the ground state:

$$\sum A_{nm} \hat{c}_n^\dagger \hat{c}_m \Rightarrow \sum A_{nm} : \hat{c}_n^\dagger \hat{c}_m :$$

- The algebra of normal-ordered (and projected) density operator is the $U(1)$ current algebra. [Azuma (94)]

Summary in (2+1)d



- Many-body approach to topological insulators and bulk-boundary correspondence
- Current algebra as spectrum-generating algebra
- For FQHE, the bulk density operator can create charge neutral excitations

Higher-dimensional generalization?

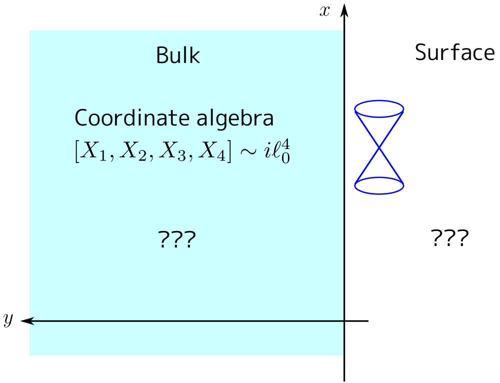
- Higher-dimensional topological insulators and Landau level models [Zhang-Hu(01); Karabali-Nair (02); Li-Wu (13), $\cdots$] E.g., in 4 spatial dimensions characterized by the 2nd Chern number

- "4-bracket structure" for *single-particle* projected position operators has been identified

$$[X_1, X_2, X_3, X_4] := \epsilon_{ijkl} \, X_i X_j X_k X_l \sim i\ell_0^4$$

  [Recall: $X_i = i\partial/\partial k_i - \mathcal{A}_i(\boldsymbol{k})$ and $[X_i, X_j] = \mathcal{F}_{ij}(\boldsymbol{k})$]
  [Neupert-Santos-SR-Chamon-Mudry (12), Estienne-Regnault-Bernevig (12), Shiozaki-Fujimoto (13), Hasebe (14-17)]

- Question: *Is there a many-body implication for this?*


x
Bulk
Surface
Coordinate algebra
$[X_1, X_2, X_3, X_4] \sim i\ell_0^4$
???
???
y

Many-body (?)

- Correspondingly to the four bracket of the position operators, for four smearing functions, $f_i(X_1, X_2, X_3, X_4)$,

$$[f_1, f_2, f_3, f_4] \sim \{f_1, f_2, f_3, f_4\} \equiv \epsilon_{ijkl}\, \partial_i f_1 \partial_j f_2 \partial_k f_3 \partial_l f_4,$$

assuming lower-brackets vanish. Here, $\{\cdots\}$ is the *Nambu bracket*. [Nambu(73)]

- One may expect (?):

$$[\hat{\rho}(f_1), \hat{\rho}(f_2), \hat{\rho}(f_3), \hat{\rho}(f_4)] \sim i\hat{\rho}[\{f_1, f_2, f_3, f_4\}] + \cdots$$

and following [Martinez-Stone] , set $f_a(x_i, w) = f_a(x_i) g(w)$

$$[\hat{\varrho}(f_1), \hat{\varrho}(f_2), \hat{\varrho}(f_2), \hat{\varrho}(f_3)] \sim -i\rho_0 \int_{\partial M_4} f_1 df_2 df_3 df_4.$$

Result 1: Bulk algebra

- For projected density operator $\hat{\rho}(\boldsymbol{q})$, consider:

$$[\hat{\rho}(\boldsymbol{q}_1), \hat{\rho}(\boldsymbol{q}_2), \cdots, \hat{\rho}(\boldsymbol{q}_d)]_{\text{mod}} \equiv \epsilon_{i_1 i_2 \cdots i_d} \hat{\rho}(\boldsymbol{q}_{i_1}) \cdots \hat{\rho}(\boldsymbol{q}_{i_d}) - (k\text{-body terms with } k > 1)$$

- Bulk density operator algebra in $d = even$ dim:

$$[\hat{\rho}(\boldsymbol{q}_1), \hat{\rho}(\boldsymbol{q}_2), \cdots, \hat{\rho}(\boldsymbol{q}_d)]_{\text{mod}} \propto (\boldsymbol{q}_1 \wedge \cdots \wedge \boldsymbol{q}_d) \sum_{a,b}^{\text{occ.}} \int \frac{\mathrm{d}^d \boldsymbol{k}}{(2\pi)^d} \epsilon^{\mu_1 \cdots \mu_d} \times [\mathcal{F}_{\mu_1 \mu_2}(\boldsymbol{k}) \cdots \mathcal{F}_{\mu_{d-1}\mu_d}(\boldsymbol{k})]^{ab} \hat{\chi}_a^\dagger(\boldsymbol{k}) \hat{\chi}_b(\boldsymbol{k}) + \cdots$$

