2024 YITP WORKSHOP

2D GBZ Theory

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arXiv: 2311.16868 (2023)

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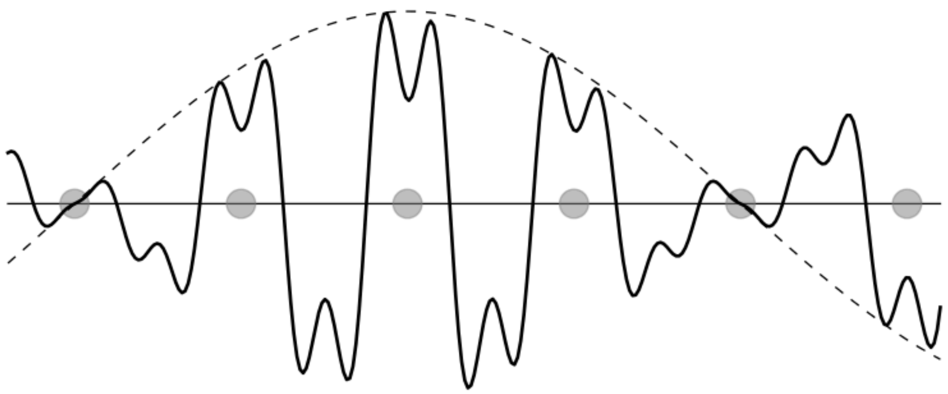
Outline

- **Introduction**
- **1D GBZ theory: review**
- **2D NHSE: numerical summary**
- **2D GBZ theory: recent developments**
- **2D GBZ theory: wave function approach**

Introduction

□ Bloch's theorem

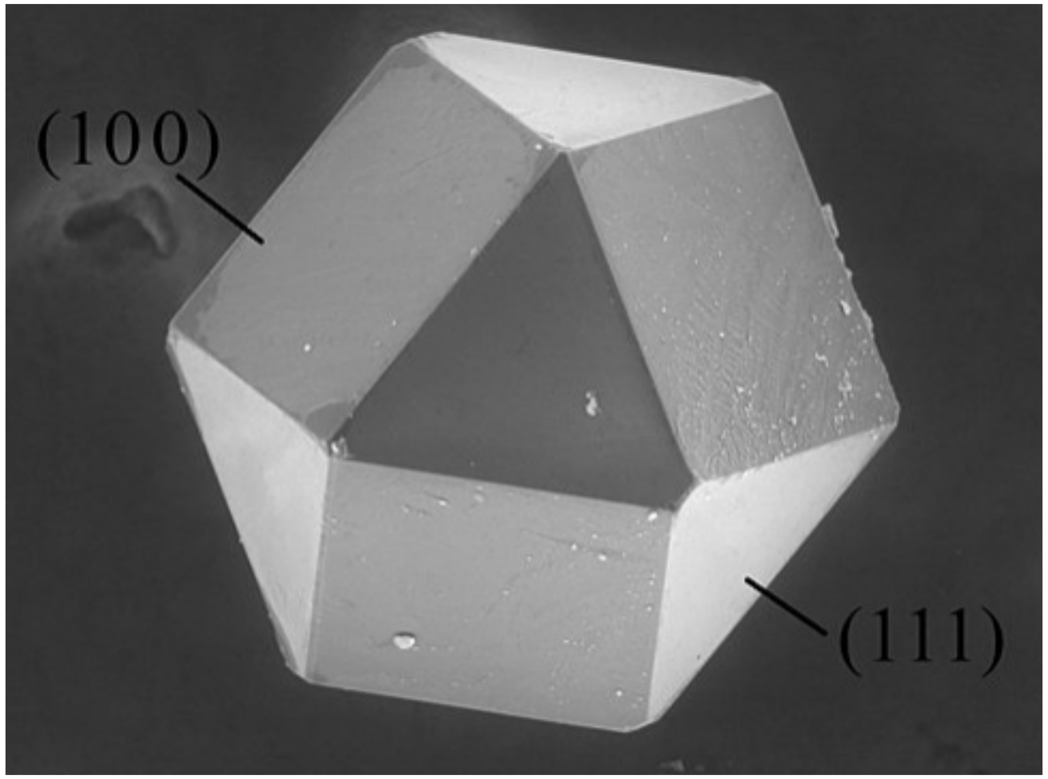
➜ Translation invariant system: eigenstate are extended Bloch waves.



$$\psi(\mathbf{r}) = \mathrm{e}^{\mathbf{ik}\cdot\mathbf{r}} u(\mathbf{r})$$

Introduction

- **Bloch's theorem**
  - → Translation invariant system: eigenstate are extended Bloch waves.
  - → Boundaries of the crystal Break the translation symmetry



Introduction

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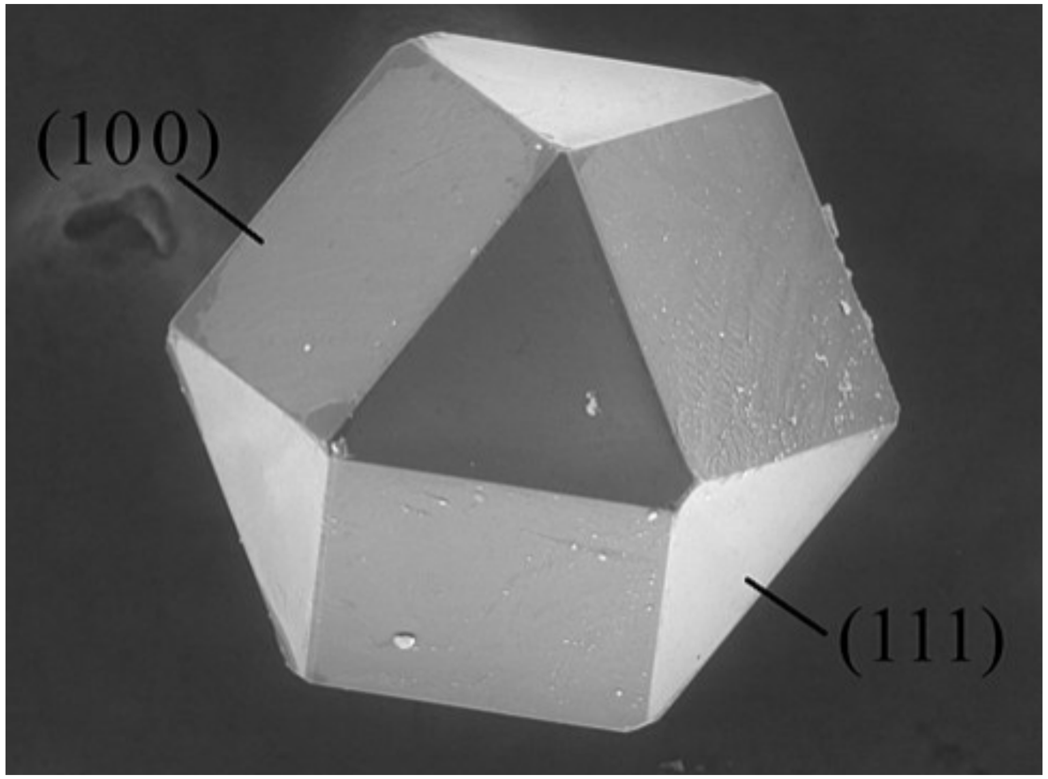

- Question: can we really use the solution of Bloch' theorem to understand the solution of OBC Hamiltonian?

Introduction

- **Thermodynamic limit**
  - → In Hermitian system, our text book tells us that the answer is yes.
  - → Thermodynamic limit argument.
  - → Large N

Introduction

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  - Large N $\longrightarrow$ Infinity N

Introduction

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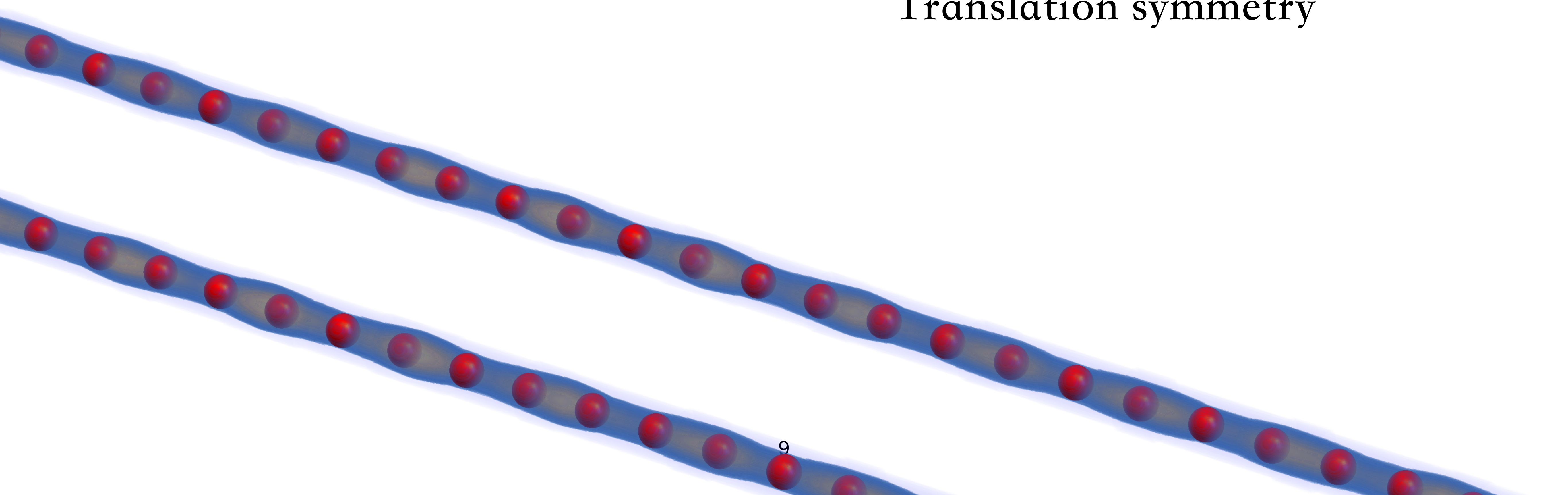
  - ➜ Large N ⟶ Infinity N ⟶ Infinity L

    ↓

    Translation symmetry



Introduction

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  - ➜ Large N ⟶ Infinity N ⟶ Infinity L ↓ Translation symmetry ⟶ k: good quantum #

Introduction

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Infinity L ↓ Translation symmetry

Translation symmetry ⟶ k: good quantum #

k: good quantum # ↓

Wave function bounded at infinity

$$|\psi(x \to \pm\infty)| < \infty$$

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Translation symmetry ⟶ k: good quantum #

k: good quantum # ↓ Wave function bounded at infinity

Wave function bounded at infinity ⟶ k is real

Introduction

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    Infinity L ⟶ Translation symmetry ⟶ k: good quantum #

    k: good quantum # ⟶ Wave function bounded at infinity ⟶ k is real ⟶ Bloch's theorem

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- **Thermodynamic limit**
  - ➜ In Hermitian system, our text book tells us that the answer is yes.
  - ➜ Thermodynamic limit argument.
  - ➜ Large N ⟶ Infinity N ⟶ Infinity L

    Infinity L ↓ Translation symmetry

    Translation symmetry ⟶ k: good quantum #

    k: good quantum # ↓ Wave function bounded at infinity

    Wave function bounded at infinity ⟶ k is real ⟶ Bloch's theorem
  - ➜ Note: no matter H is Hermitian or not.

Introduction

☐ NHSE in 1D

→ When the system has NHSE,

Yao and Wang's PRL

$$E_{\mathrm{OBC}} \neq E_{\mathrm{PBC}}, \qquad \psi_{\mathrm{OBC}}(x) \neq \psi_{\mathrm{PBC}}(x)$$

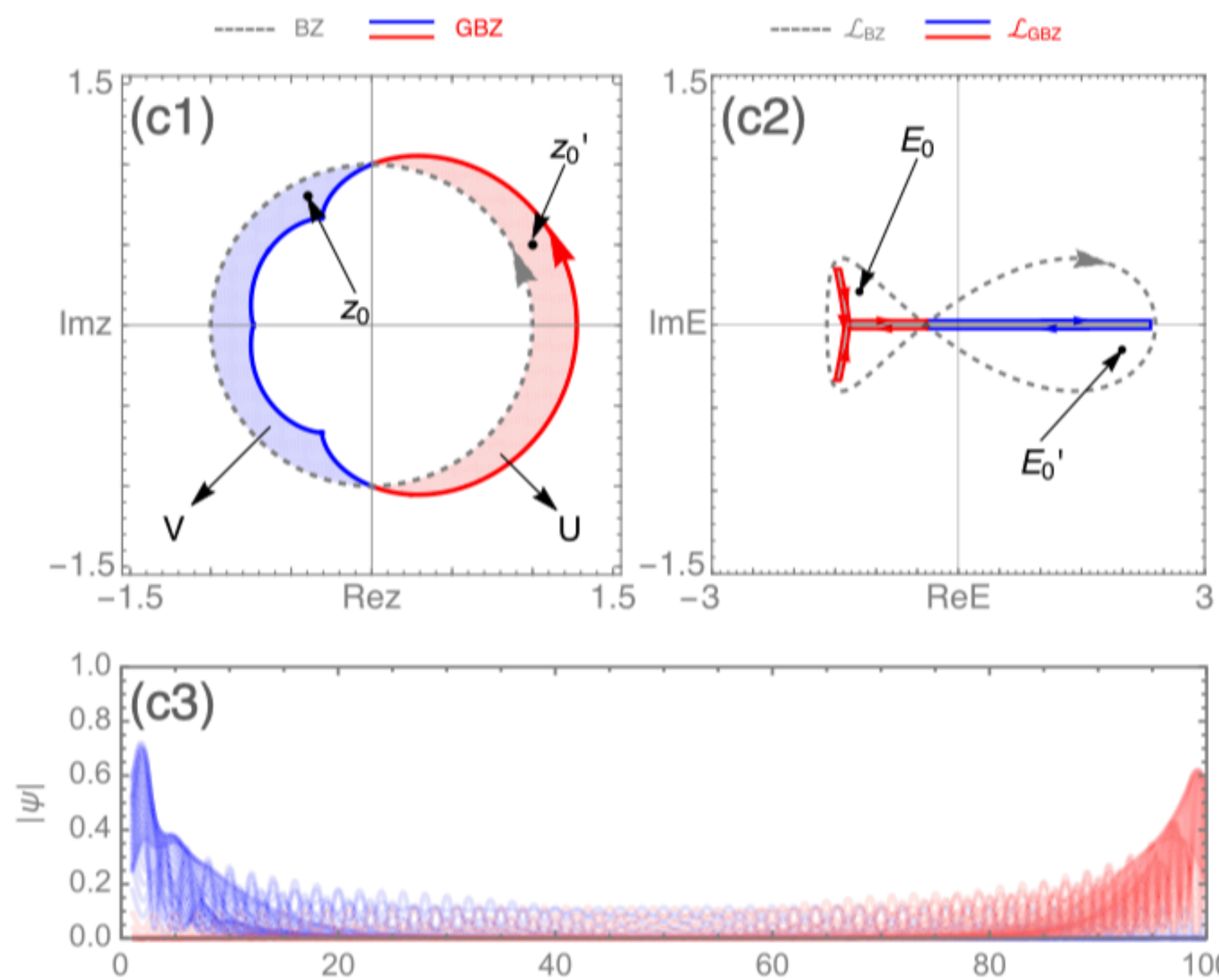


Figure from:
Zhang, ZY, and Fang,
PRL 125.126402 (2020)

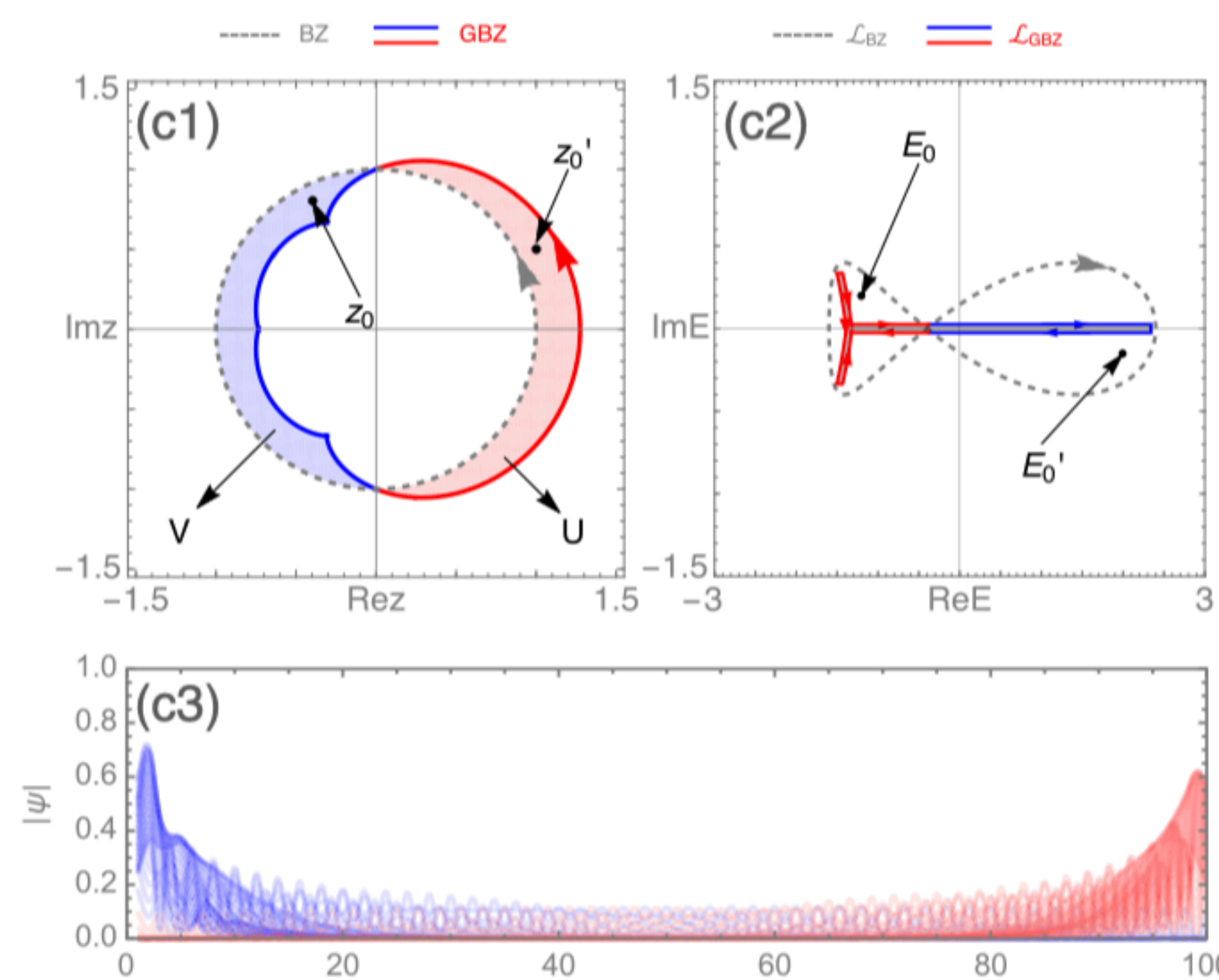
Introduction

☐ NHSE in 1D

→ When the system has NHSE, Yao and Wang's PRL

$$E_{\text{OBC}} \neq E_{\text{PBC}}, \quad \psi_{\text{OBC}}(x) \neq \psi_{\text{PBC}}(x)$$

