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Anomalous bulk-edge correspondence in topological insulators

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Recent Developments and Challenges in Topological Phases

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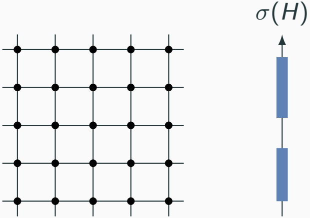
Bulk-edge correspondence

Topological insulators

original context: independent electrons in a crystal (possibly disordered)

bulk picture

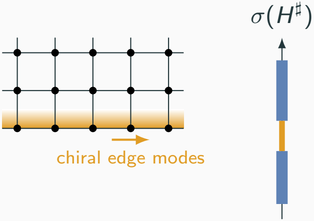
Ex: $\mathcal{H} = \ell^2(\mathbb{Z}^2), H = H^*$



$H \mapsto \mathcal{I}(H) \in \mathbb{Z}$ continuous

edge picture

$\mathcal{H}^\sharp = \ell^2(\mathbb{Z} \times \mathbb{N}), H^\sharp = H|_{\mathcal{H}^\sharp}$

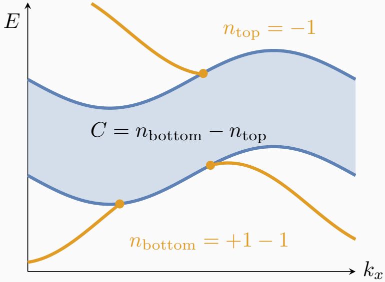


$H^\sharp \mapsto \mathcal{I}^\sharp(H^\sharp) \in \mathbb{Z}$ continuous

Bulk-edge correspondence: $\mathcal{I} = \mathcal{I}^\sharp$

Reminiscent of Atiyah-Singer '63: geometrical index = analytical index

Context: Integer Quantum Hall Effect - Independent electrons on a 2d lattice



C is the Chern number: bulk index associated to a $U(1)$-fiber bundle of a bulk band over the (magnetic) Brillouin zone $\mathbb{T}^2$.

$n_{\text{bottom/top}}$ is the number of edge mode branches below/above the band

A fruitful concept for mathematics

- Bulk-edge correspondence is a very nice interface where physical observables are turned into mathematical theorems.
- This includes various dimensions, symmetries, disorder, periodic driving (Floquet) and some interactions.
- Several approaches from functional analysis to K-theory, passing by (differential/non-commutative) geometry

Avron Seiler Simon '94, Bellissard Van Elst Schulz-Baldes '94, Kellendonk Richter Schulz-Baldes '00, Graf Elbau '02, Combes Germinet '05, Graf Porta' 13, Avila et. al '13, Essin Gurarie '11, Kubota '15, Prodan Schulz-Baldes '16, Mathai Thiang '16, Bourne Rennie '18, Graf T. '18, Drouot '19, Gomi Thiang '19, Shapiro T. '19, Cornean et al '21, Kubota '21, Bal '23, Ogata '23...

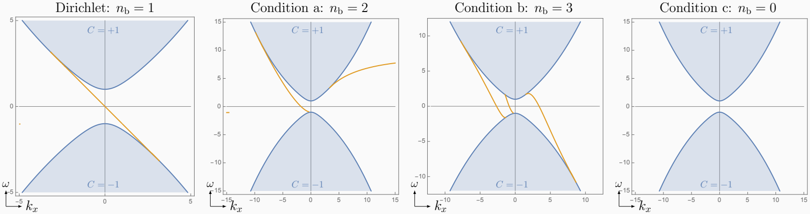
Q: can it fail?

A puzzling example

Dirac Hamiltonian in 2d (massive and regularized).

The Chern number is fixed to $C = \pm 1$ in a bulk infinite system.

Edge spectrum for various boundary conditions:



$$C_+ \neq n_b \qquad (!)$$

What is wrong here ?

- The system is closed, single-particle, translation-invariant and the boundary condition is self-adjoint (Hermitian model)
- Edge mode branches hidden above in the spectrum? Actually not
- The only difference is that bulk bands and gaps are unbounded
- Generalized bulk-edge correspondence

$$C_+ = n_\mathrm{b} + w_\infty$$

where w_∞ is dubbed "ghost topological charge".

Today's talk: what is w_∞ and what are the consequences.

