Finite Size Gaps and Low-Energy excitations in Gapless Frustration Free Systems.

Haruki Watanabe (U Tokyo)

References

[1] arXiv: 2406. 06414

[2] 2406. 06415

[3] 2310. 16881 with Hosho Katsura (U Tokyo) Jong Yeon Lee (UCB UIUC)

Rintaro Masaoka (1st year grad student)

Tomohiro Soejima (postdoc at Harvard)

Frusration Free Hamiltonians

- $\hat{H} = \sum_i \hat{H}_i$, $|GS\rangle$ minimizes all $\hat{H}_i$ simultaneously.

  finite-range int.

  i.e. $\hat{H}_i |GS\rangle = \lambda_i |GS\rangle$

  λ_i: smallest eig of $\hat{H}_i$

- $\hat{H}_i$'s do not have to commute.

- Examples (Today, we discuss only spin models)

trivial • Paramagnet $\hat{H} = - \sum_i \hat{S}_i^z$

SSB • MG model $\hat{H} = \sum_i (\vec{S}_i \cdot \vec{S}_{i+1} + \frac{1}{2} \vec{S}_i \cdot \vec{S}_{i+2})$ $S = \frac{1}{2}$

SPT • AKLT model $\hat{H} = \sum_i (\vec{S}_i \cdot \vec{S}_{i+1} + \frac{1}{3} (\vec{S}_i \cdot \vec{S}_{i+1})^2)$ $S = 1$

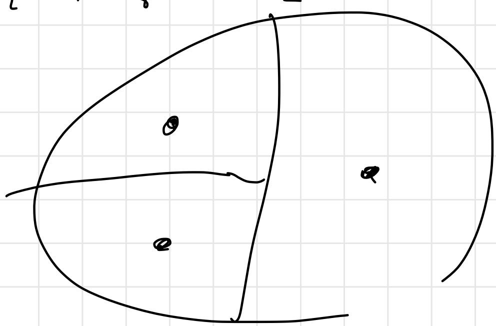
$\frac{1}{3} + \frac{1}{2} \vec{S}_i \cdot \vec{S}_{i+1} + \frac{1}{6} (\quad)^2 = P^{S=2}$

TO • Toric code.

(commuting projector)

Fracton

...



$\Rightarrow$ Understanding

- General properties of FF Hamiltonians
  limitations

- Which phases can be realized by FF Hamiltonians

are important.

Our Main Conjectures (any spatial dimension) ③

[1] If H is FF, translation-inv, Gapless

Then $\exists |\Phi_k\rangle$ s.t.

$$\hat{T}|\Phi_k\rangle = e^{-ik}|\Phi_k\rangle$$

$$\langle\Phi_k|\hat{H}|\Phi_k\rangle - E_0 = O((k-k_0)^2)$$

$\Rightarrow$ No-Go 1: Gapless phases with linear dispersion cannot be realized by FF H.

[2] If H is FF, critical

↳ GS correlation function decays power-law.

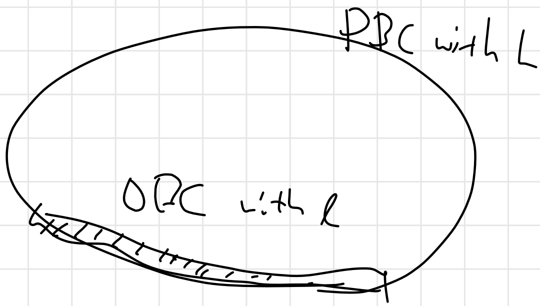
Then finite-size gap $\epsilon = O(L^{-2})$.

$\Rightarrow$ No-Go 2: low-E EFT cannot be Lorentz inv field theory.

Previous Works (Knabe type arguments)

FF, Translation inv, $\hat{T}\hat{H}_i = \hat{H}_{i+1}\hat{T}$.

$2 < \ell < L/2$



Knabe (1988)

$$\epsilon_L^{PBC} \geq \frac{\ell-1}{\ell-2}\left(\epsilon_\ell^{OBC} - \frac{1}{\ell-1}\right)$$

Gosset-Mozgunov (2016)

$$\epsilon_L^{PBC} \geq \frac{5\ell(\ell+1)}{6(\ell^2-4)}\left(\epsilon_\ell^{OBC} - \frac{6}{\ell(\ell+1)}\right)$$

Usage: ① if $\epsilon_\ell^{OBC} - \left(\begin{array}{c}\frac{1}{\ell-1} \\ \frac{6}{\ell(\ell+1)}\end{array}\right) > 0$ for a ℓ, then $\epsilon_L^{PBC} > \exists \Delta_\ell$

② if $\epsilon_L^{PBC} \to 0$ as $L \to \infty$, then $\epsilon_\ell^{OBC} < \left|\begin{array}{c}\frac{1}{\ell-1} \\ \frac{6}{\ell(\ell+1)}\end{array}\right|$

→ Applicable to OBC only.