→ Since the thermodynamic limit gives the solution in the N tending to infinity limit



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Introduction

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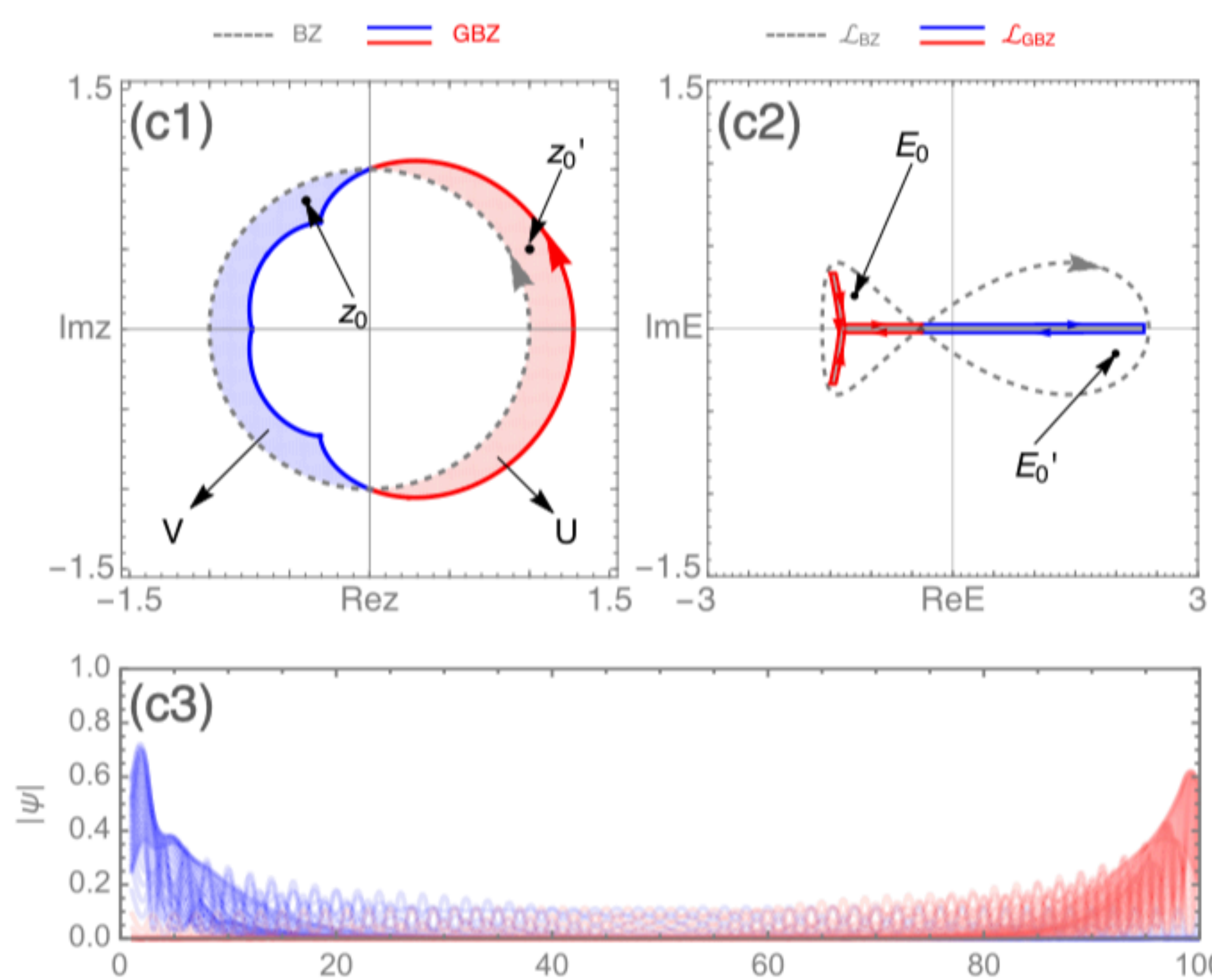
$$E_{\text{OBC}} \neq E_{\text{PBC}}, \qquad \psi_{\text{OBC}}(x) \neq \psi_{\text{PBC}}(x)$$

  - → Since the thermodynamic limit gives the solution in the N tending to infinity limit
- Question: Become identical as N increases?



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Introduction

☐ NHSE in 1D

→ When the system has NHSE,

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$$E_{\mathrm{OBC}} \neq E_{\mathrm{PBC}}, \quad \psi_{\mathrm{OBC}}(x) \neq \psi_{\mathrm{PBC}}(x)$$

→ Since the thermodynamic limit gives the solution in the N tending to infinity limit

☐ Question: Become identical as N increases?

☐ No

→ Hatano-Nelson model

$$E = t_0 + 2\sqrt{t_1 t_{-1}} \cos k,$$

$$k = \frac{\pi m}{N+1}, \quad m = 1, \ldots, N$$

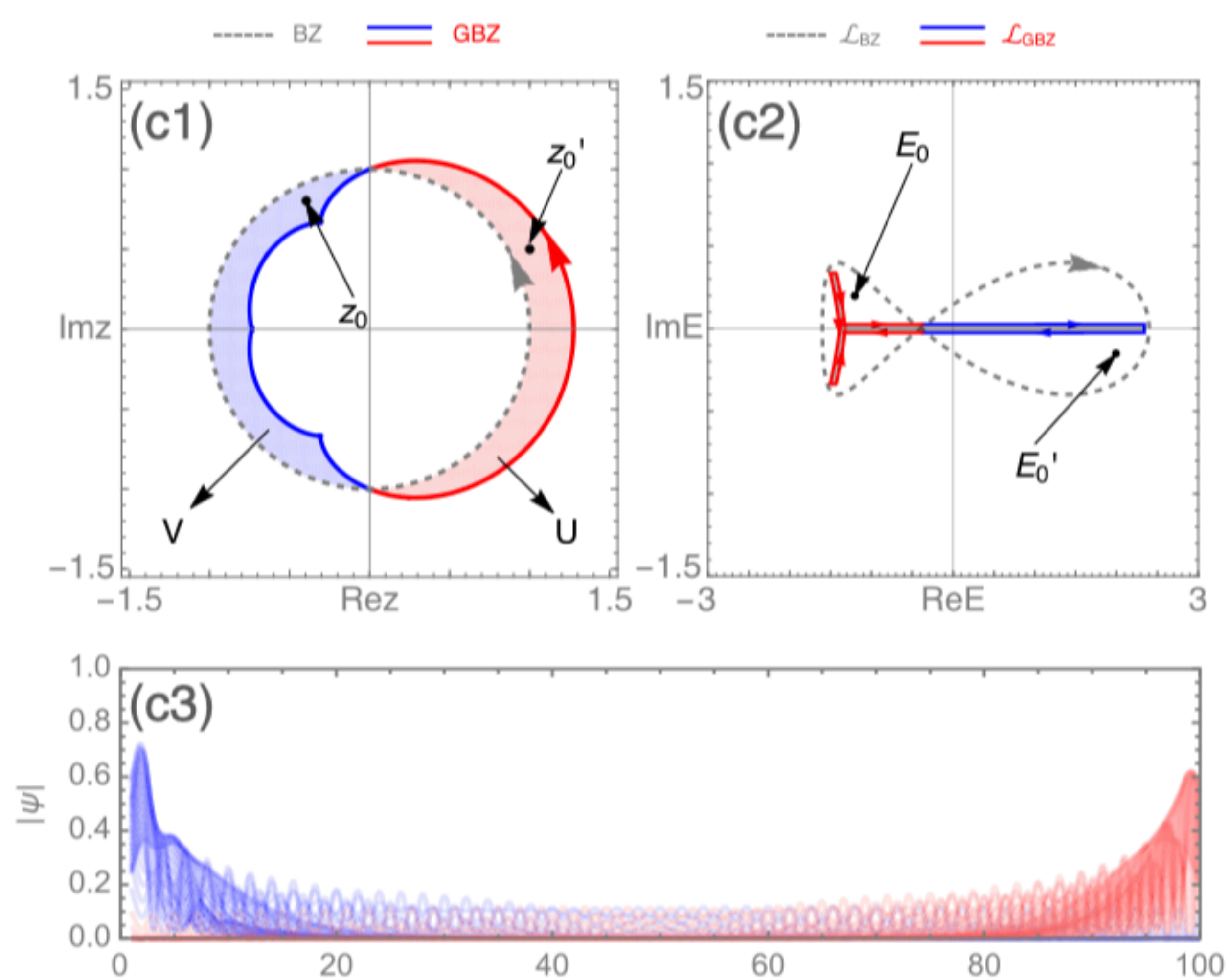


Figure from:
Zhang, ZY, and Fang, PRL 125.126402 (2020)

Introduction

- **NHSE in 1D**

What is the limit solution of the OBC Hamiltonian?

Introduction

- NHSE in 1D

What is the limit solution of the OBC Hamiltonian?

↓

Generalized Brillouin zone (GBZ) theory

Yao and Wang's PRL

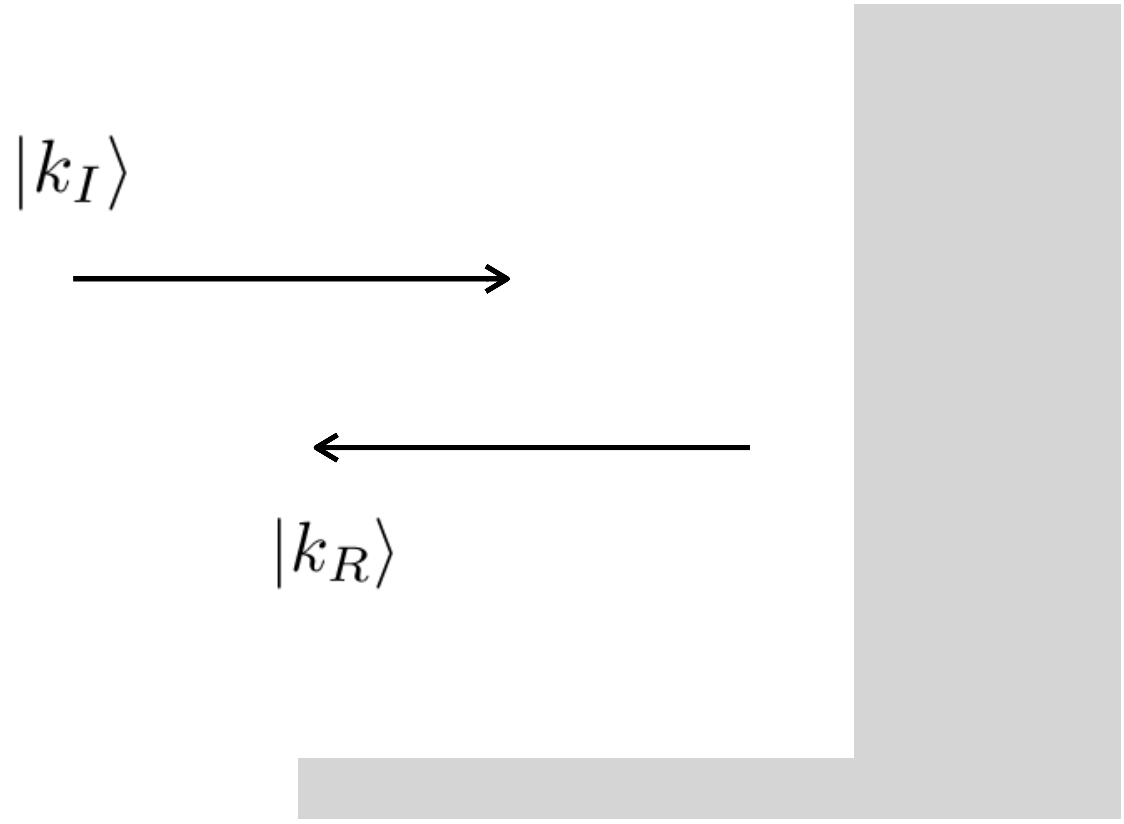
Introduction

- **Why the thermodynamic limit fails?**

Introduction

- **Why the thermodynamic limit fails?**
  - ➜ Answer: the boundary condition is changed in this limit