Anomalies in Dirac Hamiltonian

Massive and regularized Dirac Hamiltonian

$$\mathcal{H} = \begin{pmatrix} m + \epsilon(\partial_x^2 + \partial_y^2) & \mathrm{i}\partial_x + \partial_y \\ \mathrm{i}\partial_x - \partial_y & -m - \epsilon(\partial_x^2 + \partial_y^2) \end{pmatrix}$$
$$= \mathrm{i}\sigma_x\partial_x + \mathrm{i}\sigma_y\partial_y + \sigma_z(m + \epsilon(\partial_x^2 + \partial_y^2))$$

- For $\epsilon = 0$ this is a paradigmatic model describing conical intersections in graphene ($m = 0$) or gap opening mechanisms in topological insulators (e.g. Haldane, Kane Mele models)
- One should think of $0 < \epsilon \ll 1$ as a regularization term (see below)
- In Volovik'88 such a model with finite ϵ describes topological effects in a superfluid 3-He film

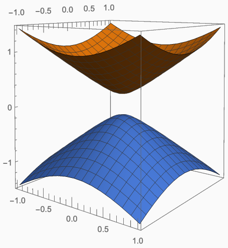
Band structure

On $L^2(\mathbb{R}^2) \otimes \mathbb{C}^2$ the system is translation invariant so that $\mathrm{i}\partial_t \Psi = \mathcal{H}\Psi$ becomes, by Bloch/Fourier theorem:

$$H\psi = \omega\psi\,, \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad H(k_x, k_y) = \begin{pmatrix} m - \epsilon k^2 & -k_x + \mathrm{i}k_y \\ -k_x - \mathrm{i}k_y & -m + \epsilon k^2 \end{pmatrix},$$

with $k^2 = k_x^2 + k_y^2$ and $k_x, k_y \in \mathbb{R}^2$ (unbounded Brillouin zone).

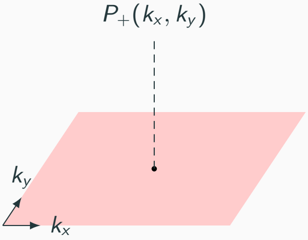
Two energy bands $\omega_\pm(k_x, k_y) = \pm\sqrt{k^2 + (m - \epsilon k^2)^2}$ separated by a spectral gap of size m.

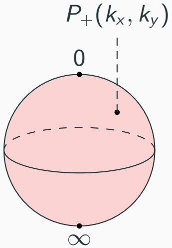


Line bundles and compactification

Upper band : $H(k_x, k_y)\psi_+(k_x, k_y) = \omega_+(k_x, k_y)\psi_+(k_x, k_y)$ with $\psi_+ \in \mathbb{C}^3$

Let $P_+ = |\psi_+\rangle\langle\psi_+|$ be the associated rank 1 eigenprojection.





For $\epsilon \neq 0$, P_+ is single-valued as $k \to \infty$. It actually defines a $U(1)$-line bundle over the **closed** manifold $\mathbb{R}^2 \cup \{\infty\} \cong S^2$.

For $\epsilon \neq 0$, P_+ and P_- are single-valued as $k \to \infty$ and each one defines a line bundle over the **closed** manifold $\mathbb{R}^2 \cup \{\infty\} \cong S^2$.

Proposition **Graf, Jud, T. '21**

For $\epsilon \neq 0$ and $P = P_\pm$, the **Chern numbers**

$$C(P) = \frac{1}{2\pi \mathrm{i}} \int_{S^2} \mathrm{d}k_x \mathrm{d}k_y \, \mathrm{tr}(P[\partial_{k_x} P, \partial_{k_y} P])$$

are topological indices with

$C(P_\pm) = \pm \frac{\mathrm{sgn}(m)+\mathrm{sgn}(\epsilon)}{2} = \pm 1$.

- $C \neq 0$ indicates the obstruction of finding a regular eigenfunction $\psi(k_x, k_y)$ over the whole S^2.
- If $\epsilon = 0$ then the r.h.s reads $\pm \frac{1}{2}\mathrm{sgn}(m)$ but $C_\pm$ is not continuous.

Proof

$H = \vec{d} \cdot \vec{\sigma}$ where $\vec{d} = (k_x, k_y, m - \epsilon k^2)$ and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$

Eigenprojections in terms of $\vec{e} = \frac{\vec{d}}{|\vec{d}|}$: $P_\pm = \frac{1}{2}\left((\vec{e} \cdot \vec{\sigma})^2 \pm \vec{e} \cdot \vec{\sigma}\right)$

For $\epsilon > 0$,

$$\vec{e} \to \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

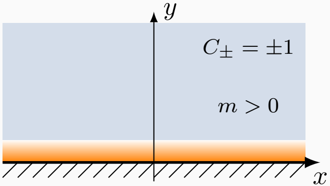
as $k \to \infty$, so that $\vec{e} : S^2 \to S^2$ and

$$C_\pm = \pm \int_{S^2} (\vec{e})^* \mathrm{vol}$$

Notice that, for $\epsilon = 0$, let $(k_x, k_y) = (r \cos\theta, r \sin\theta)$. Then

$$\vec{e} \to \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \\ 0 \end{pmatrix}$$

as $r \to \infty$. The limit depends on the direction.