- In the $\boldsymbol{q} \to 0$ limit and for constant topological density,

$$[\hat{\rho}(f_1), \cdots, \hat{\rho}(f_d)]_{\text{mod}} \sim \rho_0^{-1} (d/2)! \left(\frac{\mathrm{i}}{2\pi}\right)^{d/2} Ch_{d/2} \, \hat{\rho}(\{f_1, \cdots, f_d\})$$

Result 2: Boundary from bulk – Martinez-Stone way

- Following [Martinez-Stone], set $f_i(\boldsymbol{x}) = f_i(\mathbf{x})g(w)$ $(\boldsymbol{x} = (x, y, z, w) = (\mathbf{x}, w))$
- We then deduce the boundary algebra,

$$[\hat{\varrho}\,(f_1)\,, \cdots\,, \hat{\varrho}\,(f_d)]_{mod} \sim -(d/2)!\left(\frac{\mathrm{i}}{2\pi}\right)^{d/2} Ch_{d/2} \int_{\partial M_d} f_1 \;\mathrm{d}f_2 \cdots \mathrm{d}f_d$$

- This looks nice, but heuristic.
- Can we give a more precise derivation of this?

Result 3: Boundary from bulk – Azuma's way

- Instead of following Martinez-Stone, we follow Azuma:
- Consider (i) ground state in the presence of a boundary and then (ii) normal-ordered the (projected) density operator
- Using hybrid Wannier orbitals, the calculations reduce to purely boundary ones; We consider the boundary density operator $\varrho(\mathbf{x})$ with of the Weyl system:

$$\hat{H}_{Weyl} = \int_{\partial M_4} d^3\mathbf{x} \sum_{i=1}^{3} \hat{\psi}^\dagger i\partial_i \sigma_i \hat{\psi},$$

(when the bulk Chern number is 1.)

Result 3: Boundary from bulk – Azuma's way

- 4-bracket of *normal-ordered* density operators:

$$
\begin{aligned}
&[\hat{\varrho}(f_1), \hat{\varrho}(f_2), \hat{\varrho}(f_3), \hat{\varrho}(f_4)] \\
&= \epsilon^{i_1 i_2 i_3 i_4} \hat{\varrho}(f_{i_1}) \hat{\varrho}(f_{i_2}) \hat{\varrho}(f_{i_3}) \hat{\varrho}(f_{i_4}) \\
&= \hat{\varrho}([f_1, f_2, f_3, f_4]) + b_4(f_1, f_2, f_3, f_4) + \epsilon^{ijkl} \hat{Q}_2(f_i f_j \otimes f_k f_l),
\end{aligned}
$$

  - $\hat{Q}_2$: some four fermion operator.
  - b_4: c-number part; "Schwinger term"

- The c-number part b_4

$$
b_4\left([f_1, f_2, f_3, f_4]\right) = \mathrm{Tr}\left([f_1, f_2, f_3, f_4]\, P_-\right)
$$

- C.f. in (1+1)d,

$$
[\hat{\varrho}(f_1), \hat{\varrho}(f_2)] = \hat{\varrho}([f_1, f_2]) + \underbrace{b_2(f_1, f_2)}_{\text{Schwinger term}}
$$

Result 3: Boundary from bulk – Azuma's way

- The c-number part b_4 can be split into two parts: $b_4 = R_4 + S_4$, where

$$R_4(f_1, f_2, f_3, f_4) = \frac{1}{16}\epsilon^{\alpha\beta\gamma\delta}\,\mathrm{Tr}\left(F\{F, f_\alpha\}\{F, f_\beta\}[F, f_\gamma][F, f_\delta]\right)$$
$$S_4(f_1, f_2, f_3, f_4) = -\frac{1}{32}\epsilon^{\alpha\beta\gamma\delta}\,\mathrm{Tr}\left(F[F, f_\alpha][F, f_\beta][F, f_\gamma][F, f_\delta]\right)$$

where

$$F(\mathbf{p}) = \sum_{i=1}^{3}\frac{p_i\sigma_i}{|\mathbf{p}|} \quad \text{:“the grading operator”}$$

- For reasonable f_α, S_4 is well-defined (cutoff independent) since it is given as the trace of trace class operators. On the other hand, R_4 is cutoff dependent.