$$E(k_I) = E(k_R)$$



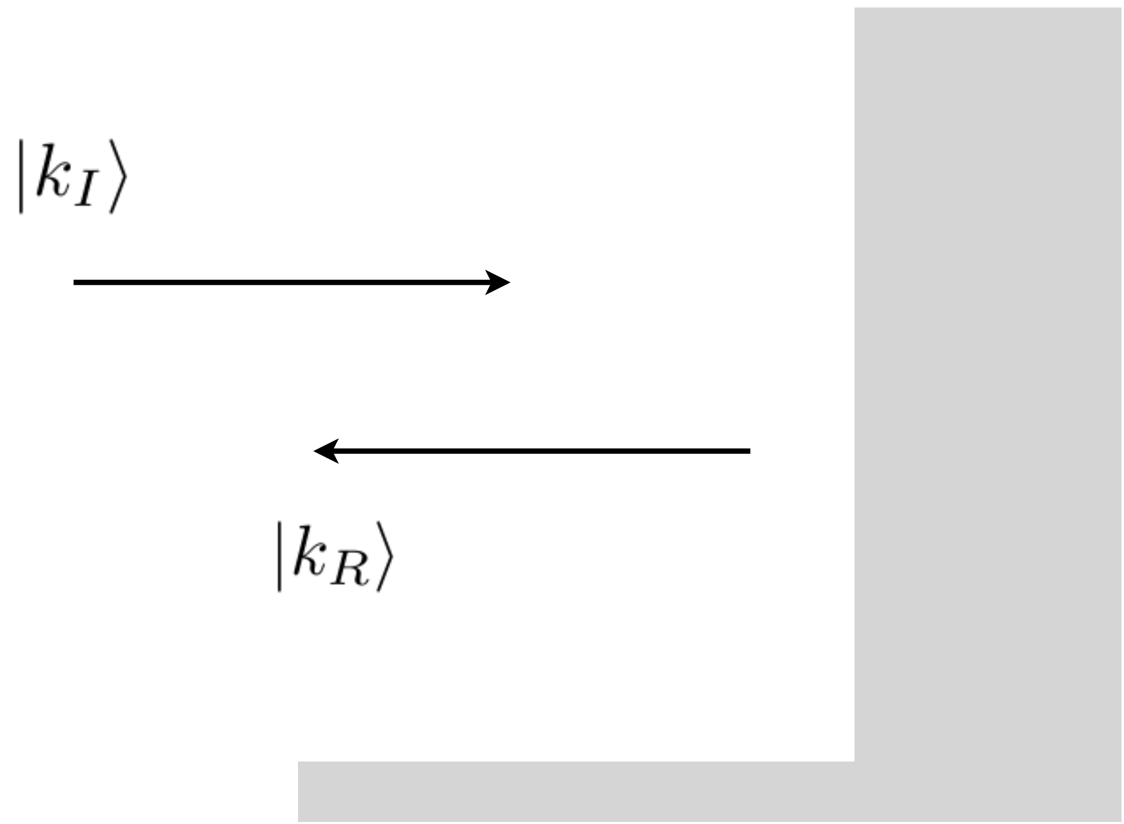
Introduction

- **Why the thermodynamic limit fails?**

  ➜ Answer: the boundary condition is changed in this limit

$$E(k_I) = E(k_R)$$

$$|\psi_{\rm OBC}\rangle = c_1|k_I\rangle + c_2|k_R\rangle + ...$$



  ➜ Standing wave solution

Introduction

- **Why the thermodynamic limit fails?**

  → Push the boundary to infinity

$|k_I\rangle$

$|k_R\rangle$ disappear

Introduction

- **Why the thermodynamic limit fails?**

  ➜ Push the boundary to infinity

$|k_I\rangle$

$|k_R\rangle$ disappear

$$\lim_{N,L\to\infty} |\psi_{\text{OBC}}\rangle = |k_I\rangle$$

↓

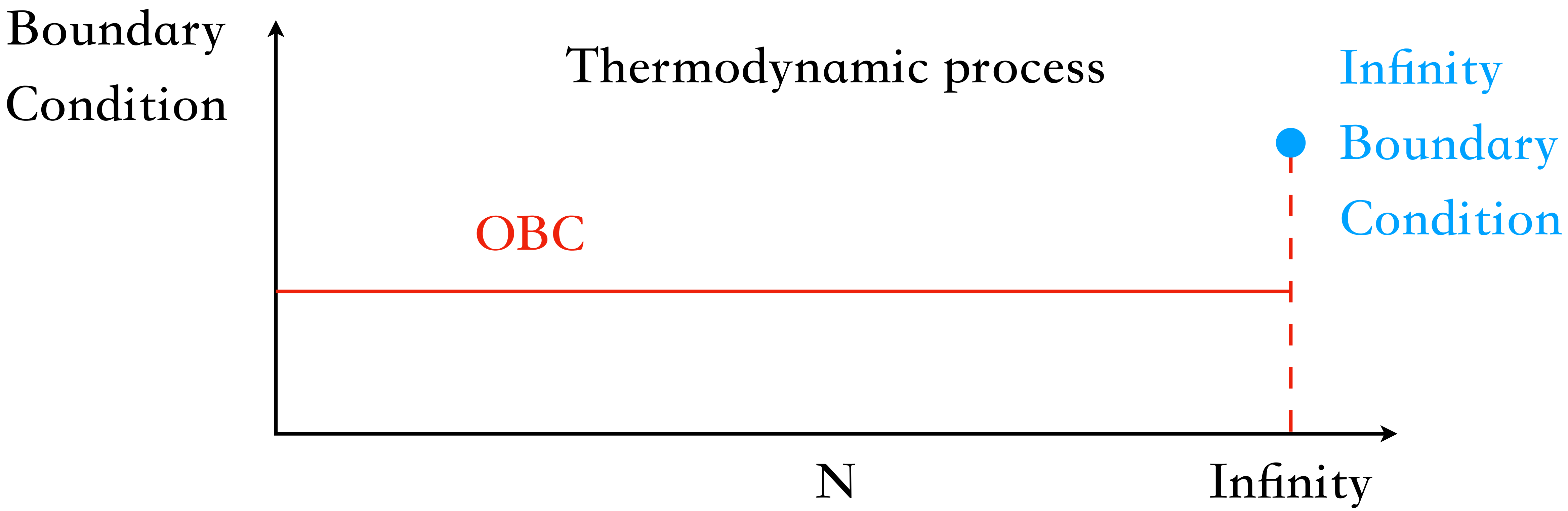
Preserve translation symmetry

➜ Traveling wave solution

Introduction

- **Why the thermodynamic limit fails?**

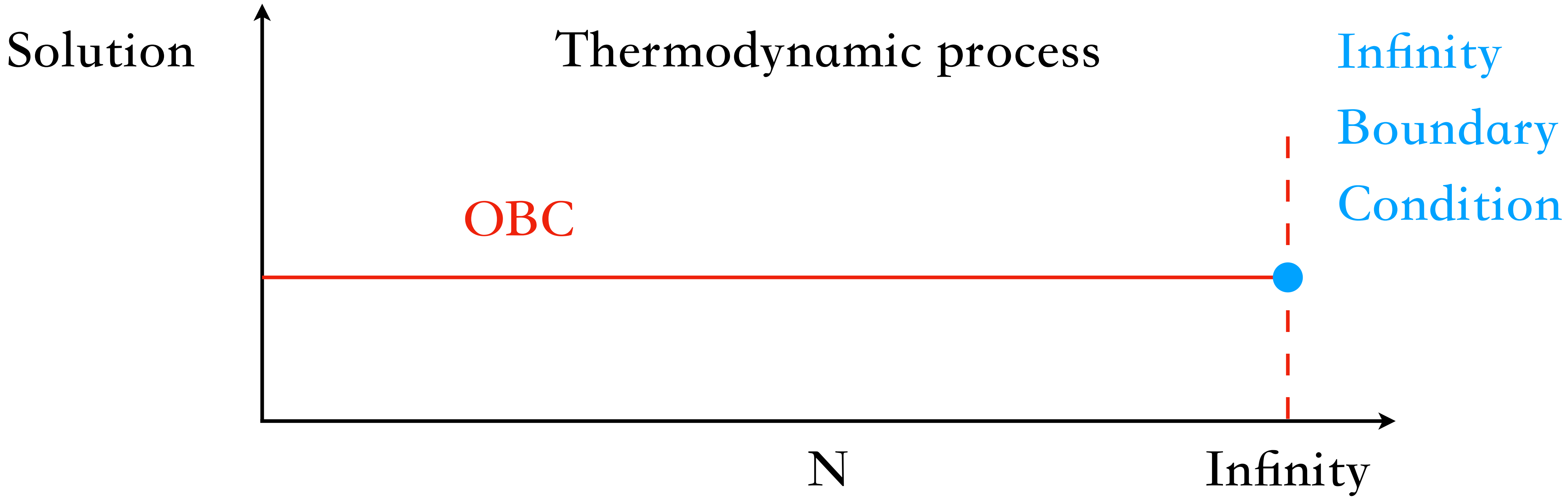
  → Discontinue



Introduction

- **Why the thermodynamic limit fails?**

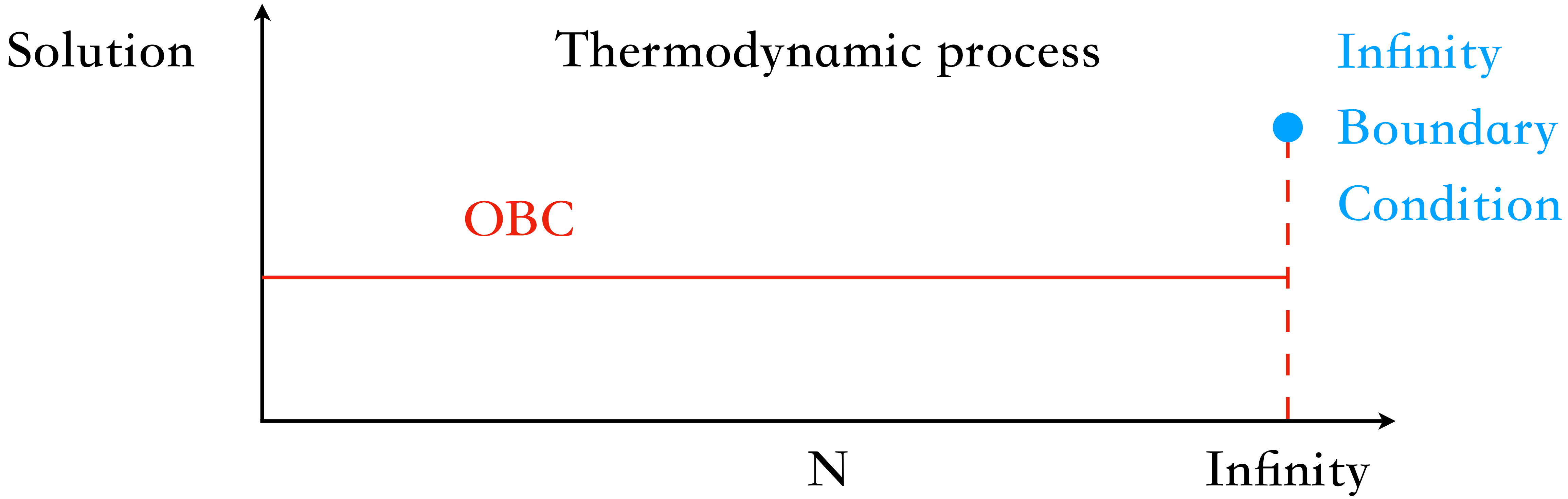
  → No NHSE



Introduction

- **Why the thermodynamic limit fails?**

  → With NHSE



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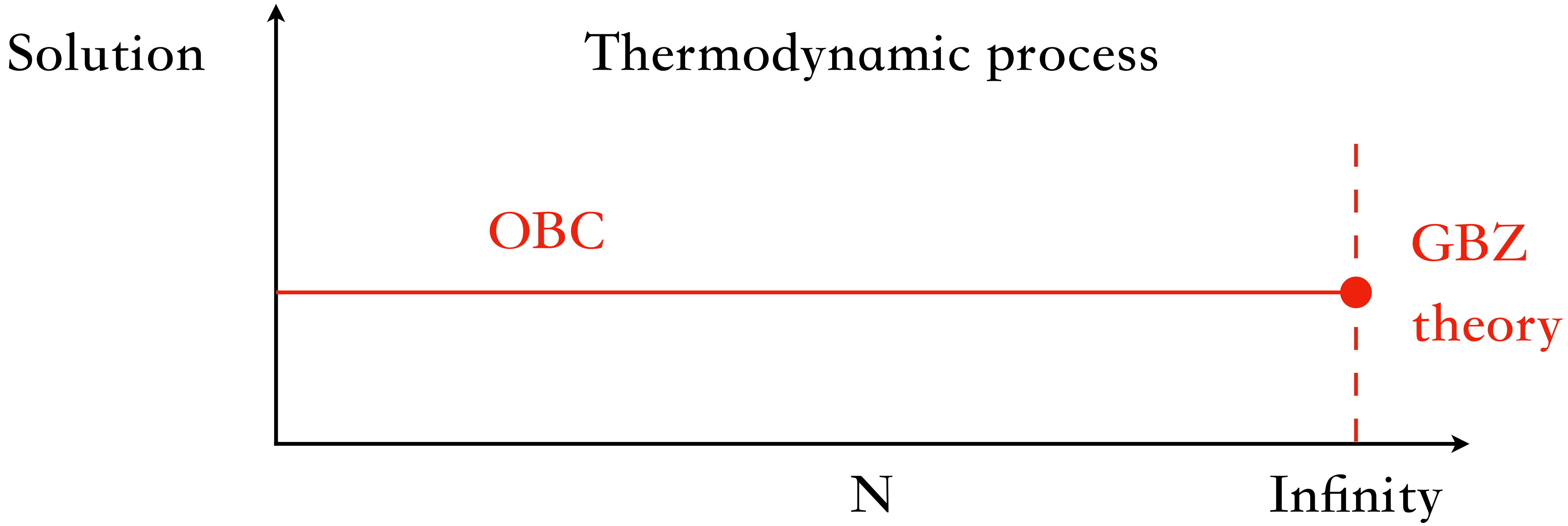
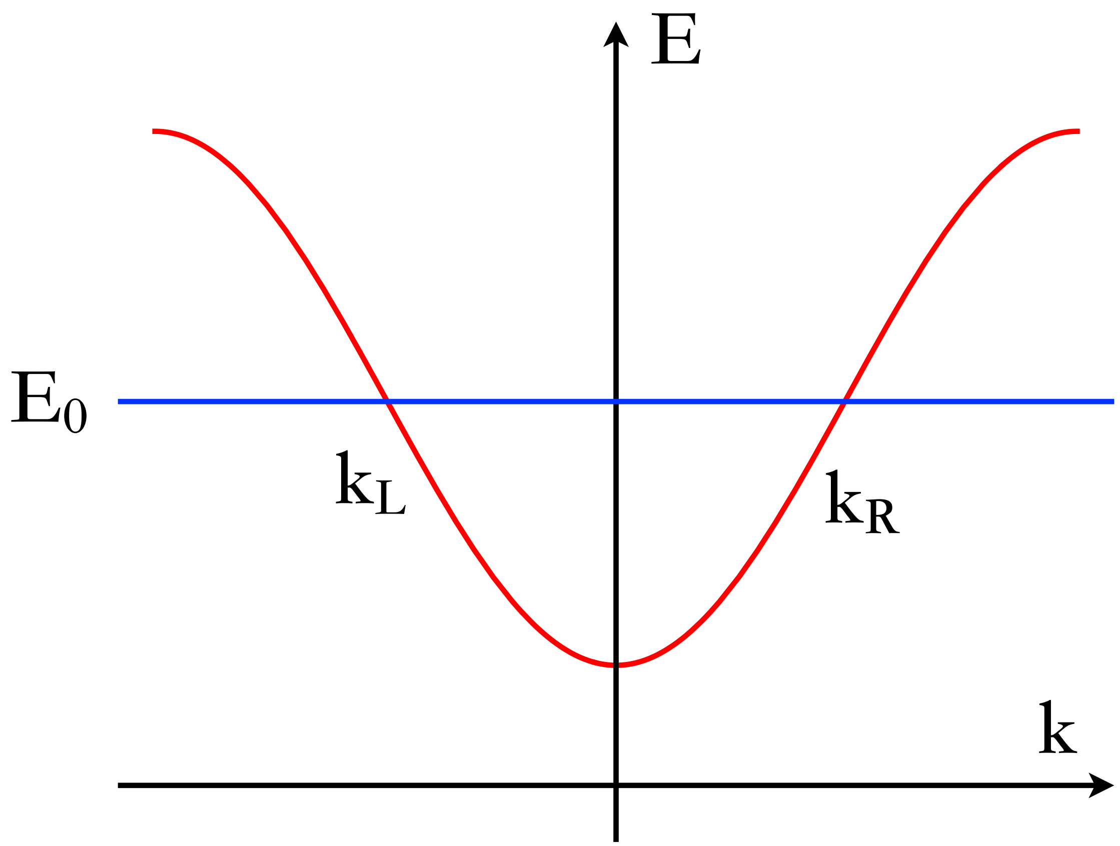
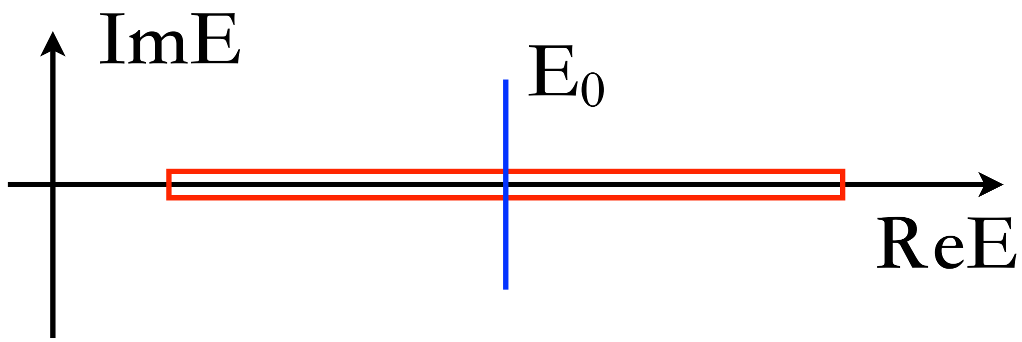


Introduction

- **Dynamical degeneracy splitting**
  - Why the BC is important?
    - Hermitian case
      - Reflection channels
      - Fermi doubling theorem





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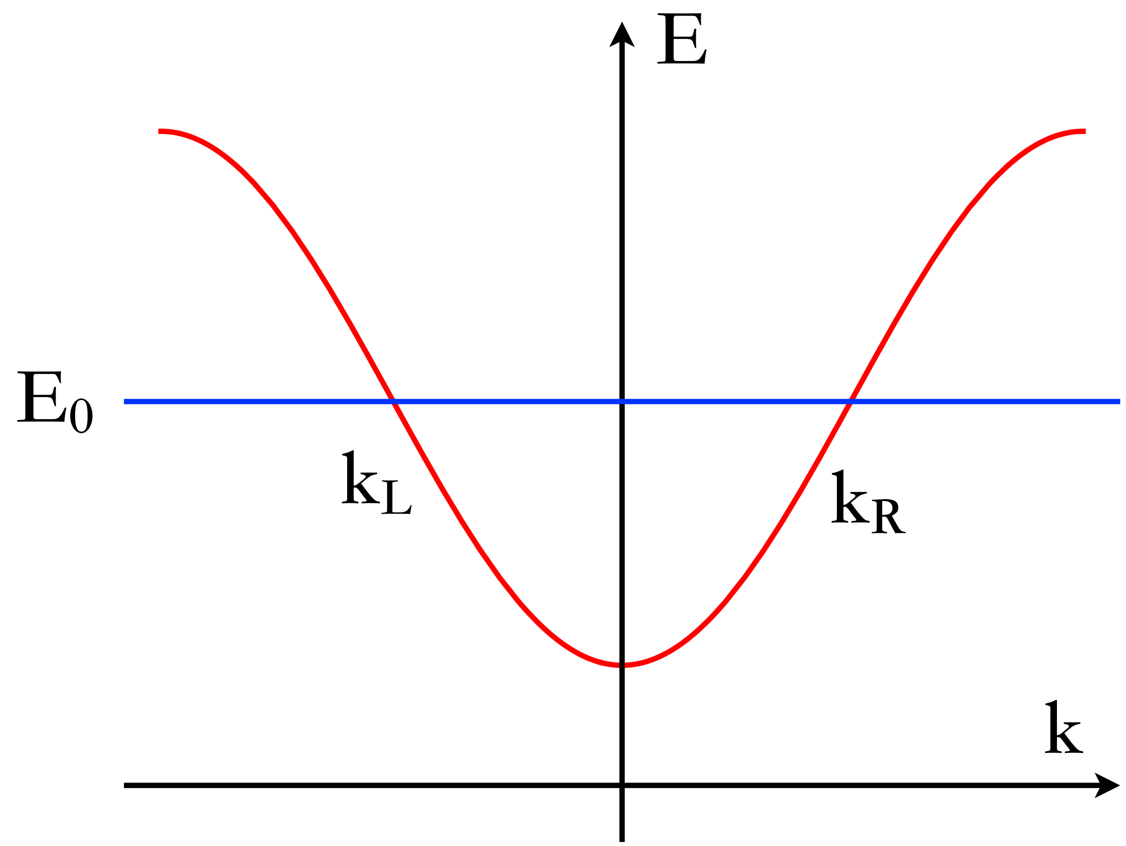
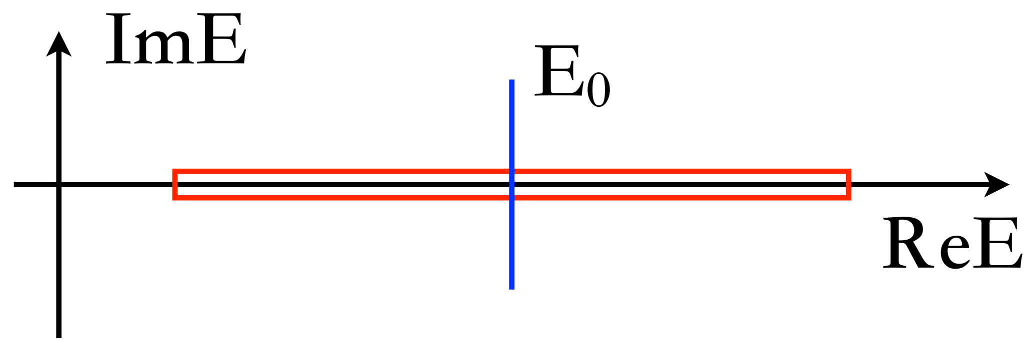
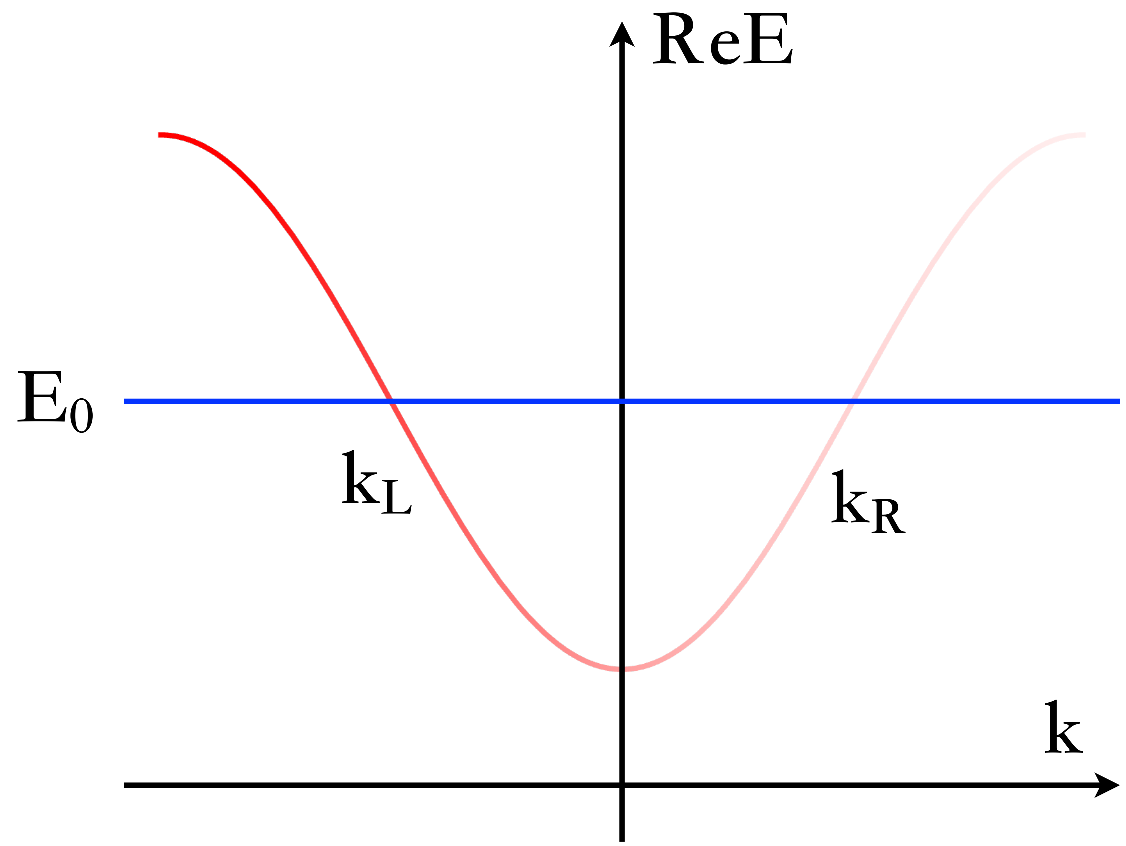
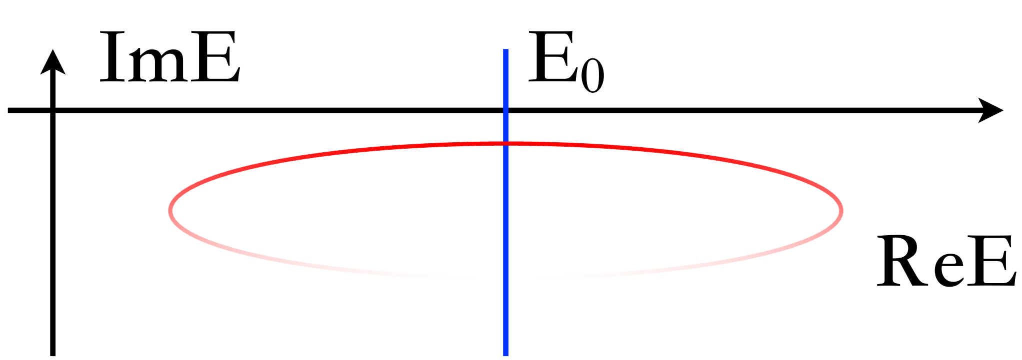