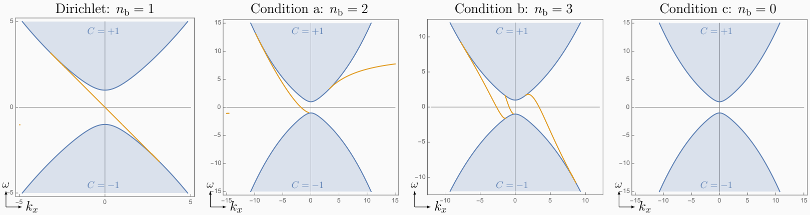
Solve $i\partial_t \Psi = \mathcal{H}\Psi$ on $L^2(\mathbb{R} \times \mathbb{R}^+)$

$$H^\sharp \widetilde{\psi} = \omega \widetilde{\psi}, \qquad H^\sharp(k_x) = \begin{pmatrix} m - \epsilon k_x^2 + \epsilon \partial_y^2 & -k_x + \partial_y \\ -k_x - \partial_y & -m + \epsilon k_x^2 - \epsilon \partial_y \end{pmatrix}$$

Add a self-adjoint boundary condition. ODE problem to solve for each value of k_x and ω. Either: oscillating solution (bulk mode), solution decaying away from the boundary (edge mode) or no solution. Leads to edge spectrum.

Counting edge modes

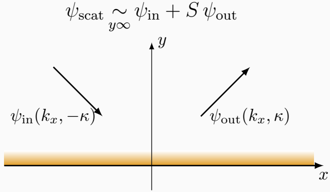
Focus on the upper band with $C_+ = 1$



Dirichlet	Condition (a)	Condition (b)	Condition (c)
$\widetilde{\psi}_1(0)=0$	$\widetilde{\psi}_1(0)=0$	$(A+\mathrm{i}k_x B)\widetilde{\psi}(0)$	$\widetilde{\psi}_1(0)=0$
$\widetilde{\psi}_2(0)=0$	$\widetilde{\psi}_2'(0)+k_x\psi_2(0)=0$	$+C\widetilde{\psi}'(0)=0$	$\widetilde{\psi}_2'(0)-k_x\psi_2(0)=0$
$\rightarrow n_\mathrm{b} = 1$	$\rightarrow n_\mathrm{b} = 2$	$\rightarrow n_\mathrm{b} = 3$	$\rightarrow n_\mathrm{b} = 0$

with $A=\begin{pmatrix} 1 & 1 \\ 9 & 1 \end{pmatrix}$, $B=\begin{pmatrix} 4\mathrm{i} & -\mathrm{i} \\ \mathrm{i} & -\mathrm{i} \end{pmatrix}$, $C=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Scattering amplitude



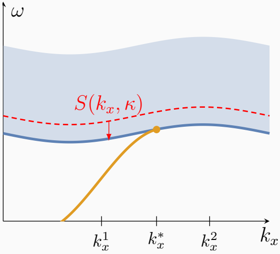
- Choose a boundary condition at $y = 0$. Fix $k_x \in \mathbb{R}$, $k_y = \kappa > 0$.
- Let $\omega = \omega_+(k_x, \kappa)$, and notice that $\omega(k_x, -\kappa) = \omega$.
- In the bulk $\mathbb{R}^2$, there are two solutions (plane waves) ψ_{out} and ψ_{in} travelling at momentum $(k_x, \pm\kappa)$ and same energy ω.
- It exists S such that

$$\psi_{\text{scat}} = \psi_{\text{in}} + S\psi_{\text{out}} + \underset{y\infty}{o}(1)$$

satisfies the boundary condition of the edge problem.

- $S(k_x, \kappa) \in U(1)$ is the scattering (reflection) amplitude. It exists for any $(k_x, \kappa) \in \mathbb{R} \times \mathbb{R}_+^*$, with $\omega = \omega_+(k_x, \kappa)$.

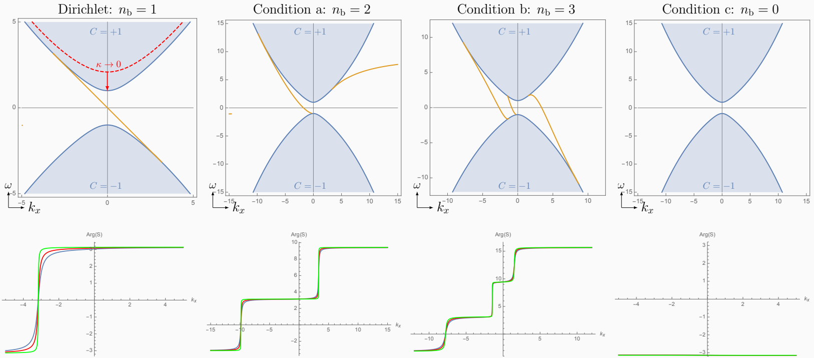
Edge modes and levinson's theorem



Theorem **Graf, Porta '13**

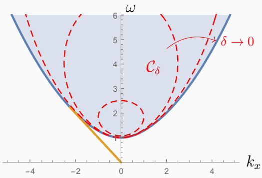
$$\lim_{\kappa \to 0} \frac{1}{2\pi} \int_{k_x^1}^{k_x^2} (S^{-1} \partial_{k_x} S) \mathrm{d}k_x = n_{\mathrm{b}}$$

Some examples



$S^{-1}\partial_{k_x} S = \arg[S(k_x, \kappa)]$ for $\kappa = 0.1$, 0.05 and 0.01.