Edge states ??

Bulk excitations may be gapped ...

Definition of Gapped vs Gapless.

Assume finite L (open BC / periodic BC)

Diagonalize $\hat{H}$

$$E_1 \le E_2 \le \cdots \le E_{N_{deg}}$$

Gap: $\exists N_{deg}(L)$ s.t.

$$\lim_{L\to\infty}(\underbrace{E_{N_{deg}} - E_1}_{\text{finite size splitting}}) = 0$$

$$\lim_{L\to\infty}(E_{N_{deg}+1} - E_{N_{deg}}) = \Delta \neq 0 \quad \text{excitation gap.}$$

$$\boxed{\text{Conjecture 1: Absence of finite size splitting for FF.}}$$

→ Gap can be determined by $\epsilon = E_{\tilde{N}_{deg}+1}$, $\not\to 0$ gapped, $\to 0$ gapless.

$$E_1 = E_2 = \cdots = E_{\tilde{N}_{deg}} = 0.$$

non FF

TFIM ($0 < h < \infty$)

Haldane

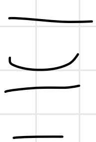
MG ($J_2 \sim \frac{1}{2}J_1$ but $J_2 \neq \frac{1}{2}J_1$)

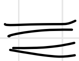
FF

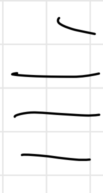
IM ($h = 0$)

AKLT

MG ($J_2 = \frac{1}{2}J_1$)







→ finite size gap.

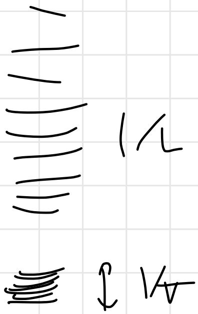
exact → degeneracy

Relation to Hohenberg-Mermin-Wagner

no continuous SSB in 2D at $T>0$

1D at $T=0$.

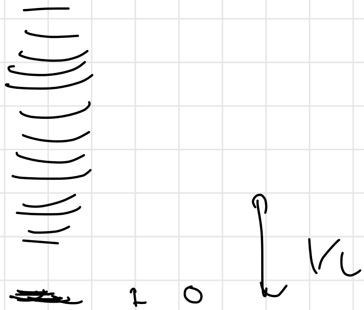
Continuous SSB



Goldstone excitation

Degenerate ToS

in 1D no distinction. $\rightarrow$ no SSB



$$\hat H(\Delta) = \sum_i \hat H_i$$

$$H_i := -\frac{J}{\Delta}\left(\hat S_i^x \hat S_{i+1}^x + \hat S_i^y \hat S_{i+1}^y + \Delta \hat S_i^z \hat S_{i+1}^z\right) + \frac{J}{\Delta^2}\left[1-(1-\Delta)(\hat S_i^z)^2\right]\left[1-(1-\Delta)(\hat S_{i+1}^z)^2\right]$$

	$[\hat H, \hat M]$	FF	SSB
$\Delta = 0$	$= 0$	✓	✓
$\Delta \neq 0$	$\neq 0$	✓	✓

Symmetry Generator

$$\hat Q = \sum_i \hat S_i^z$$

Order parameter

$$\hat M = \sum_i \hat S_i^x .$$

$$\hat M = [i\hat Q, \hat X]$$

Bogoliubov ineq.

(Schwartz ineq.)

$$\hat Q_k := \sum_{r\in\Lambda} \hat Q_r e^{ik\cdot r}$$

$\omega(k) = O(|k|^2)$??

$$\frac{1}{V^2}\sum_k \langle \hat X_k^\dagger \hat X_k + \hat X_k \hat X_k^\dagger \rangle \geq \frac{1}{V}\sum_k \frac{\left\{\frac{\omega(k)}{2T}\right\} \frac{|\langle [i\hat Q_k^\dagger, \hat X_k]\rangle|^2}{V^2}}{\frac{\langle [\hat Q_k, [\hat H(\Delta), \hat Q_k^\dagger]]\rangle}{V}}$$

$\frac{2}{V}\langle \sum_{r\in\Lambda} \hat X_r^2 \rangle = O(1)$

$\propto \left|\frac{\langle \hat M\rangle}{V}\right|^2$

$\propto k^2$

(c.f. 'pole'

Bravyi-Gosset

(17)

$S=1/2$. NN, translation inv. $\hat{H}=\sum_x \hat{Q}_{x,x+1}$

Rank 1

$\hat{Q}_{x,x+1}=|\psi\rangle\langle\psi|$ with $|\psi\rangle=(\alpha+i\beta)|01\rangle+(\alpha+i\beta)|10\rangle+\delta|11\rangle$

$(2\alpha^2+2\beta^2+\delta^2=1)$ $\quad |\psi\rangle=(\alpha+i\beta)|01\rangle+(\alpha-i\beta)|10\rangle+\delta|11\rangle$

→ Ferromag model if $\alpha=\delta=0$, $\beta=\frac{1}{\sqrt{2}}$.