Introduction

- **Dynamical degeneracy splitting**
  - → Why the BC is important?
    - → Non-Hermitian case
      - → No longer reflection channels
      - → Dynamical version of chiral anomaly





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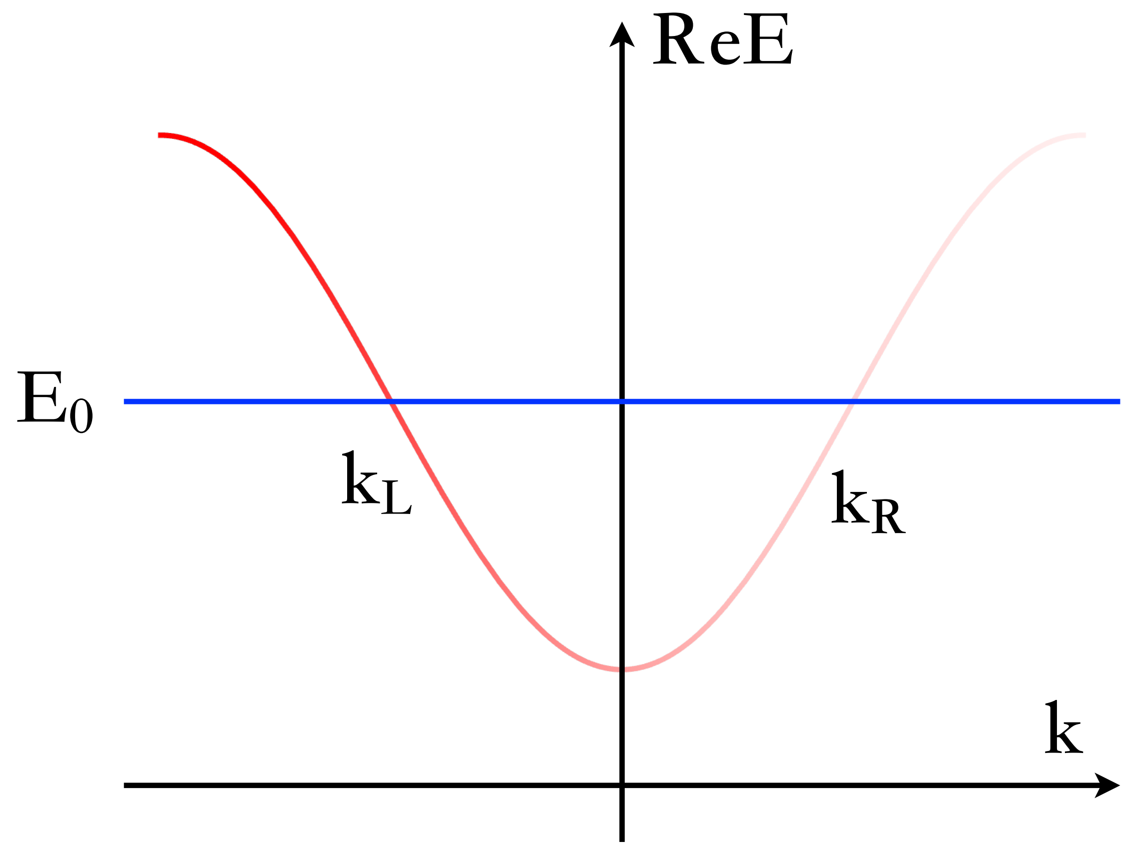
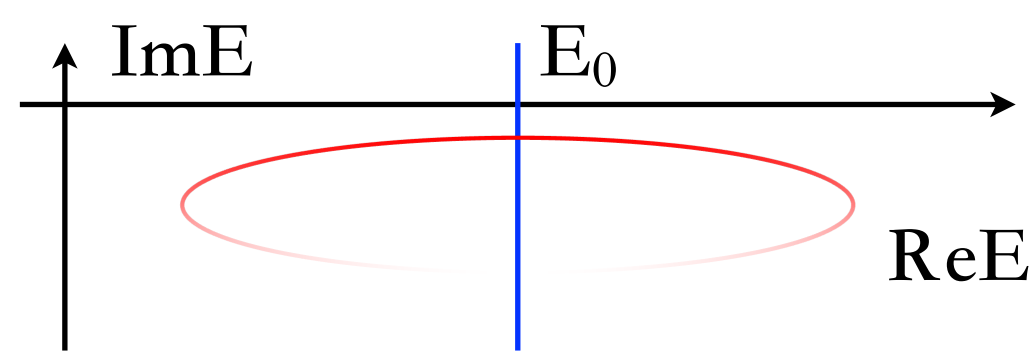


- **Emergence of NHSE**
  - → Band criteria of NHSE

    Zhang, Fang and ZY, PRL 131.036402 (2023)



Introduction

- **NHSE in 1D**
  - Symmetry and onsite dissipation:

    Yi and ZY, PRL 125.186802 (2020)

$$\mathcal{H}_{\mathrm{RM}}(k) = (t_1 + t_2 \cos k)\sigma_x + t_2 \sin k\sigma_y + \mu\sigma_z + i\gamma\sigma_z,$$

  - No NHSE

Introduction

☐ NHSE in 1D

➔ Symmetry and onsite dissipation:

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$$\mathcal{H}_{\mathrm{RM}}(k) = (t_1 + t_2 \cos k)\sigma_x + t_2 \sin k\sigma_y + \mu\sigma_z + i\gamma\sigma_z,$$

➔ No NHSE

➔ Two way to NHSE: Break TRS

$$\lambda \sin k\sigma_z$$

Add SOC

$$\lambda_I \sin k\sigma_z s_z - \lambda_R \sigma_y (s_x - \sqrt{3} s_y)/2,$$

Outline

- Introduction
- 1D GBZ theory: review
- 2D GBZ theory: recent developments
- 2D NHSE: numerical summary
- 2D GBZ theory: wave function approach

1D GBZ theory: review

- **GBZ condition**
  - → Question: How to calculate the OBC solution?

1D GBZ theory: review

GBZ condition

→ Question: How to calculate the OBC solution?

→ Three steps:

1. Find all the bulk solutions

$$\det[E_0 - H(\beta)] = 0$$

$$|\beta_1| \le \dots \le |\beta_p| \le |\beta_{p+1}| \le \dots \le |\beta_{p+s}|$$ p: order of the pole

2. Take linear superposition

$$|\psi(E_0)\rangle = \sum_{i=1}^{p+s} c_i |\beta_i\rangle$$

3. Determine the solution via boundary conditions

1D GBZ theory: review

☐ GBZ condition

Non-Bloch Band Theory of Non-Hermitian Systems

Kazuki Yokomizo and Shuichi Murakami
Phys. Rev. Lett. **123**, 066404 – Published 7 August 2019

➜ OBC eigenvalue

$$|\beta_p(E_0)| = |\beta_{p+1}(E_0)|$$

➜ No OBC eigenvalue

$$|\beta_p(E_0)| \neq |\beta_{p+1}(E_0)|$$

1D GBZ theory: review

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- Note

→ No spinful anomalous time reversal symmetry

Yi and ZY, PRL 125.186802 (2020)
Kawabata et. al. PRB 101, 195147 (2020)

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- **Note**

→ No spinful anomalous time reversal symmetry

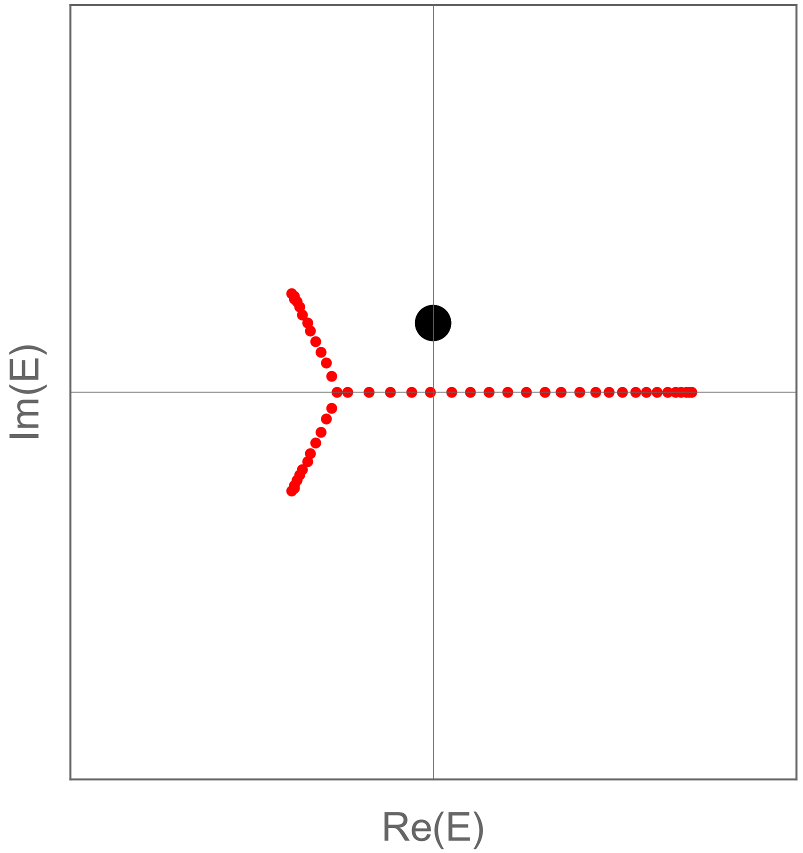
→ GBZ spectrum, not including topological boundary states

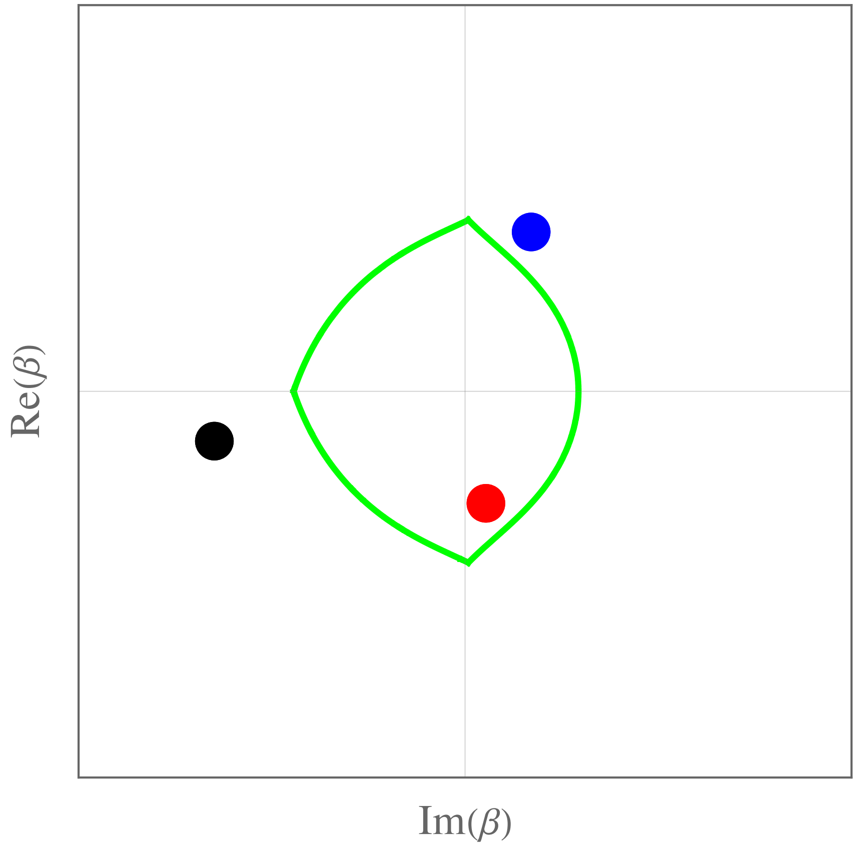
1D GBZ theory: review

- **Zero winding number condition**

  → Geometry interpretation

$$\frac{1}{\beta} + 2\,\beta + 3\,\beta^2$$





1D GBZ theory: review

- **Zero winding number condition**

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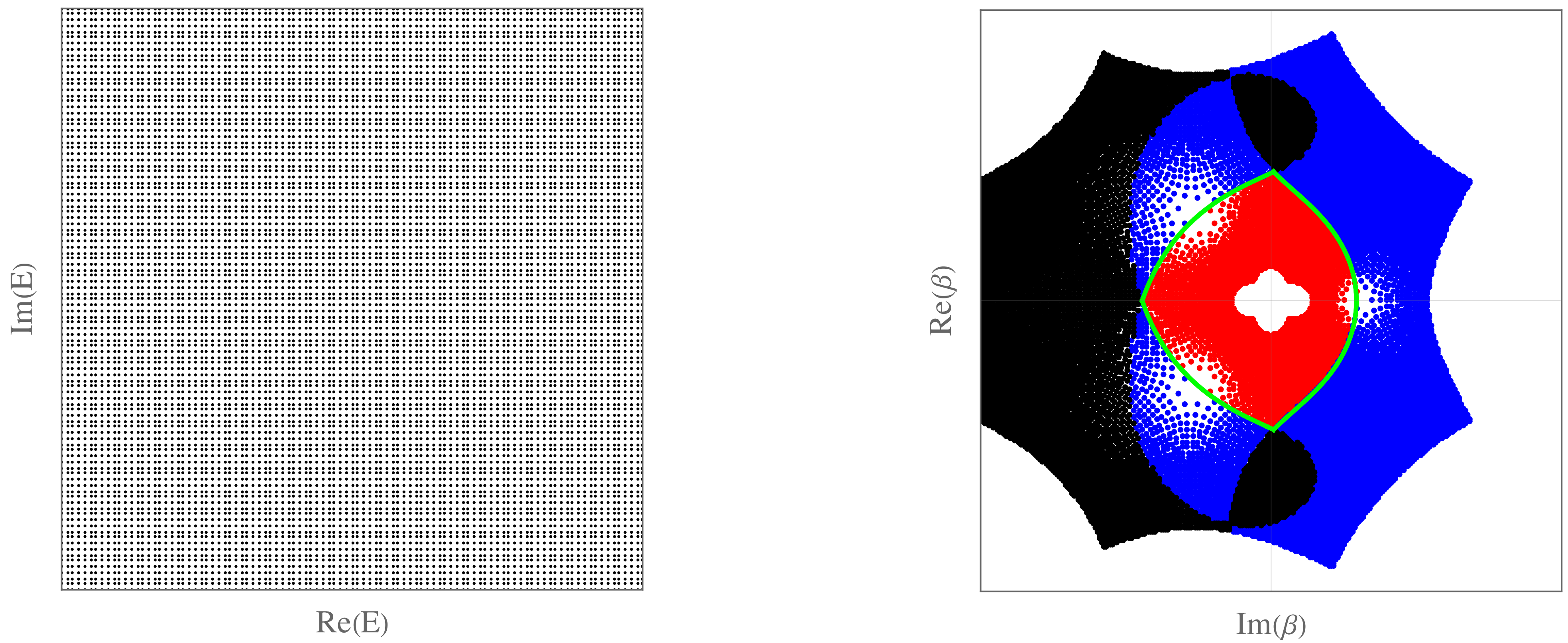


→ GBZ encloses all the first roots (since p=1 in this model).