Bulk-scattering correspondence



Theorem **Graf, Jud, Tauber '21**

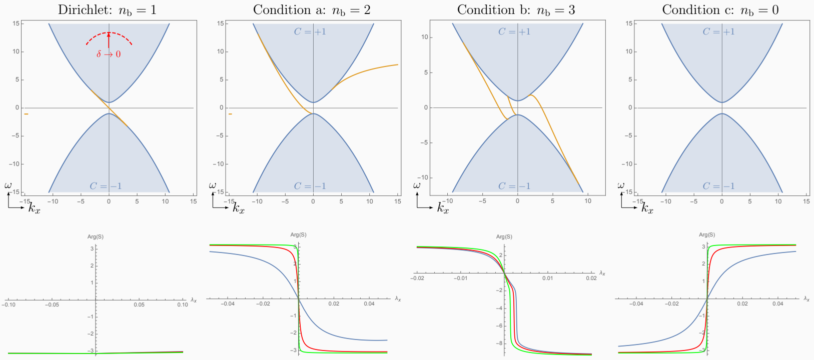
For any closed, anti clockwise and not self-intersecting curve $\mathcal{C}_\delta$ inside the upper bulk band

$$\frac{1}{2\pi}\int_{\mathcal{C}_\delta} S^{-1}\mathrm{d}S = C_+$$

Proof: S is also a transition function between two bulk sections.

In the limit $\delta \to 0$ we recover n_b and, possibly, some contribution at ∞

Ghost topological charge



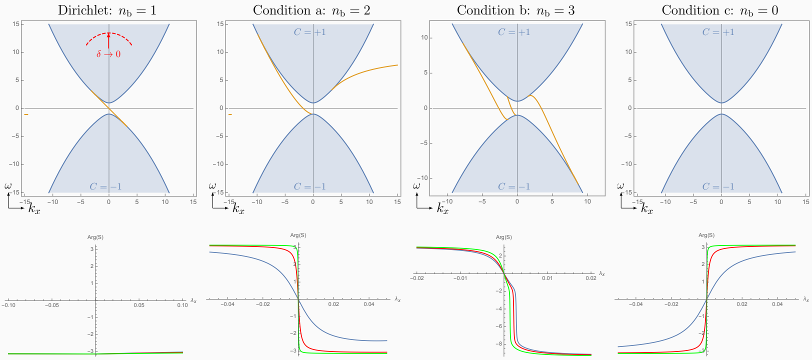
Dual variables λ_x, δ with

$$k_x = -\frac{\lambda_x}{\lambda_x^2+\delta^2}, \quad \kappa = \frac{\delta}{\lambda_x^2+\delta^2}$$

so that $\delta \to 0$ and $\lambda_x = 0^\pm$ explores $(k_x, \kappa) = (0^\mp, \infty)$.

Compute $\int S^{-1}\partial_{\lambda_x} S(\lambda_x, \delta)$ for $1 \gg \delta_1 > \delta_2 > \delta_3 > 0$

Ghost topological charge



$w_\infty = 0$ $\quad$ $w_\infty = -1$ $\quad$ $w_\infty = -2$ $\quad$ $w_\infty = 1$

$$C_+ = n_b + w_\infty \ (!)$$

w_∞ is interpreted as a “ghost” topological charge at infinity

Partial summary

Partial summary

- The scattering amplitude is a pivotal tool for bulk edge-correspondence.
- It detects n_{b} via Levinson's theorem
- Its winding on a closed curve is the Chern number C_+
- In the case where $C_+ \neq n_{\mathrm{b}}$, an additional winding contribution w_∞ appears exactly at infinity, even though there are no edge modes there:

$$C_+ = n_b + w_\infty$$

  In particular, C_+ and w_∞ fix the value of n_b.
- This formalism also exists in other 2D continuous (see below) or discrete tight-binding models (Graf, Porta '13).
- It would probably work in arbitrary dimension. However it strongly relies on translation invariance.

When do we have $w_\infty \neq 0$?

Classification: boundary conditions

- General local and x-translation invariant boundary condition

$$(B_0 + ik_x B_1)\widetilde{\psi} + B_2\widetilde{\psi}'\Big|_{y=0} = 0$$

with $B_0, B_1, B_2 \in M_2(\mathbb{C})$. Or equivalently

$$A\Psi|_{y=0} = 0, \qquad \Psi = \begin{pmatrix} \widetilde{\psi} \\ \widetilde{\psi}' \end{pmatrix} \in \mathbb{C}^4$$

with $A := A_0 + \mathrm{i}k_x A_1 \in M_{2,4}(\mathbb{C}), A_0 := [B_0|B_2], A_1 := [B_1|0]$

- $A \sim GA$ for $G \in GL_2(\mathbb{C})$. The GL_2-invariance reduces the problem to $\mathrm{Gr}_{2,4}(\mathbb{C})$ (Schubert cell decomposition)
- Self-adjoint condition $\langle \phi, H^\sharp(k_x)\psi\rangle = \langle H^\sharp(k_x)\phi, \psi\rangle$ imposes further constraints on A.