Rank 2

$$\hat{Q}_{x,x+1}=\frac{1}{2}-\frac{1}{2}\left(e^{i\theta}\hat{S}_x^+\hat{S}_{x+1}^-+e^{-i\theta}\hat{S}_x^-\hat{S}_{x+1}^+\right)-\frac{1}{2}\left(\hat{S}_x^z+\hat{S}_{x+1}^z\right)$$

これらの場合は Gapped.

We show, under PBC,

all above cases have $|\Phi_0\rangle=|0\cdots0\rangle$ as a GS

$$|\Psi_k\rangle=\sum_x \frac{e^{ikx}}{\sqrt{L}}\hat{S}_x^-|\Phi_0\rangle$$ is a variational state

with $\langle\Psi_k|\hat{H}|\Psi_k\rangle=O((k-k_0)^2)$

Min-Max theorem

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$\hat{O} \geq 0$ if $\hat{O}^\dagger = \hat{O}$ & $\left(\begin{array}{l} \langle v|\hat{O}|v\rangle \geq 0 \text{ for all } |v\rangle. \\ \text{all e-val of } \hat{O} \geq 0. \end{array} \right.$

If, $\hat{V} := \hat{H}' - \hat{H} \geq 0$

$$E_j' \geq E_j$$

trivial if $\hat{U}\hat{H}\hat{U}^\dagger = \begin{pmatrix} E_1 & & \\ & E_2 & \\ & & \ddots \end{pmatrix}$

$\hat{U}\hat{H}'\hat{U}^\dagger = \begin{pmatrix} E_1' & & \\ & E_2' & \\ & & \ddots \end{pmatrix}$

by the same $\hat{U}$.

- Gapped under OBC $\rightarrow$ Gapped under PBC.
- Gapped in 1D $\rightarrow$ Gapped in d Dim.

Sufficient to consider Gapless cases.

- Gapless case

$|\Phi_0\rangle = |0 \cdots 0\rangle$ is GS

$|\Psi_k\rangle = \sum_r e^{ik\cdot r} \hat{S}_r^- |\Phi_0\rangle$ is good variational state

with $\langle \Psi_k|\hat{H}|\Psi_k\rangle = \sum_i E(k_i)$

Gosset – Huang (PRL 2016)

F. F.

$$\langle \Psi | \hat{O}_x^{\dagger} (1 - \hat{G}) \hat{O}_y' | \Psi \rangle$$

GS projector.

c.f.

Hastings.

Hastings Koma (2006) $\le C' e^{-g'' |x-y| \varepsilon}$

$\to \xi = O(1/\varepsilon)$

Gosset-Huang $\le C \exp\left(-g' \cdot |x-y| \cdot \sqrt{\frac{\varepsilon}{g^2+\varepsilon}}\right)$

$\to \xi = O(1/\sqrt{\varepsilon})$.

When LHS $\sim O(L^{-\Delta})$.

contradiction if $\varepsilon = \frac{c}{L^2}$.

Examples (GHZ uncle)

$$\hat{H} = \sum_{x=1}^{L} \hat{Q}_{x,x+1,x+2}$$

$$\hat{Q}_{x,x+1,x+2} = \frac{1}{2} - \hat{S}^z_{x+1}(\hat{S}^z_x + \hat{S}^z_{x+2}) - 2\hat{S}^x_{x+1}\left(\frac{1}{4} - \hat{S}^z_x S^z_{x+2}\right)$$

$$|+\rangle = |1\rangle + |0\rangle$$

$$\ker \hat{Q} = \mathrm{span}\{ |000\rangle \; |111\rangle \; |1+0\rangle \; |0+1\rangle \}$$

GS

$$|0\cdots0\rangle \quad |1\cdots1\rangle \quad \sum_{n=1}^{L-1} |n\rangle, \quad \sum_{n=1}^{L-1} |\bar{n}\rangle$$

$$\mapsto 0 \leftrightarrow 1$$

$$|n\rangle = |\underbrace{0\cdots0}_{L-n}\,\underbrace{1\cdots1}_{n}\rangle$$

$$\langle \Psi_k | \hat{H} | \Psi_k \rangle = 2$$

$$|\Psi_{k,l}\rangle = \frac{\sqrt{2}}{L} \sum_{n=1}^{L-1} \sum_{m=0}^{L-1} e^{ikm} e^{-ikn/2} \sin\left(\frac{\pi n l}{L}\right) \hat{T}^m |n\rangle.$$

$$E_l(k) = 2\left[1 - \cos\frac{k}{2}\cos\left(\frac{\pi l}{L}\right)\right]$$

$$= 4\sin^2\frac{\pi l}{2L} + 4\cos\left(\frac{\pi l}{L}\right)\sin^2\left(\frac{k}{4}\right)$$

$$\Rightarrow \begin{cases} \text{exp decay} \\ \text{gapless.} \end{cases}$$