1D GBZ theory: review

- **Zero winding number condition**

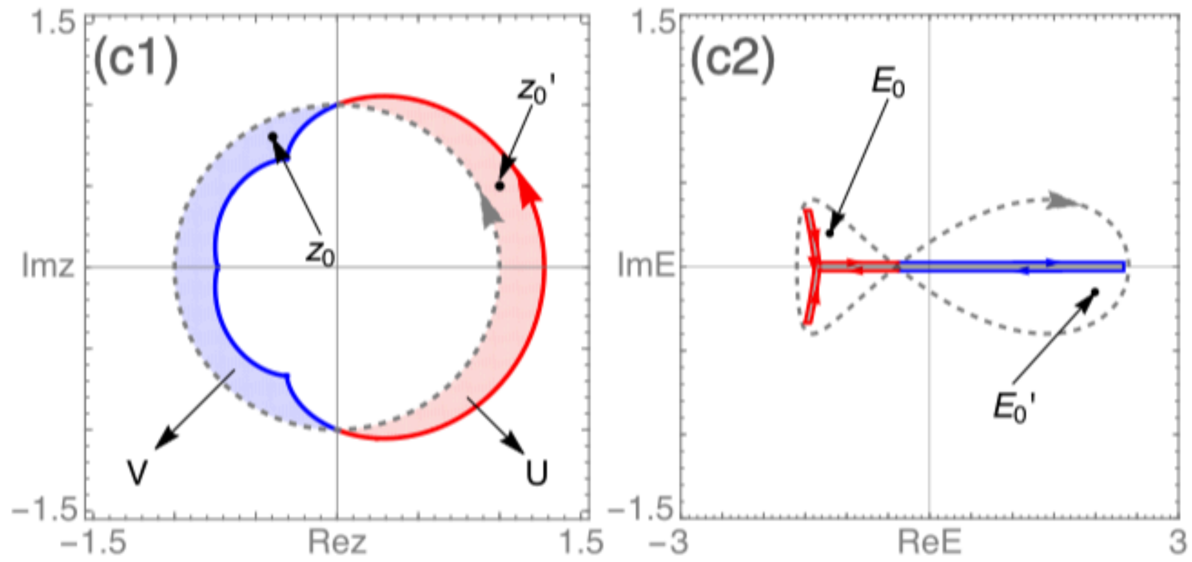
  → Geometry interpretation

  Zero spectral winding number of the OBC spectrum

$$w_{\mathcal{C},E_b} := \frac{1}{2\pi}\oint_{\mathcal{C}} \frac{d}{dz}\arg[H(z)-E_b]dz.$$

$$= N_{\text{zeros}} - N_{\text{poles}},$$

→ $\forall E_b \in \mathbb{C}$

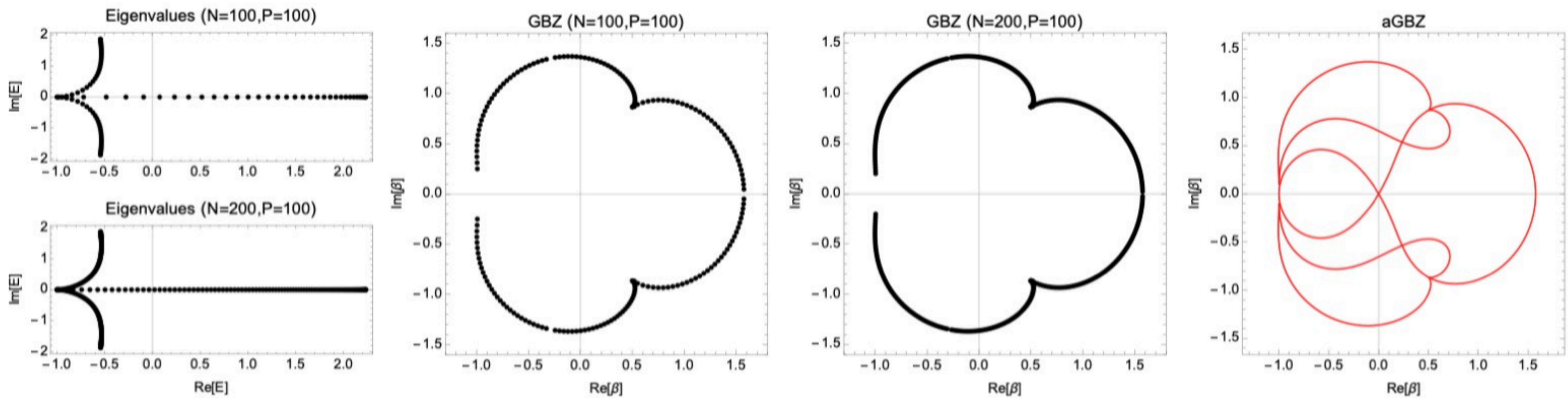


Zhang, ZY, and Fang PRL 125.126402 (2020)
Okuma's PRL

1D GBZ theory: review

- 1D GBZ calculation: numerical method

  → Finite size effect

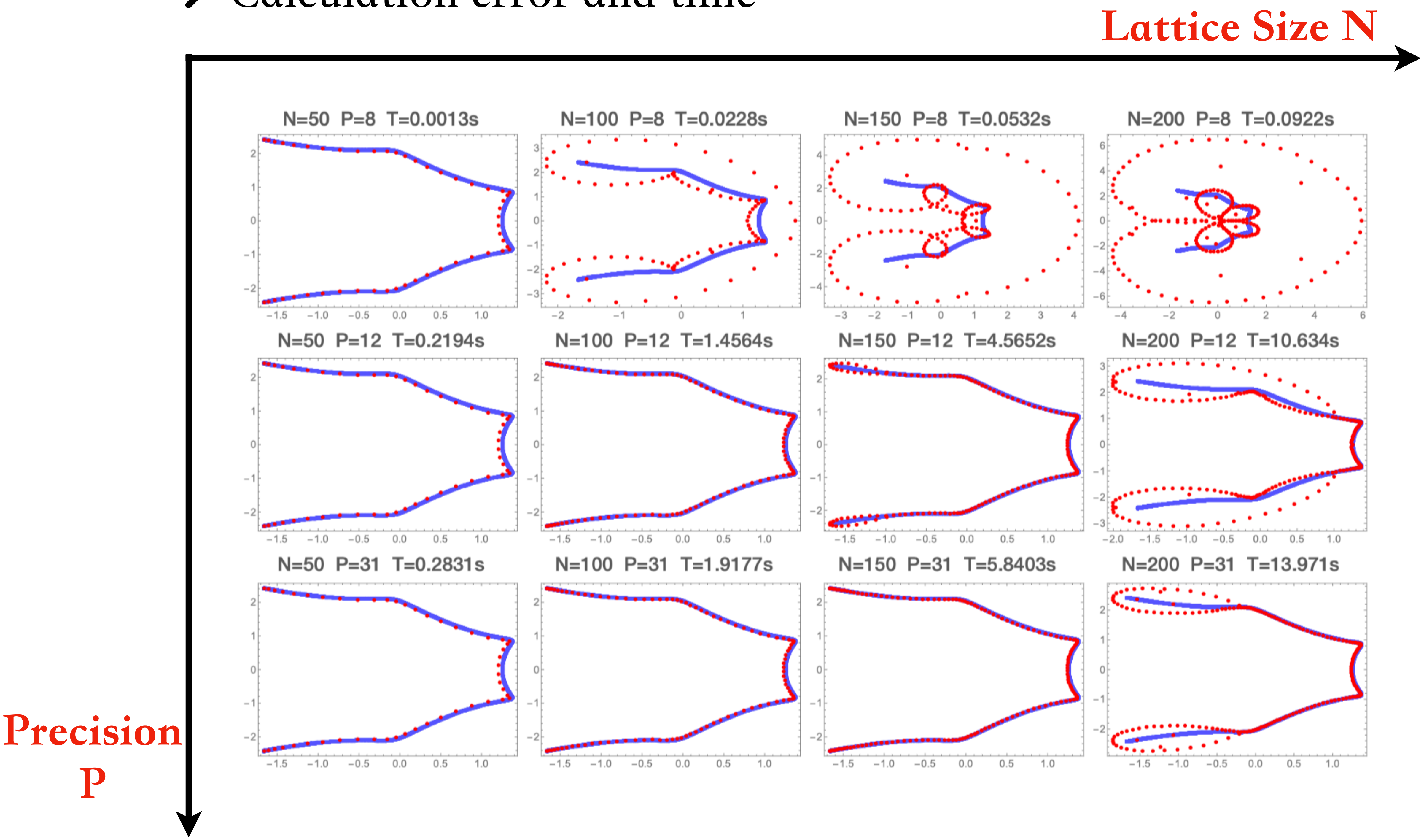


$$\mathcal{H}(\beta) = \beta + 1/\beta^2 + 1/\beta^3.$$

1D GBZ theory: review

☐ 1D GBZ calculation: numerical method

→ Calculation error and time



1D GBZ theory: review

☐ 1D GBZ calculation: analytic method

➜ Relax the GBZ condition

$$|\beta_p| = |\beta_{p+1}| \to |\beta_i| = |\beta_j|,\ 1 \le i, j \le p+s$$

$$\downarrow$$

$$f(E,\beta) = f(E,\beta e^{i\theta}) = 0,\ \theta \in \mathbb{R}$$

ZY, Zhang, Fang, Hu PRL 125.226402 (2020)

1D GBZ theory: review

☐ 1D GBZ calculation: analytic method

➜ Relax the GBZ condition

$$|\beta_p| = |\beta_{p+1}| \to |\beta_i| = |\beta_j|,\ 1 \le i, j \le p+s$$

$$\downarrow$$

$$f(E, \beta) = f(E, \beta e^{i\theta}) = 0,\ \theta \in \mathbb{R}$$

➜ Eliminate variables, E and theta

$$F_{\mathrm{aGBZ}}(\mathrm{Re}\beta, \mathrm{Im}\beta) = \sum_{i,j} c_{ij} (\mathrm{Re}\beta)^i (\mathrm{Im}\beta)^j = 0.$$

Auxiliary GBZ equation

ZY, Zhang, Fang, Hu PRL 125.226402 (2020)

1D GBZ theory: review

☐ 1D GBZ calculation: analytic method

→ Example $\frac{1}{\beta} + 2\beta + 3\beta^2$

$$\left(x^4 - 2x^6 - 6x^7 + 2x^2y^2 - 6x^4y^2 - 18x^5y^2 + y^4 - 6x^2y^4 - 18x^3y^4 - 2y^6 - 6xy^6\right)$$
$$\left(576x^8 + 1152x^{10} + 3456x^{11} + 6912x^{13} + 5184x^{14} + 10\,368x^{16} + 2304x^6y^2 + 5760x^8y^2 + 17\,280x^9y^2 + 41\,472x^{11}y^2 + 15\,552x^{12}y^2 + 82\,944x^{14}y^2 + 3456x^4y^4 + 11\,520x^6y^4 + 34\,560x^7y^4 + 103\,680x^9y^4 - 15\,552x^{10}y^4 + 290\,304x^{12}y^4 + 2304x^2y^6 + 11\,520x^4y^6 + 34\,560x^5y^6 + 138\,240x^7y^6 - 129\,600x^8y^6 + 580\,608x^{10}y^6 + 576y^8 + 5760x^2y^8 + 17\,280x^3y^8 + 103\,680x^5y^8 - 233\,280x^6y^8 + 725\,760x^8y^8 + 1152y^{10} + 3456xy^{10} + 41\,472x^3y^{10} - 202\,176x^4y^{10} + 580\,608x^6y^{10} + 6912xy^{12} - 88\,128x^2y^{12} + 290\,304x^4y^{12} - 15\,552y^{14} + 82\,944x^2y^{14} + 10\,368y^{16}\right)$$

Auxiliary GBZ equation

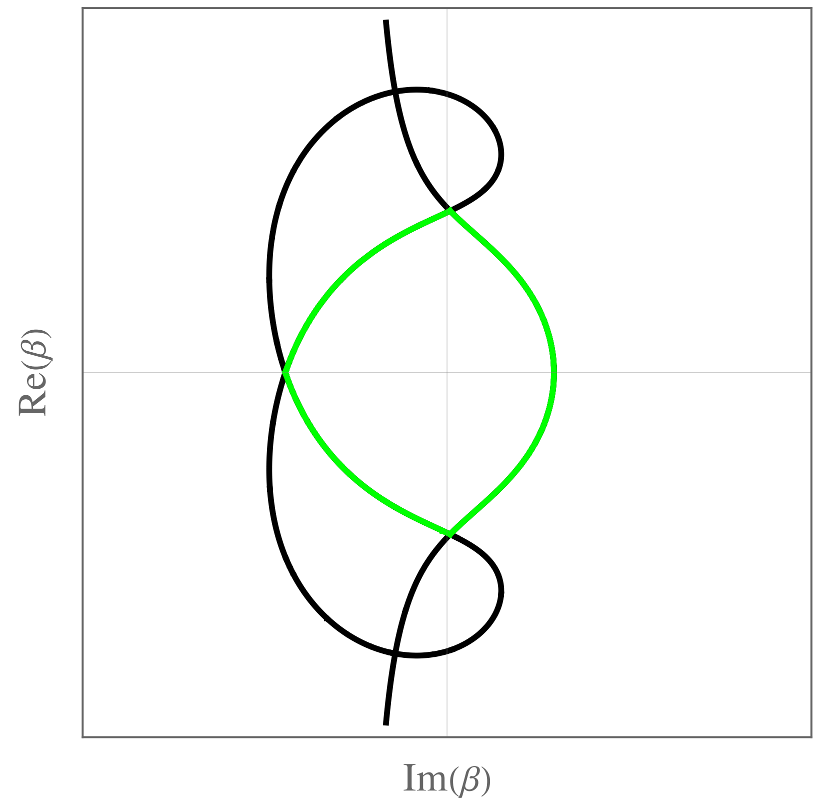
ZY, Zhang, Fang, Hu PRL 125.226402 (2020)

1D GBZ theory: review

☐ 1D GBZ calculation: analytic method

→ Example $\frac{1}{\beta} + 2\beta + 3\beta^2$

$$\begin{aligned}
&(x^4 - 2x^6 - 6x^7 + 2x^2y^2 - 6x^4y^2 - 18x^5y^2 + y^4 - 6x^2y^4 - 1 \\
&\quad(576x^8 + 1152x^{10} + 3456x^{11} + 6912x^{13} + 5184x^{14} + 10\,368 \\
&\quad\quad 5760x^8y^2 + 17\,280x^9y^2 + 41\,472x^{11}y^2 + 15\,552x^{12}y^2 + \\
&\quad\quad 11\,520x^6y^4 + 34\,560x^7y^4 + 103\,680x^9y^4 - 15\,552x^{10}y^4 \\
&\quad\quad 2304x^2y^6 + 11\,520x^4y^6 + 34\,560x^5y^6 + 138\,240x^7y^6 - 1 \\
&\quad\quad 580\,608x^{10}y^6 + 576y^8 + 5760x^2y^8 + 17\,280x^3y^8 + 103\,6 \\
&\quad\quad 725\,760x^8y^8 + 1152y^{10} + 3456xy^{10} + 41\,472x^3y^{10} - 202 \\
&\quad\quad 6912xy^{12} - 88\,128x^2y^{12} + 290\,304x^4y^{12} - 15\,552y^{14} + 8
\end{aligned}$$



ZY, Zhang, Fang, Hu PRL 125.226402 (2020)

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2D GBZ theory: numerical summary

- **The role of OBC geometry**
  - → Infinity types of OBC geometry
    - → 1D OBC

2D GBZ theory: numerical summary

- **The role of OBC geometry**
  - → Infinity types of OBC geometry
    - → 1D OBC
    - → 2D OBC

…

2D GBZ theory: numerical summary

- **The role of OBC geometry** Zhang, ZY, and Fang, Hu NC 13, 2496 (2022).

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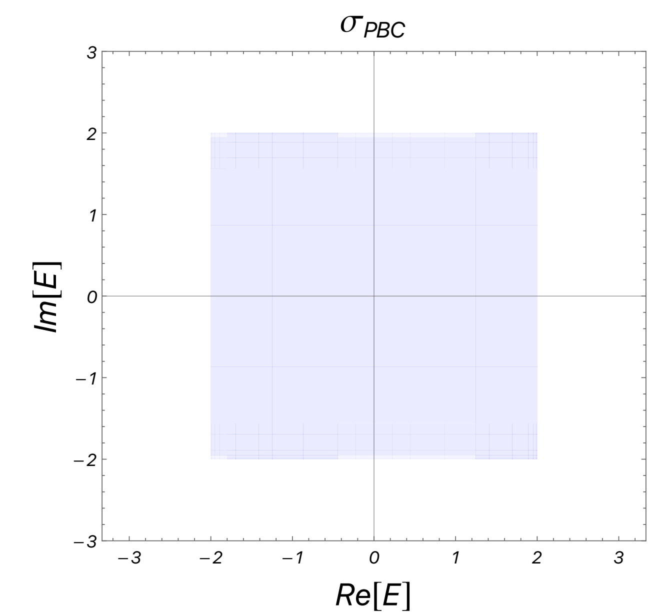

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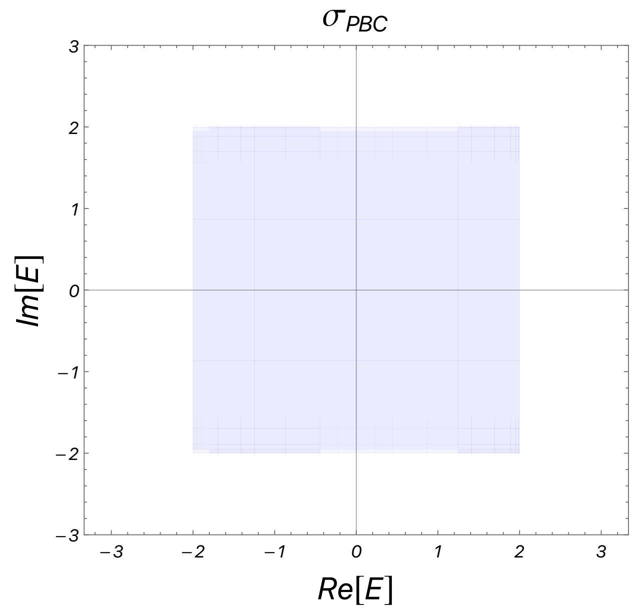
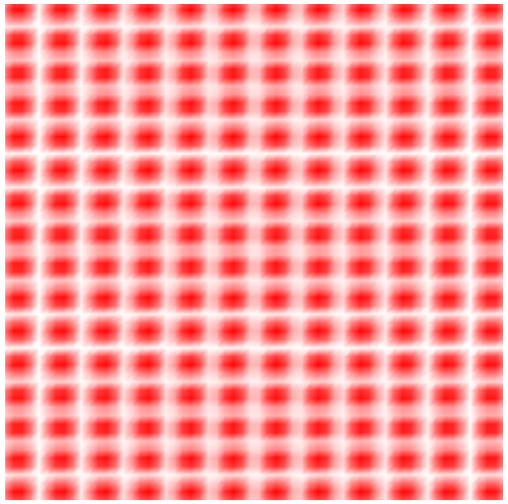
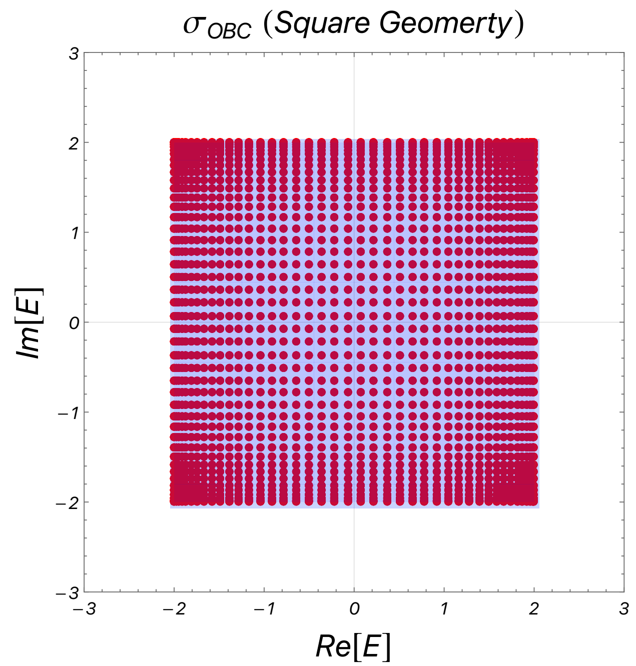