Result: Exhaustive classification of local, x-translation invariant and self-adjoint boundary conditions.

Class	A_0	A_1
$\mathfrak{A}_{1,2}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} b_{11} & b_{12} & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 \end{pmatrix}$
$\mathfrak{A}_{1,4}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 1 \end{pmatrix}$ $\alpha \in \mathbb{R}$	$\begin{pmatrix} b_{11} & 0 & 0 & 0 \\ b_{21} & \mathrm{i}\beta & 0 & 0 \end{pmatrix}$ $\beta \in \mathbb{R}$
$\mathfrak{A}_{2,3}$	$\begin{pmatrix} 0 & 1 & 0 & 0 \\ \alpha & 0 & 1 & 0 \end{pmatrix}$ $\alpha \in \mathbb{R}$	$\begin{pmatrix} 0 & b_{12} & 0 & 0 \\ \mathrm{i}\beta & b_{22} & 0 & 0 \end{pmatrix}$ $\beta \in \mathbb{R}$
$\mathfrak{A}_{2,4}$	$\begin{pmatrix} a_{11} & 1 & 0 & 0 \\ a_{21} & 0 & (a_{11}^*)^{-1} & 1 \end{pmatrix}$ $a_{11} \in \mathbb{C} \setminus \{0\}, \quad a_{21} = \alpha a_{11} + \epsilon^{-1}, \quad \alpha \in \mathbb{R}$	$\begin{pmatrix} b_{11} & b_{11}(a_{11})^{-1} & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 \end{pmatrix}$ $b_{22} - b_{21} a_{11}^{-1} = \mathrm{i}\beta, \quad \beta \in \mathbb{R}$
$\mathfrak{A}_{3,4}$	$\begin{pmatrix} \alpha_1 & a_{12} & 1 & 0 \\ \epsilon^{-1} - a_{12}^* & \alpha_2 & 0 & 1 \end{pmatrix}$ $\alpha_1, \alpha_2 \in \mathbb{R}$	$\begin{pmatrix} \mathrm{i}\beta_1 & b_{12} & 0 & 0 \\ b_{12}^* & \mathrm{i}\beta_2 & 0 & 0 \end{pmatrix}$ $\beta_1, \beta_2 \in \mathbb{R}$
$\mathfrak{B}$	$\begin{pmatrix} a_1 & a_2 & \mathrm{i}\alpha & -\mathrm{i}\mu^*\alpha \\ \mu a_1 & \mu a_2 & \mathrm{i}\mu\alpha & -\mathrm{i}\lvert\mu\rvert^2\alpha \end{pmatrix}$ $\alpha \in \mathbb{R}, \ \alpha\big(\alpha\,\mathrm{Im}(\mu) - \epsilon\,\mathrm{Re}(a_1 - a_2\mu)\big) = 0$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$
$\mathfrak{C}$	$\begin{pmatrix} a_1 & a_2 & 0 & a_4 \\ \mu a_1 & \mu a_2 & 0 & \mu a_4 \end{pmatrix}$ $(a_2, a_4) \neq 0, \mu \in \mathbb{C} \setminus \{0\}, \mathrm{Im}\big(a_2 a_4^*\big) = 0$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Table 1: Classification of local self-adjoint boundary conditions from Theorem 2, up to GL_2-invariance. All unconstrained parameters are arbitrary complex numbers: $a_{ij}, b_{ij}, a_i \in \mathbb{C}$.

Dirichlet $\in \mathfrak{A}_{1,2}$, Conditions a/b/c $\in \mathfrak{A}_{1,4}, \mathfrak{A}_{3,4}, \mathfrak{A}_{1,4}$

Classification: anomalies

- For each boundary condition A, the scattering amplitude S can be computed and expanded near ∞ via dual variables.
- w_∞ can be computed analytically for each class.
- Quite tedious but doable with a formal computer software like Mathematica (some expressions have about 70 terms, from which one needs to extract the leading order).
- Exhaustive classification of anomalies

Classification: anomalies

Theorem 11. *w_∞ can be systematically computed for almost every self-adjoint boundary condition. Its value is given in Table 2. Each expression for $\mathcal{S}$ can be found in Section 5.*