Schwinger term S_4

- The contribution S_4 is evaluated as [Langmann (95)].

$$S_4(f_1, f_2, f_3, f_4) = -\frac{1}{48\pi^2} \int_{\partial M_4} \epsilon^{ijkl} f_i df_j df_k df_l$$

Recall: $[\hat{\varrho}(f_1), \hat{\varrho}(f_2)] = \frac{-i\nu}{2\pi} \int_{\partial M_2} f_1 df_2$

- Appears in noncommutative geometry
  - $i[F, \cdot]$ is a generalization of exterior derivative;
  - Tr_c is a generalization of integral;

$$\mathrm{Tr}_c \epsilon^{ij\cdots}(f_i[F, f_j]\cdots) \longleftrightarrow \int f_i df_j \cdots$$

- Also appears in the noncommuative geometry approach to the (mostly integer) quantum Hall effect. [Bellissard-Schulz-Baldes-van Elst (94), Prodan-Leugn-Bellissard (13), ...]

Further result

- We focused on the density operator algebra, as they are directly related to the coordinate algebra. Also, naively, density is a generator of $U(1)$ symmetry.
- However, it may also be important to include currents.
- Indeed, if we try to make connections with quantum anomalies, it would be natural to consider currents. Usually, anomaly states non-conservation of currents in the presence of background gauge field. One may "convert" background by current operators.
- For the boundary (non-Abelian) current operators, we showed

$$\frac{1}{8}\epsilon^{i_0 i_1 i_2 i_3}\left[\left[\left[\hat{J}_0\left(f_{i_0}\right), \hat{J}_1\left(f_{i_1}\right)\right], \hat{J}_2\left(f_{i_2}\right)\right], \hat{J}_3\left(f_{i_3}\right)\right]$$

$$= \mathrm{i}\hat{J}_0\left([f_0, f_1, f_2, f_3]\right) + b_4\left([f_0, f_1, f_2, f_3]\right)$$

where $\hat{J}_\mu(f) = \int_{\partial M_4} \mathrm{d}^3\mathbf{x}\, f_{ab}(\mathbf{x}) \,:\hat{\psi}_a^\dagger(\mathbf{x})\sigma_\mu\hat{\psi}_b(\mathbf{x}):$ $(\mu = 0, \ldots, 3)$.

- The repeated commutator – may be related to non-linear response.

More speculations

- Quantum field theory descriptions? Noncommutative Chern-Simons theory [Susskind (01)]

$$S = \frac{1}{2\pi}\int BdA + \frac{k}{4\pi}\int dtd^2x\,\epsilon^{\mu\nu\lambda}\underbrace{\left[A_\mu \star \partial_\nu A_\lambda + i\frac{2}{3}A_\mu \star A_\nu \star A_\lambda\right]}_{\sim A_\mu \partial_\nu A_\lambda + \frac{\theta}{3}\{A_\mu, A_\nu\}A_\lambda}$$

  The BF term appears due to the functional bosonization [Cf. Chan-Hughes-SR-Fradkin (13)]

- Perhaps:

$$S \sim \frac{1}{2\pi}\int BdA + \int dtd^4x\,\varepsilon^{\mu\nu\lambda}\Big[\text{5d Chern-Simons term} + \theta\varepsilon^{\mu\nu\lambda\rho\kappa\sigma}\{A_\mu, A_\nu, A_\lambda, A_\kappa\}A_\sigma\Big]$$

- It may be also related to other areas, such as string theory.

Comments

- Overall, the role of (higher) non-commutative coordinates in the many-body context is still unclear.
- The renormalization or subtraction of R_4?
- Our calculations are many-body but non-interacting. The field theory formulation may be useful: "Hydrodynamic" or "bosonization"approach to interacting topological states.
- Boundary collective excitations; gravitons? ["Collective excitations at the boundary of a 4D quantum Hall droplet" [Hu-Zhang (01)] [Elvang-Polchinski (02)]]
- Similar structure for Fermi surfaces: [Pok Man Tam-Kane(23)]

$$s_M(\mathbf{q}_1, \ldots, \mathbf{q}_{M-1}) = \int \frac{d^D \mathbf{q}_M}{(2\pi)^D} \left\langle \rho_{\mathbf{q}_1} \rho_{\mathbf{q}_2} \ldots \rho_{\mathbf{q}_M} \right\rangle_c ,$$

$$s_4(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) = \frac{|\mathbf{q}_1 \cdot (\mathbf{q}_2 \times \mathbf{q}_3)|}{(2\pi)^3} \chi_F$$