OBC eigenstate





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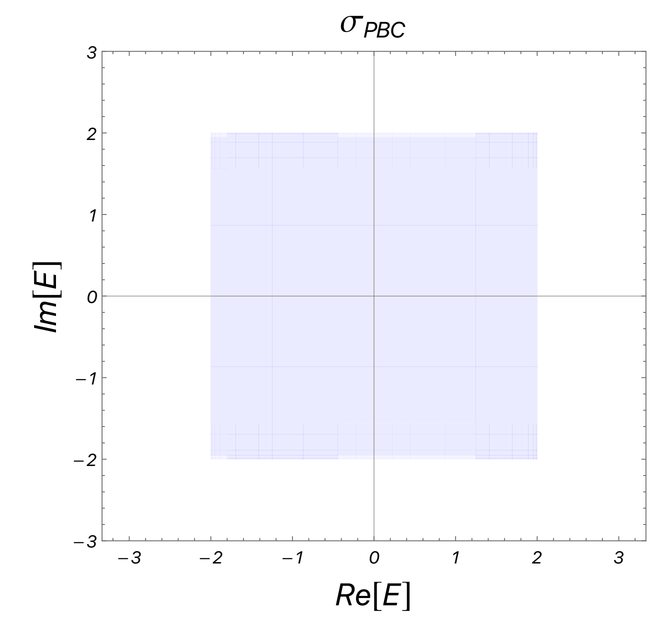

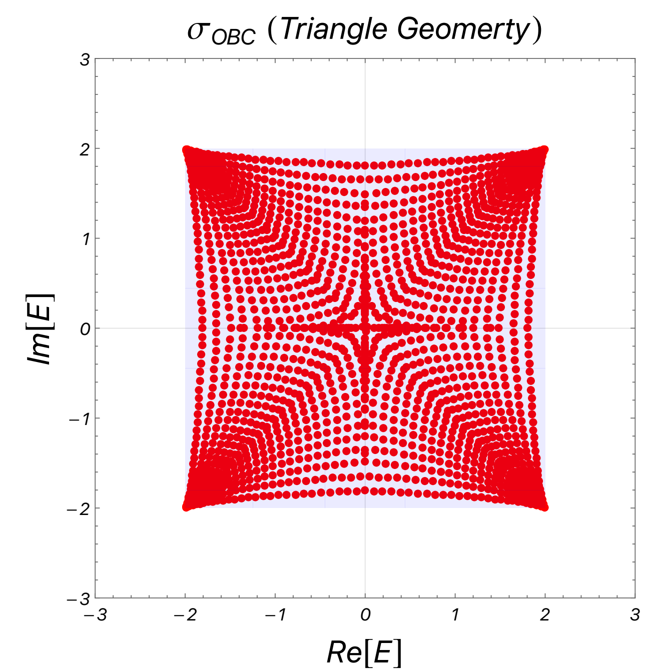

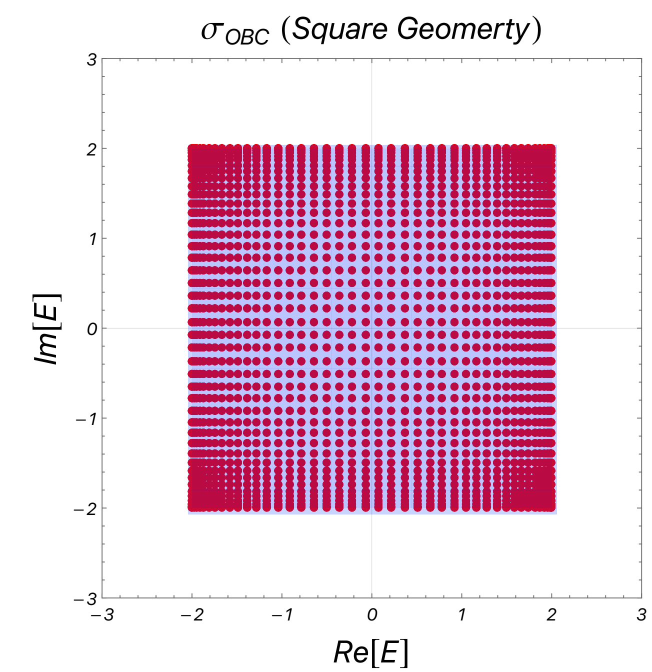

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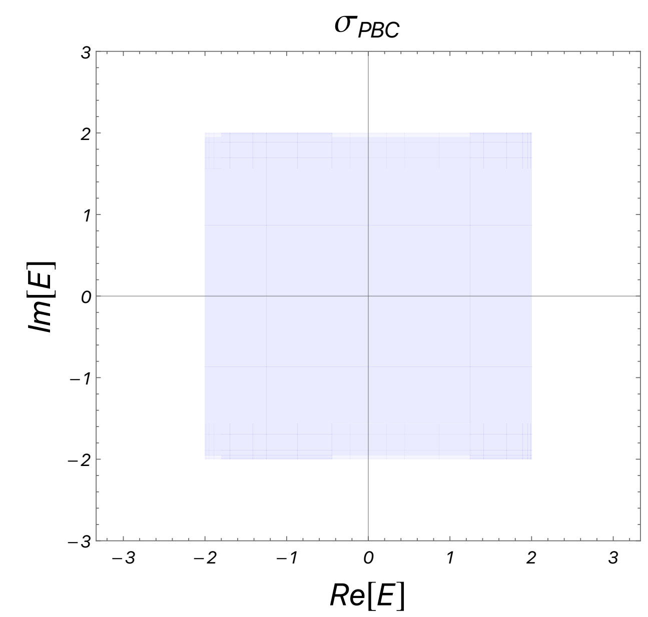

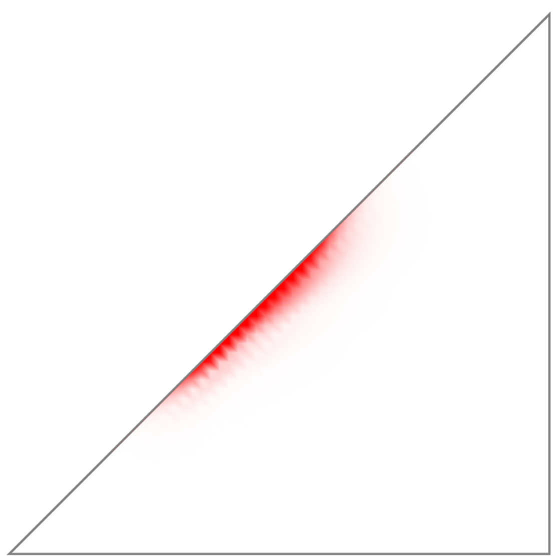

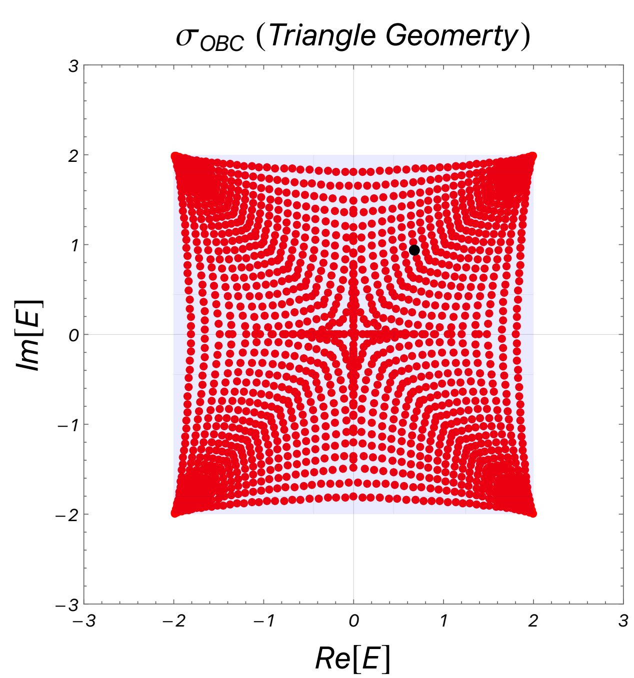

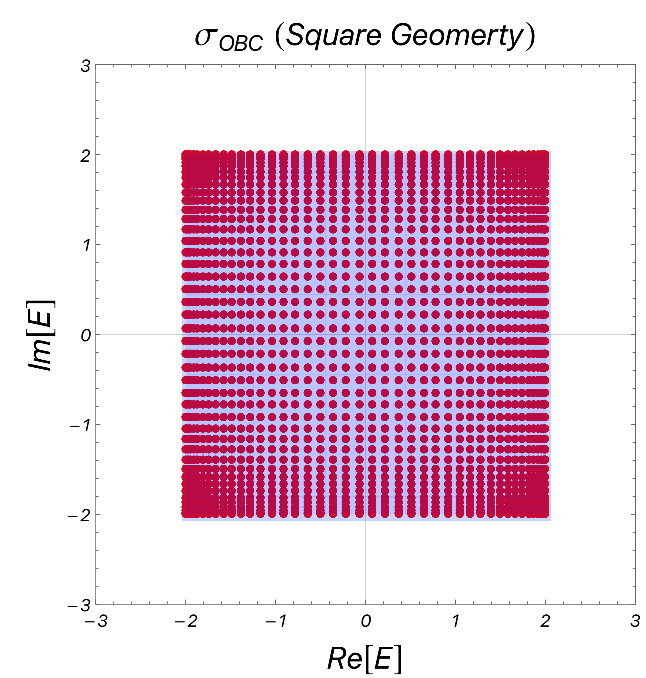

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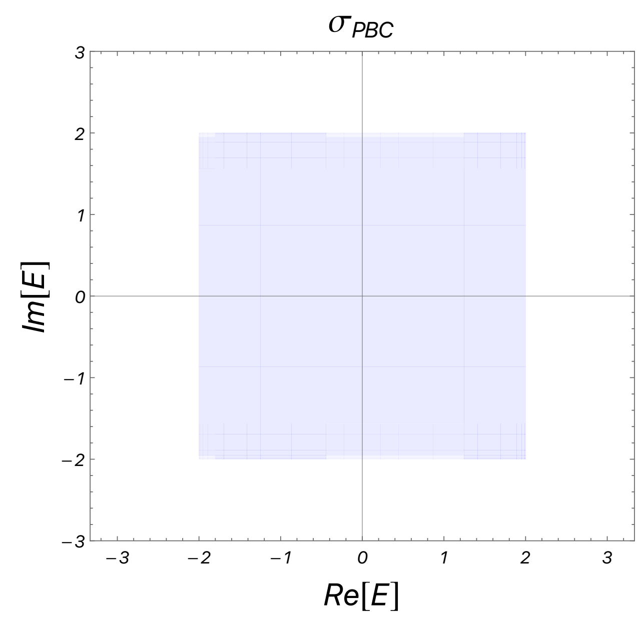

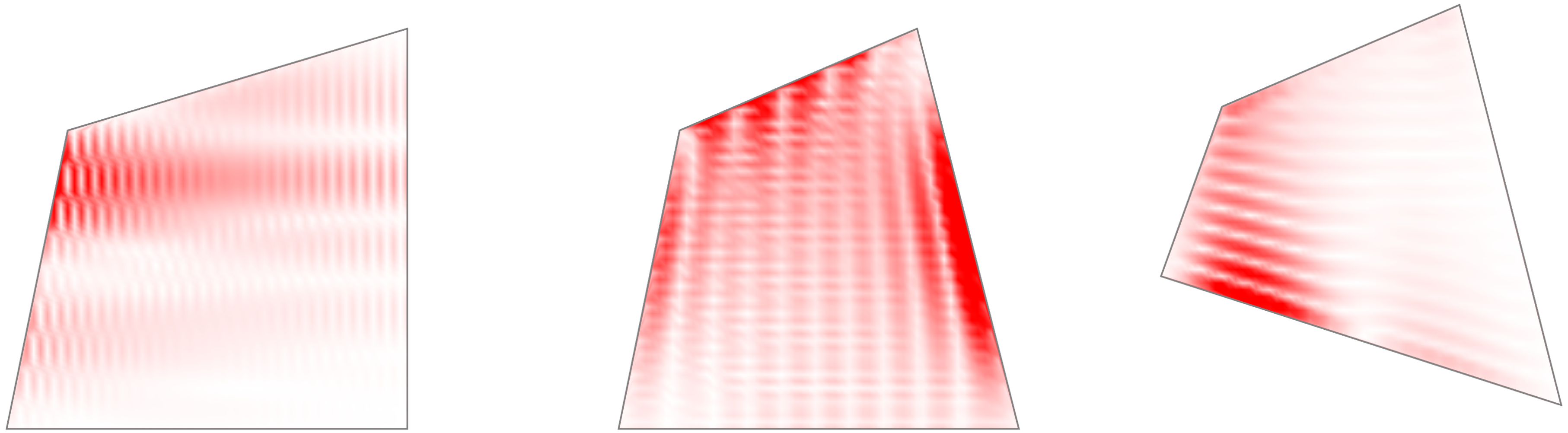

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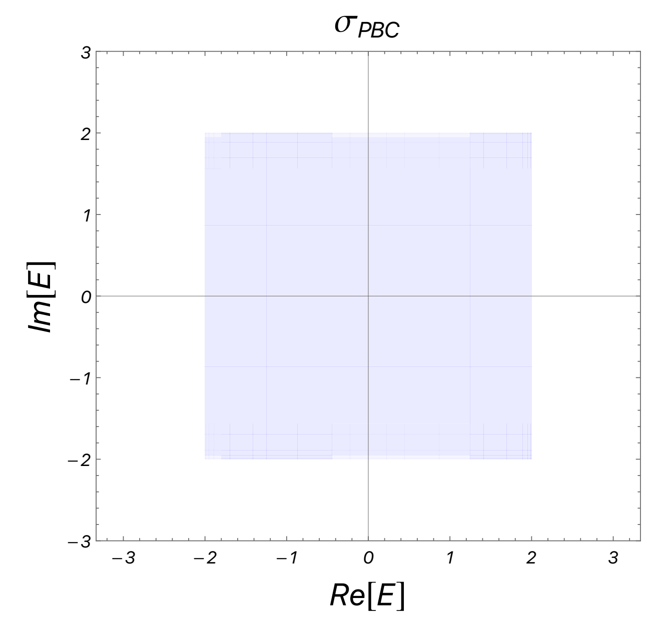

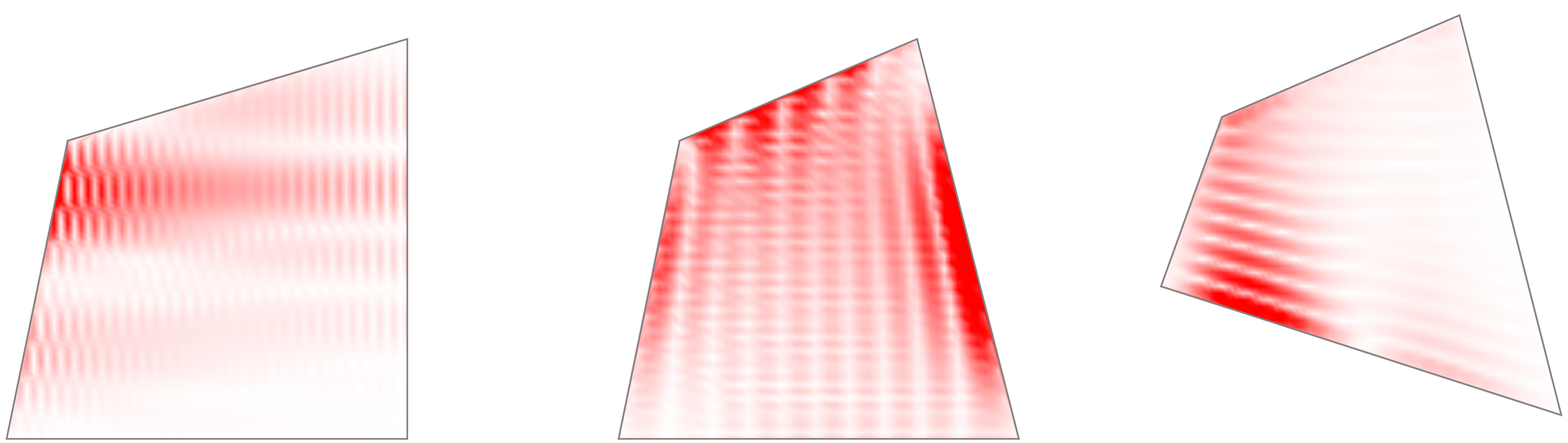

→ Skin modes number: all

2D GBZ theory: numerical summary

- **The role of OBC geometry.**

  → 2D GBZ highly depends on the OBC geometry ??