Class	w_∞	Condition
$\mathfrak{A}_{1,2}$	0	None
$\mathfrak{A}_{1,4}$	$\text{sign}_0(\beta)$	None
$\mathfrak{A}_{2,3}$	$-\text{sign}(\beta)$	$\beta \neq 0$
	1	$\beta = 0$ and $\lvert\alpha - \epsilon^{-1}\rvert < 1/\sqrt{2}$
	0	$\beta = 0$ and $\lvert\alpha - \epsilon^{-1}\rvert > 1/\sqrt{2}$
$\mathfrak{A}_{2,4}$	$\text{sign}(\beta)$	$\beta\lvert a_{11}\rvert^2 > \sqrt{2}$ or $\beta\lvert a_{11}\rvert^2 < -\sqrt{2}$
	0	$-\sqrt{2} < \beta\lvert a_{11}\rvert^2 < \sqrt{2}$
$\mathfrak{A}_{3,4}$	$\text{sign}(B_+)$	$b_{12} \neq 0$ and $B_+B_- < 0$
	0	$b_{12} \neq 0$ and $B_\pm > 0$
	$2 \times \text{sign}(\sqrt{2} - \beta_1)$	$b_{12} \neq 0$ and $B_\pm < 0$
	$\text{sign}_0(\beta)$	$b_{12} = 0$ and $\beta_1^2 \neq 2$
$\mathfrak{B}$	0	None
$\mathfrak{C}$	1	$a_2 = 0$
	0	$a_2 \neq 0$

Table 2: Anomaly classification for almost any self-adjoint boundary condition. The classes and parameters refer to Table 1. For class $\mathfrak{A}_{3,4}$ we also define $B_\pm := \beta_2(\beta_1 \pm \sqrt{2}) + \lvert b_{12}\rvert^2$. We distinguish the sign function $\text{sign} : \mathbb{R} \setminus \{0\} \to \{\pm 1\}$ from its extended version $\text{sign}_0 : \mathbb{R} \to \{0, \pm 1\}$ which sends 0 to 0.

Anomalies are everywhere! $w_\infty \in \{0, \pm 1, \pm 2\}$

Digression: shallow-water waves

Motivation

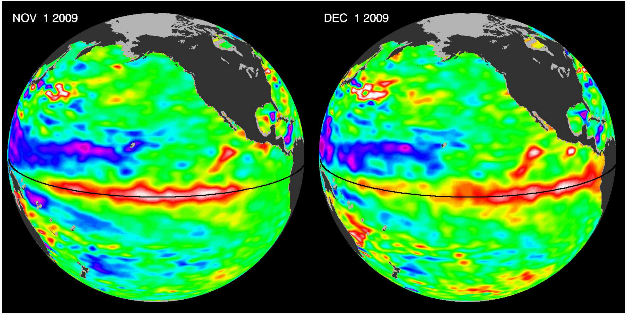
Topological edge modes have been actually observed in many classical wave systems (acoustic, optics, fluids,....).

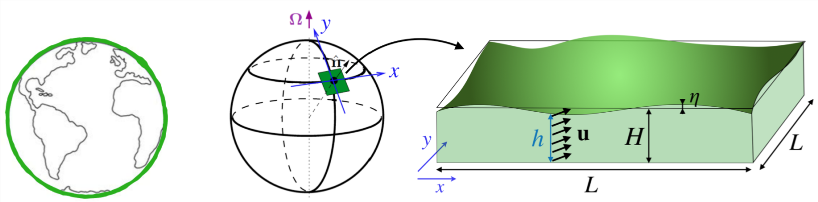
What is the biggest topological insulator on Earth?

Topological edge modes have been actually observed in many classical wave systems (acoustic, optics, fluids,....).

What is the biggest topological insulator on Earth?

The Earth itself!





$$\partial_t \eta = -H\,\vec{\nabla}\cdot\vec{u} \qquad \text{(mass conservation)}$$

$$\partial_t \vec{u} = -g\vec{\nabla}\eta - f\,\vec{u}^{\perp} + \nu\nabla^2\vec{u}^{\perp} \qquad \text{(momentum conservation)}$$

with $\vec{u} = (u, v)$ and $\vec{u}^{\perp} = \hat{n}\times\vec{u} = (-v, u)$,

$-g\vec{\nabla}\eta$: gravity pressure,

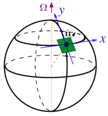
$-f\,\vec{u}^{\perp}$: Coriolis effect with $f = 2\vec{\Omega}\cdot\hat{n} = 2\Omega\sin(y)$

$\nu\nabla^2\vec{u}^{\perp}$: odd-viscous regularizing term with $0 < \nu \ll 1$

The bulk picture

f-plane approximation — Thomson 1880

Tangent **plane**: $\Omega = \mathbb{R}^2$ and Coriolis f is a **constant**



The system is formally analogous to $\mathrm{i}\partial_t \psi = \mathcal{H}\psi$ with

$$\psi = \begin{pmatrix} \eta \\ u \\ v \end{pmatrix}, \qquad \mathcal{H} = \begin{pmatrix} 0 & p_x & p_y \\ p_x & 0 & -\mathrm{i}(f - \nu p^2) \\ p_y & \mathrm{i}(f - \nu p^2) & 0 \end{pmatrix}$$

where $p_x = -\mathrm{i}\partial_x$, $p_y = -\mathrm{i}\partial_y$ and $p^2 = p_x^2 + p_y^2$.

Prop: $\mathcal{H}$ is a (densely defined) self-adjoint operator on $L^2(\mathbb{R}^2, \mathbb{C}^3)$.

By **translation invariance**, normal modes $\psi = \hat{\psi} e^{\mathrm{i}(\omega t - k_x x - k_y y)}$ reduce to

$$H\hat{\psi} = \omega\hat{\psi},$$

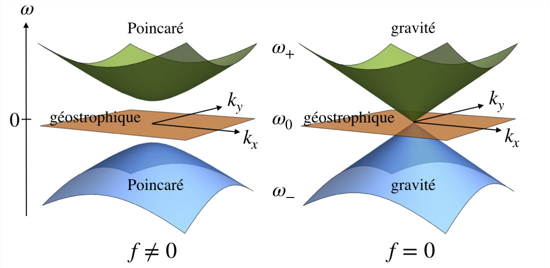
with $H(k_x, k_y) \in M_3(\mathbb{C})$. Both frequency $\omega \in \mathbb{R}$ and momentum $k_x, k_y \in \mathbb{R}^2$ are unbounded.

Band structure and bulk index

Eigenvalues of H:

$$\omega_\pm(k_x, k_y) = \pm\sqrt{k^2 + (f - \nu k^2)^2}, \qquad \omega_0(k_x, k_y) = 0$$

with $k_x, k_y \in \mathbb{R}^2$ and $k^2 = k_x^2 + k_y^2$.



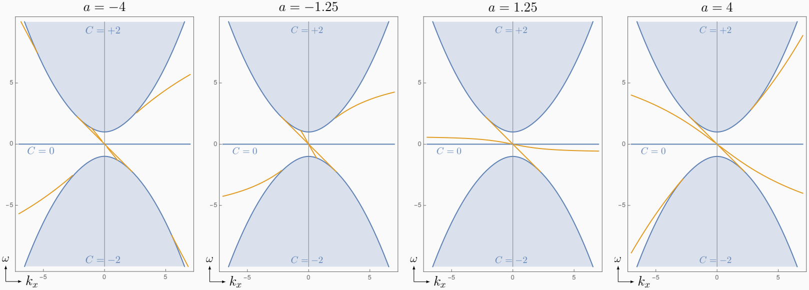
For $f \neq 0$ the three bands are separated by two spectral gaps.

Eigenprojection $P_\pm, P_0 : \mathbb{R}^2 \to M_3(\mathbb{C})$ define a line bundle over S^2 when $\nu \neq 0$.

Chern number $C_\pm = \pm 2$ and $C_0 = 0$.

Anomalous bulk edge correspondence

Boundary condition $v|_{y=0} = 0, (\partial_x u + a\partial_y v)|_{y=0} = 0$ with $a \in \mathbb{R}$.



$n_b = 2$ $w_\infty = 0$

$n_b = 3$ $w_\infty = -1$

$n_b = 1$ $w_\infty = 1$

$n_b = 2$ $w_\infty = 0$

$$C_+ = n_b + w_\infty$$

- Dirac: $H = \vec{d} \cdot \vec{\sigma}$ where $\vec{d} = (k_x, k_y, m - \epsilon k^2)$ and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

  $C(P_\pm) = \pm 1$

- Shallow-water: $H = \vec{d} \cdot \vec{S}$ where $\vec{d} = (k_x, k_y, f - \nu k^2)$ and

$$S_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad S_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad S_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}.$$

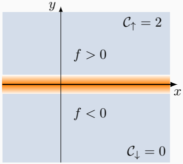
  $C(P_\pm) = \pm 2,\ C_0 = 0$

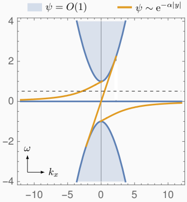
Very likely to work for any $H = \vec{d} \cdot \vec{S}$ where $\vec{S}$ is a spin-s representation, with $C(P_m) = 2m,\ m \in \{-s, -s+1, \ldots, s-1, s\}$.

Towards new physics?

Curing the anomaly

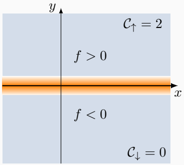
- Topological interface instead of sharp wall

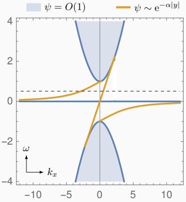




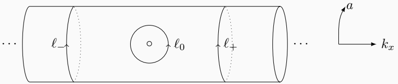
Curing the anomaly

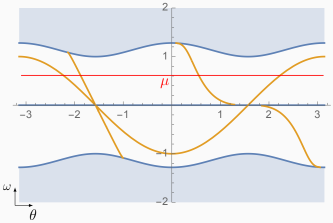
- Topological interface instead of sharp wall





- Avoid infinite spectrum. Recall $v|_{y=0} = 0, (\partial_x u + a\partial_y v)|_{y=0} = 0$