2D GBZ theory: numerical summary

- ☐ **The role of OBC geometry.**
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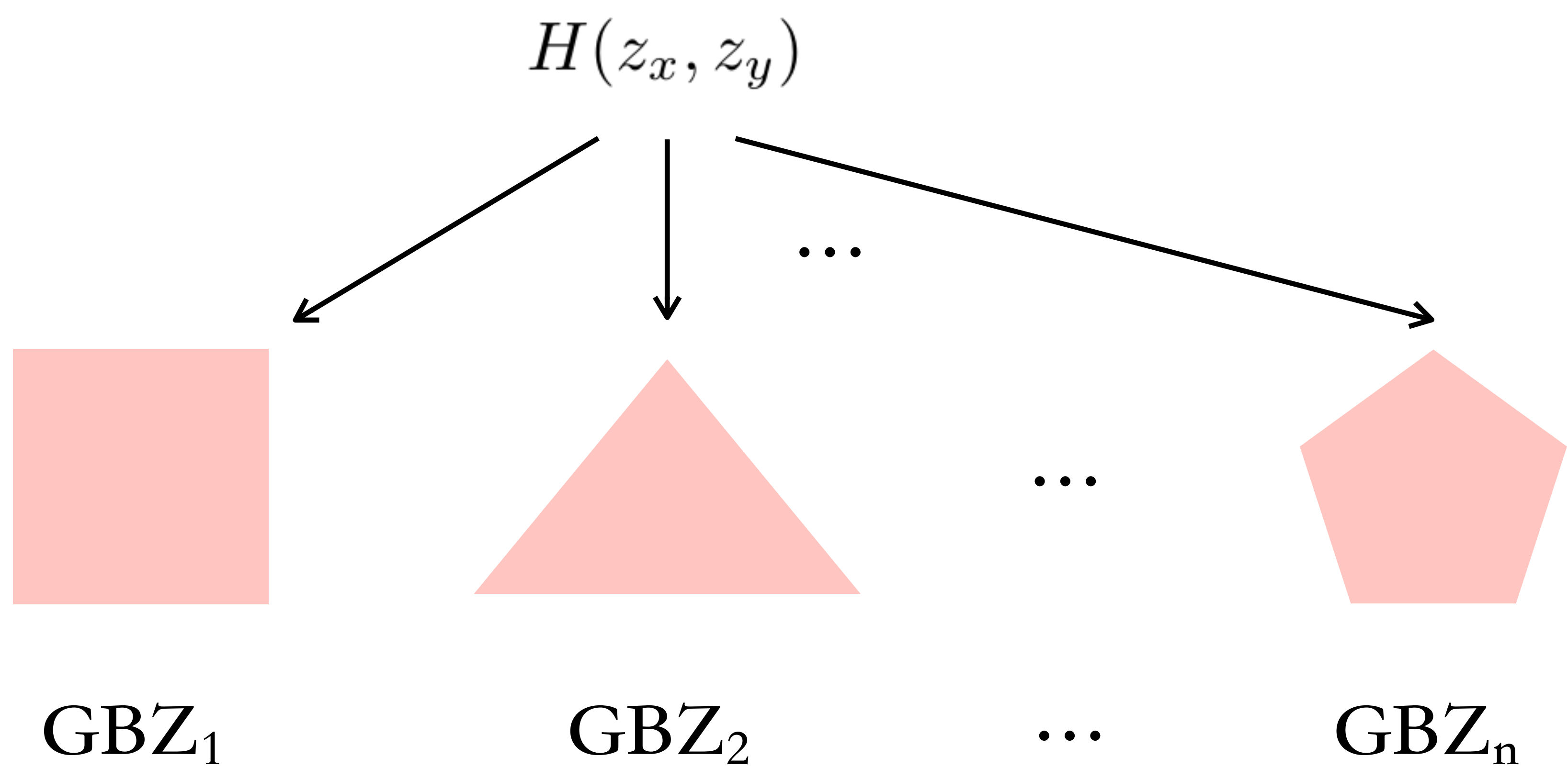

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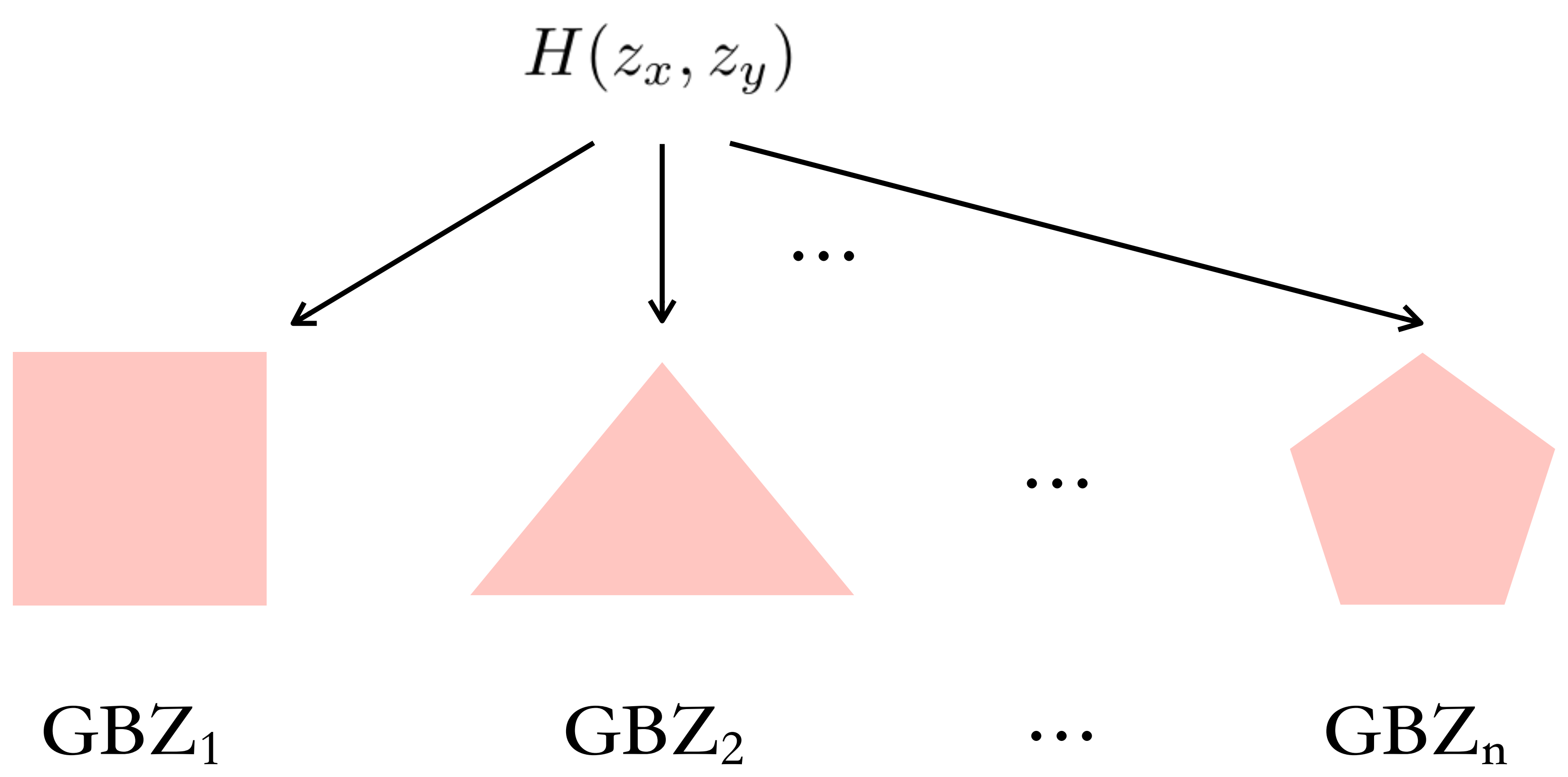

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2D GBZ theory: numerical summary

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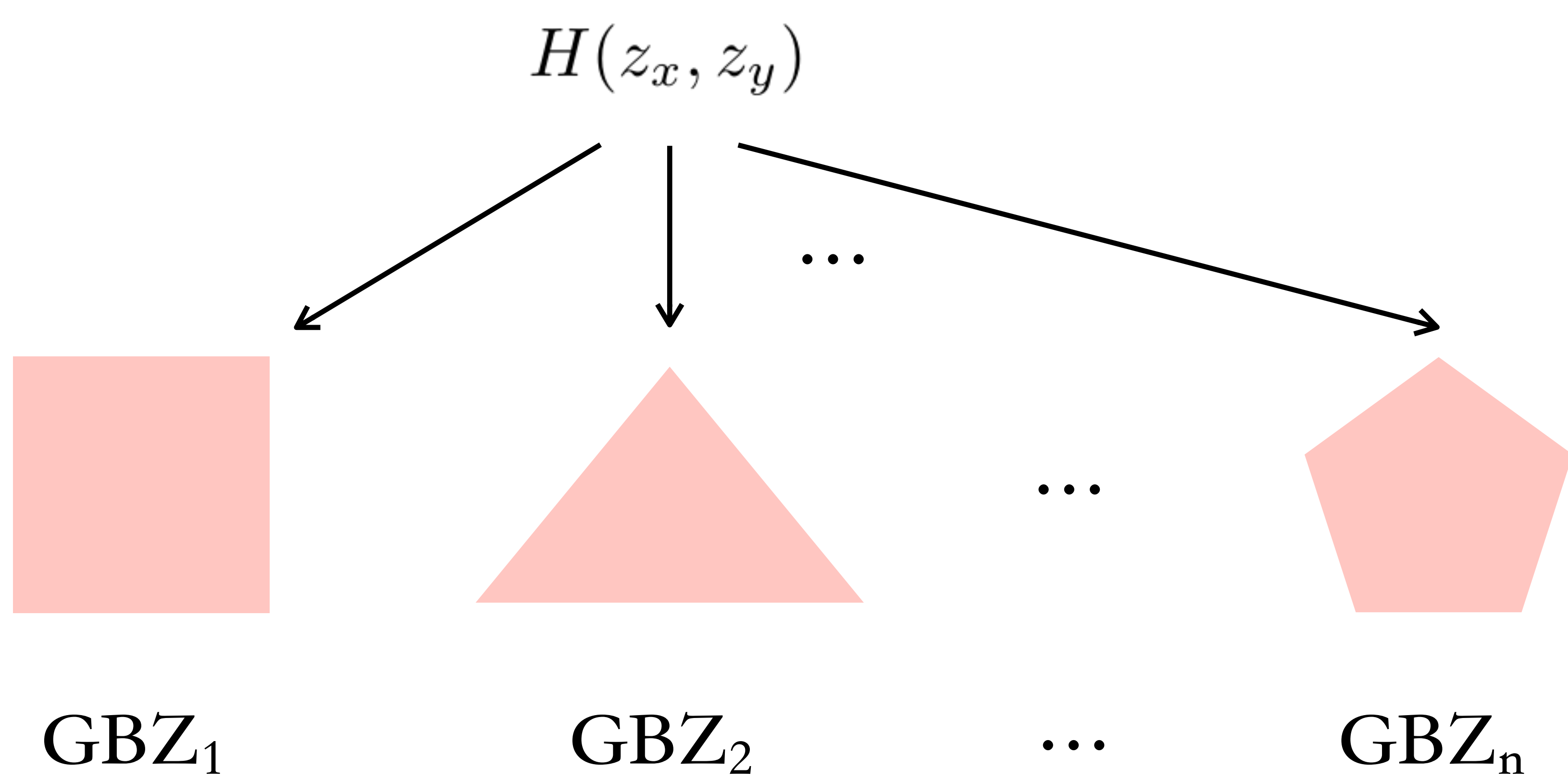


→ The role of bulk Hamiltonian ??

→ Bulk contribution to the GBZ ??

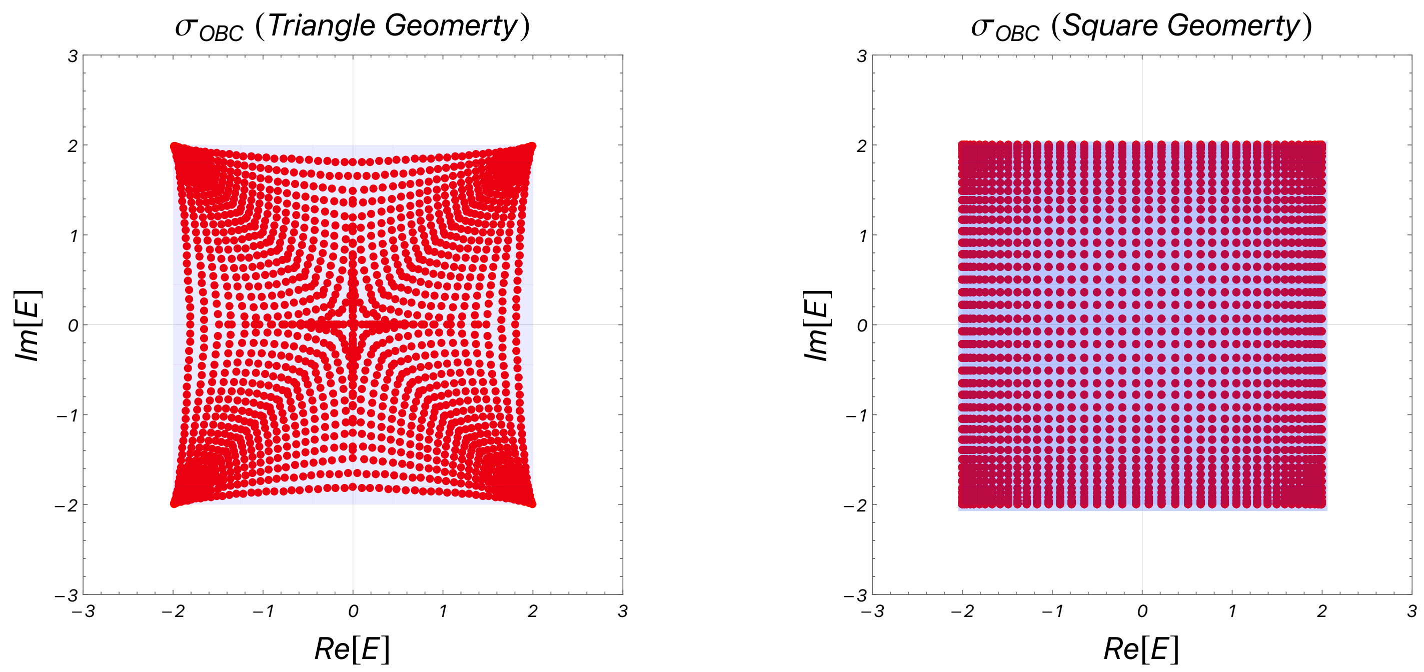
2D GBZ theory: numerical summary

- **Geometry independent quantities**

2D GBZ theory: numerical summary

- **Geometry independent quantities**

  → Hint 1: Coverage region of OBC spectrum

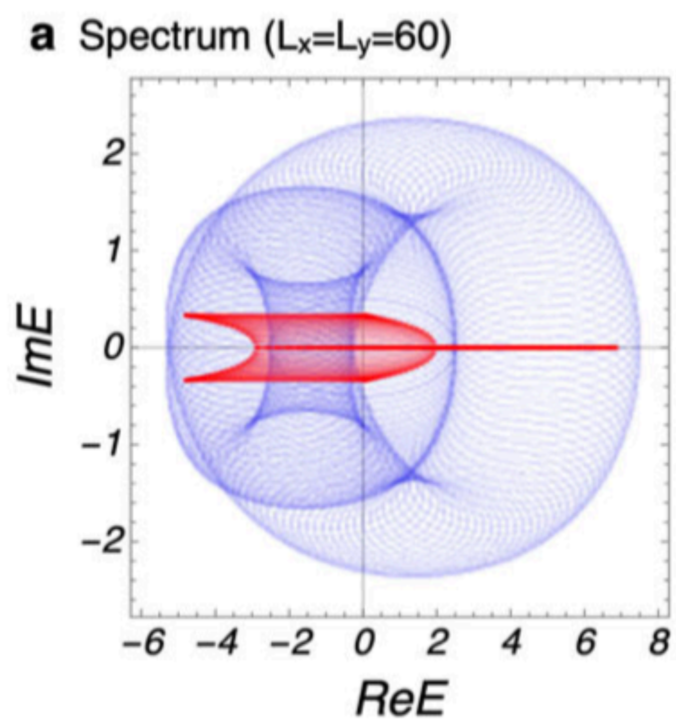


→ Seems for all models ??

2D GBZ theory: numerical summary

- ☐ **GRSE v.s. NRSE** Zhang, ZY, and Fang, Hu NC 13, 2496 (2022).