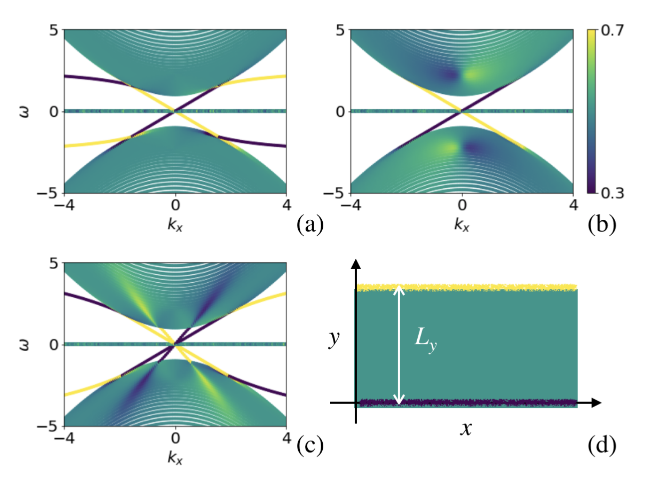


$$\mathcal{C}_R = \{(k_x, a) = (R\cos(\theta), R\sin(\theta)), \theta \in [-\pi, \pi]\} \sim \ell_0$$

$$n(\mathcal{C}_R, \mu) = C_+ - C_0$$

Channel geometry

Schallow-water waves for (a) Dirichlet and (b/c) $v|_{y=0} = 0, (\partial_x u \pm \partial_y v)|_{y=0} = 0$



(a) $n_b = 2,\ w_\infty = 0$ (b) $n_b = 1,\ w_\infty = 1$ (c) $n_b = 3,\ w_\infty = -1$

Local density of states

Back on the upper half-plane: $(x, y) \in \mathbb{R} \times \mathbb{R}^+$. Consider

$$\rho(y,\omega) = \int \mathrm{d}k_x \mathrm{d}\kappa \, |\psi_{\mathrm{scat}}(k_x, \kappa, y)|^2 \, \delta(\omega - \omega_+(k_x, \kappa))$$

After some a algebra we get

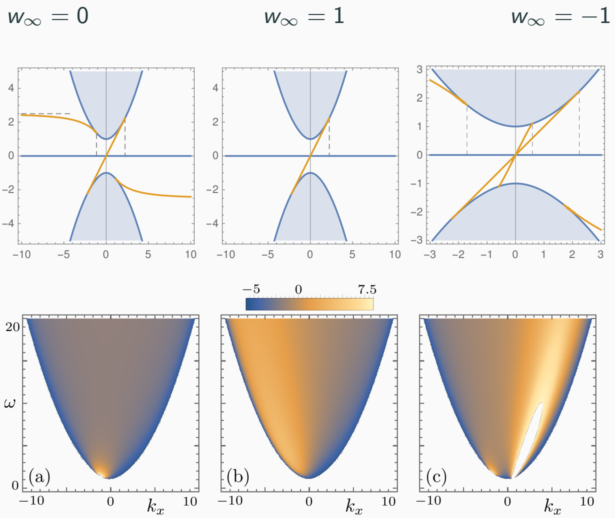
$$\rho(y,\omega) = \rho_0(\omega) + \int_{-k_{\max}}^{k_{\max}} \mathrm{d}k_x \, \mathcal{R}(y, k_x, \omega)$$

where $\rho_0(\omega)$ is indep. of y and the boundary condition, and

$$\mathcal{R}(y, k_x, \omega) = \frac{g(\omega)}{\kappa_{\mathrm{out}}} \Big(2\mathrm{Re}\big(\langle \psi_{\mathrm{in}}, S\psi_{\mathrm{out}} \rangle + \langle \psi_{\mathrm{in}}, T\psi_{\mathrm{ev}} \rangle + \langle S\psi_{\mathrm{out}}, T\psi_{\mathrm{ev}} \rangle\big) + |T\psi_{\mathrm{ev}}|^2 \Big)$$

Lets plot $\mathcal{R}(0, k_x, \omega)$.

Local density of states



Lets plot $\mathcal{R}(0, k_x, \omega)$.

Conclusion

Conclusion and perspective

- The bulk-edge correspondence does not hold for the (massive and regularized) Dirac Hamiltonian and shallow-water waves
- Generalized relation

$$C_+ = n_b + w_\infty$$

where w_∞ is interpreted as a "ghost" topological charge.

- This is not a fine-tuned effect (cf. anomaly classification and shallow water waves)
- Scattering amplitude is the key concept.

Perspectives:

- Could be investigated in any d or with symmetries.
- What about $\epsilon = 0$?
- The main challenge remains to find a physical interpretation of w_∞.

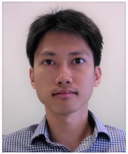
Further reading


P. Delplace


A. Venaille


G. M. Graf


H. Jud


G. C. Thiang

- Tauber, Delplace, Venaille, J. Fluid Mech. 868, R2 (2019)
- Tauber, Delplace, Venaille, Phys. Rev. Research 2(1) 013147 (2020)
- Graf, Jud, Tauber, Communications in Mathematical Physics, 383(2), 731-761 (2021)
- Tauber, Thiang, Annales Henri Poincaré, 24, 107-132 (2023)
- Jud, Tauber, arXiv:2403.04465 (2024)

Thank you!