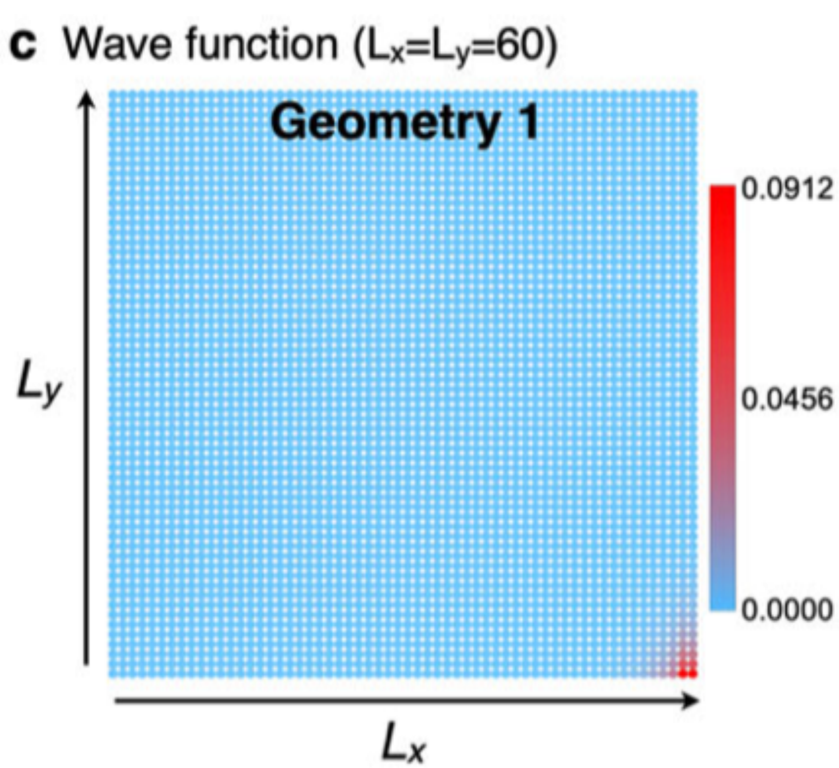
→ Two types of NHSE: non-reciprocal skin effect



→ Different coverage regions

$$\sigma^{\mathrm{PBC}} \neq \sigma^{\mathrm{OBC}}$$

→ Corner localization



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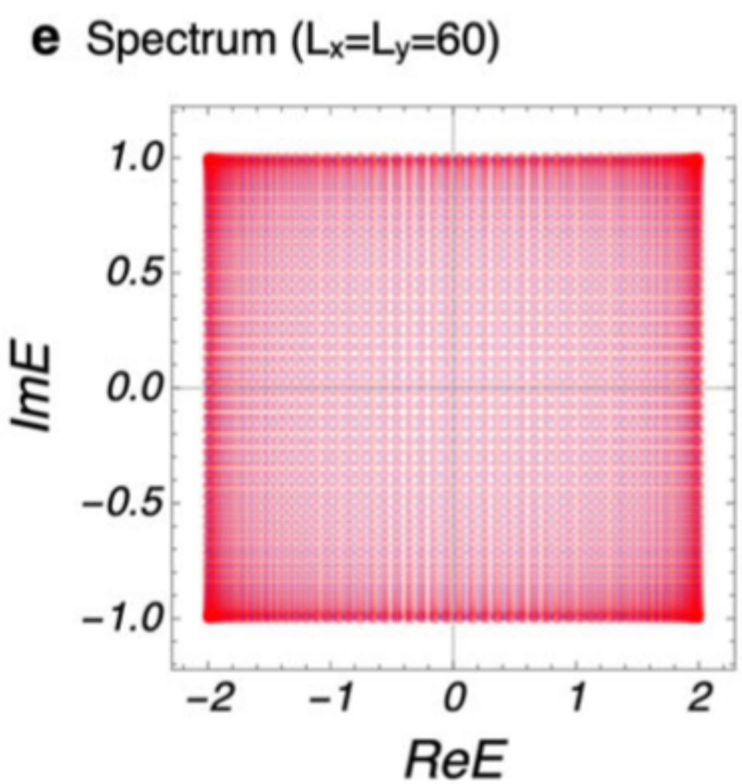
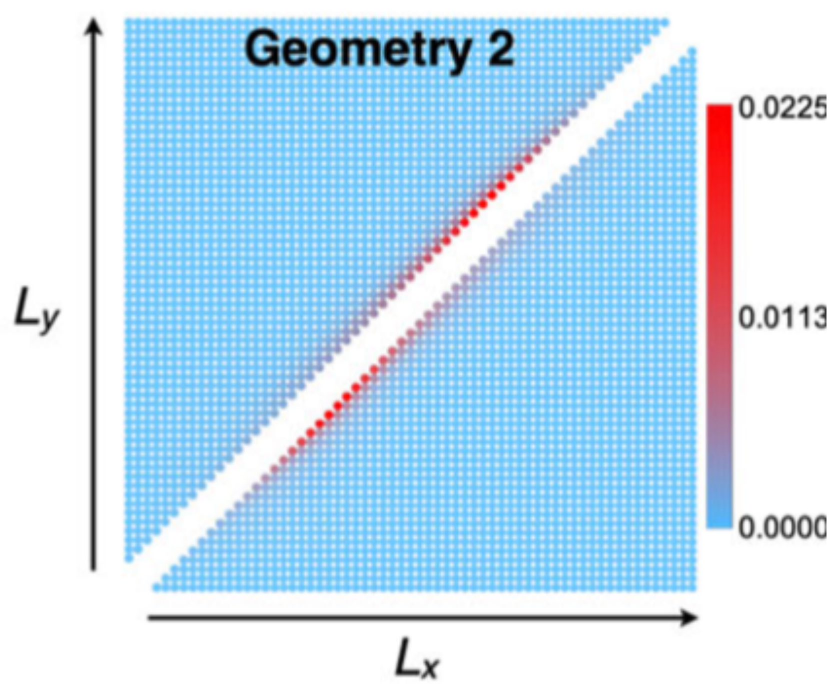


→ Common coverage regions

$$\text{GRSE}: \sigma^{\text{PBC}} = \sigma^{\text{OBC}}$$

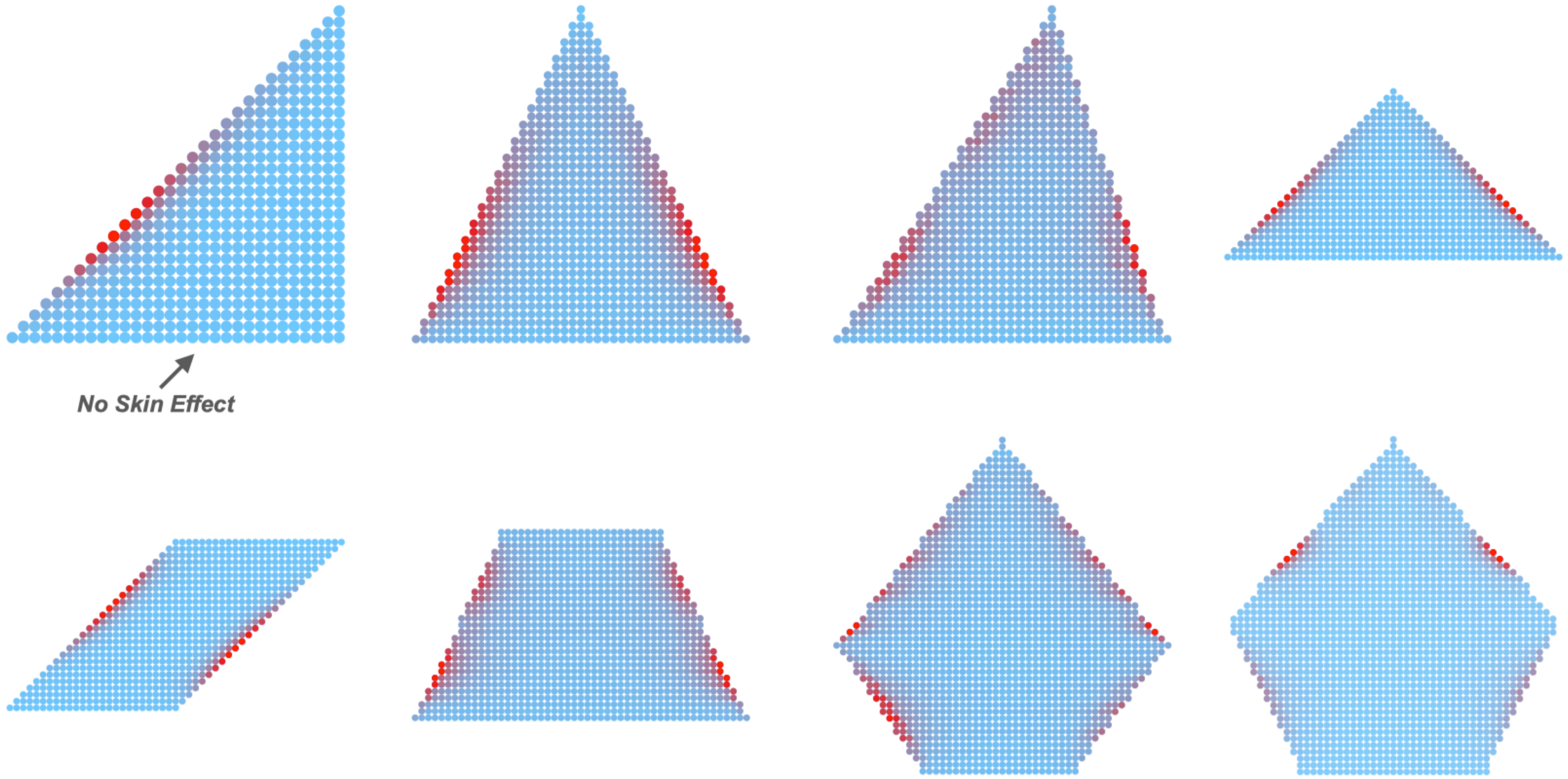
→ Edge localization



2D GBZ theory: numerical summary

□ **Geometry independent quantities** Zhang, ZY, and Fang, Hu NC 13, 2496 (2022).

→ Hint 2: Particular edge



2D GBZ theory: numerical summary

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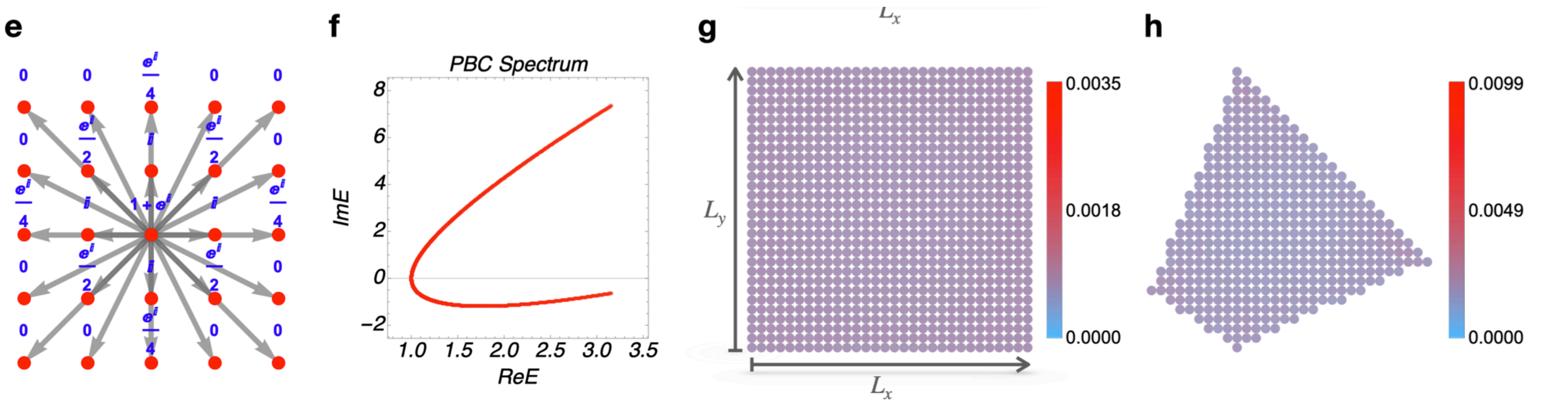
→ Hint 3: The case no skin effect

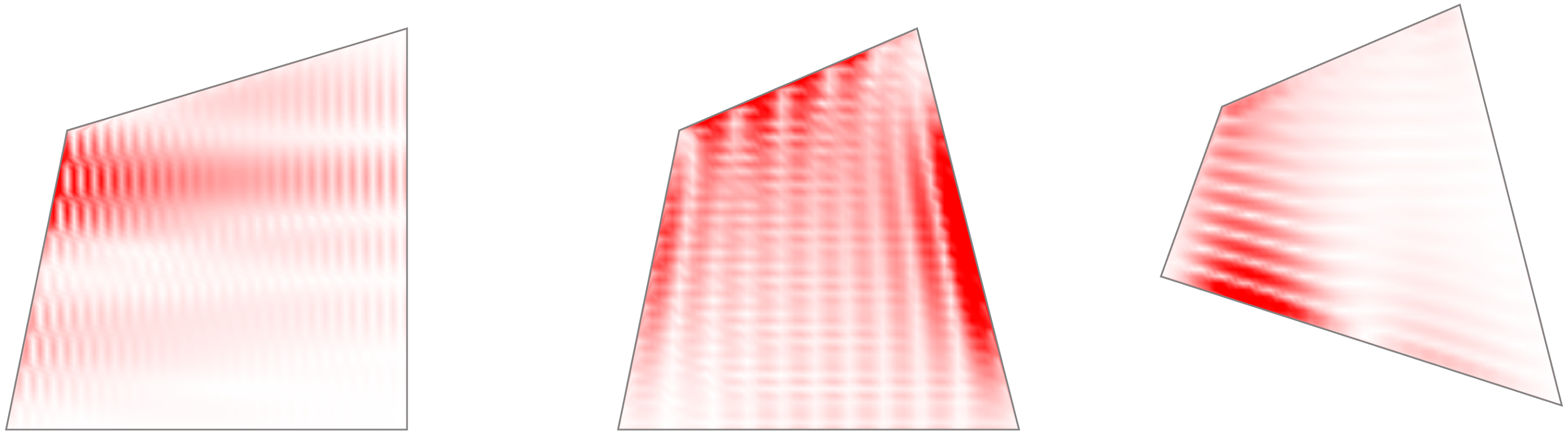


→ Robust to OBC geometry

2D GBZ theory: numerical summary

- **Geometry independent quantities**

  → Hint 4: Universal edge localization ?



  → No universal localization direction

2D GBZ theory: numerical summary

- **Summary**

  → Our basic physical picture on the 2D GBZ

2D GBZ

2D GBZ theory: numerical summary

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Depend on the OBC geometry shapes

Determine: DoS of OBC spectrum,

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2D GBZ

Geometry part

↓

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↓

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↓

Determine: coverage region of the OBC spectrum,
Robust edge localizations

2D GBZ theory: numerical summary

- **Questions related to the 2D GBZ**

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4. when the OBC geometry G_0 undergoes changes, how do the above three quantities change accordingly, and is there a fundamental rule to identify the corresponding changes?

Outline

- Introduction
- 1D GBZ theory: review
- 2D NHSE: numerical summary
- 2D GBZ theory: recent developments
- 2D GBZ theory: wave function approach

2D GBZ theory: developments

- **Summary of the previous works**
  - → 2D GBZ condition

Dimensional Transmutation from Non-Hermiticity

Hui Jiang and Ching Hua Lee
Phys. Rev. Lett. **131**, 076401 – Published 17 August 2023

Amoeba Formulation of Non-Bloch Band Theory in Arbitrary Dimensions

Hong-Yi Wang, Fei Song, and Zhong Wang
Phys. Rev. X **14**, 021011 – Published 16 April 2024

2D GBZ theory: developments

- **Winding number condition**

Non-Bloch Band Theory of Non-Hermitian Systems

Kazuki Yokomizo and Shuichi Murakami
Phys. Rev. Lett. **123**, 066404 – Published 7 August 2019

The construction of the generalized Brillouin zone can be extended to higher dimensions as well. In two-dimensional (2D) systems, we introduce the two parameters $\beta^x \left(= e^{ik_x}\right)$ and $\beta^y \left(= e^{ik_y}\right)$. Then the eigenvalue equation $\det\left[\mathcal{H}\left(\beta^x, \beta^y\right) - E\right] = 0$, where $\mathcal{H}\left(\beta^x, \beta^y\right)$ is a 2D generalized Bloch Hamiltonian, is an algebraic equation for β^x and β^y. If we fix β^y (β^x), this system can be regarded as a 1D system, and the criterion is given by $\left|\beta^x_{M_x}\right| = \left|\beta^x_{M_x+1}\right|$ $\left(\left|\beta^y_{M_y}\right| = \left|\beta^y_{M_y+1}\right|\right)$, where $2M_x$ $(2M_y)$ is the degree of the eigenvalue equation for β^x (β^y). Thus, we can get the conditions for the continuum bands. Nevertheless, it is still an open question how to determine the generalized Brillouin zone in higher dimensions.

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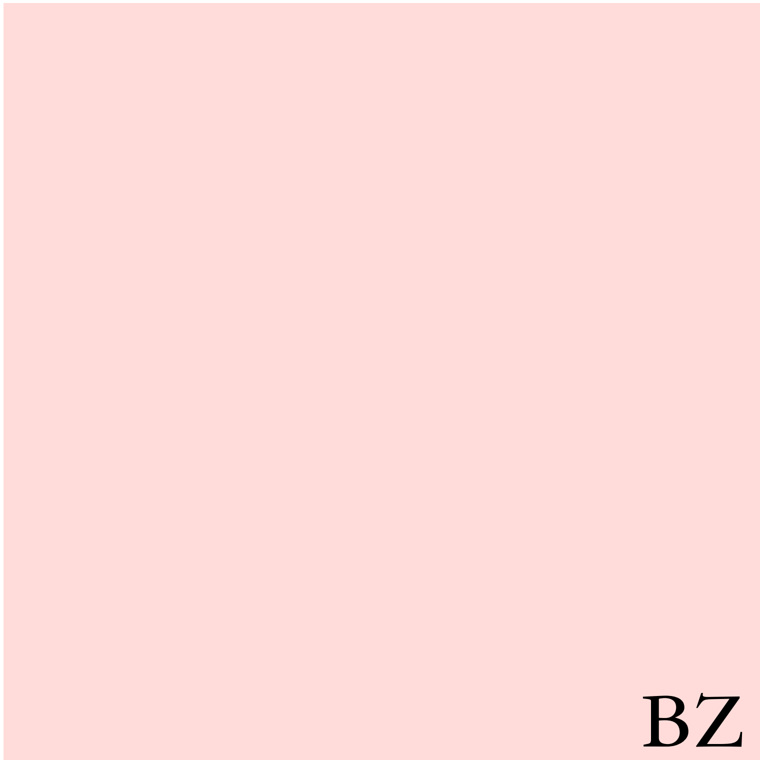

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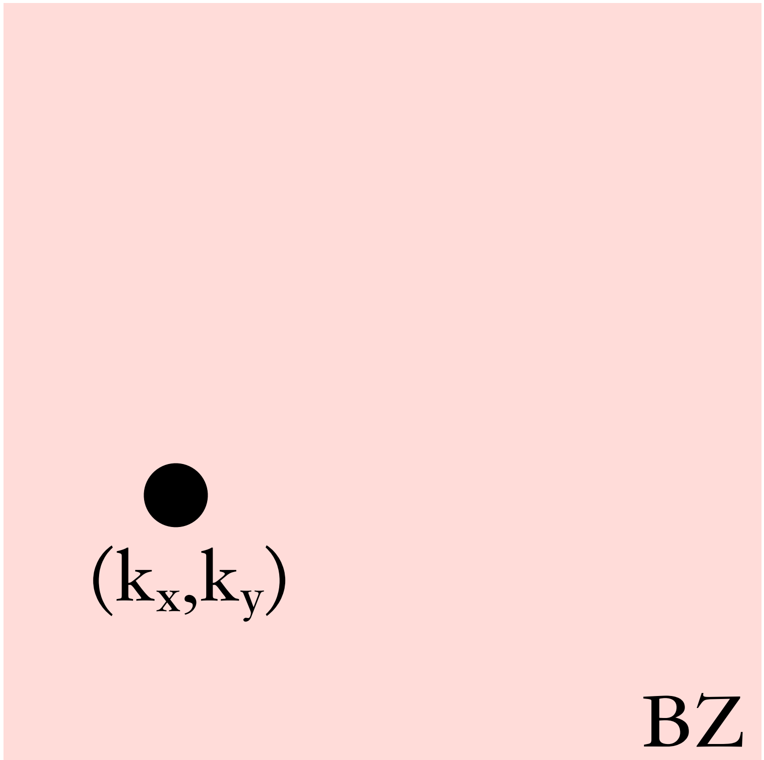

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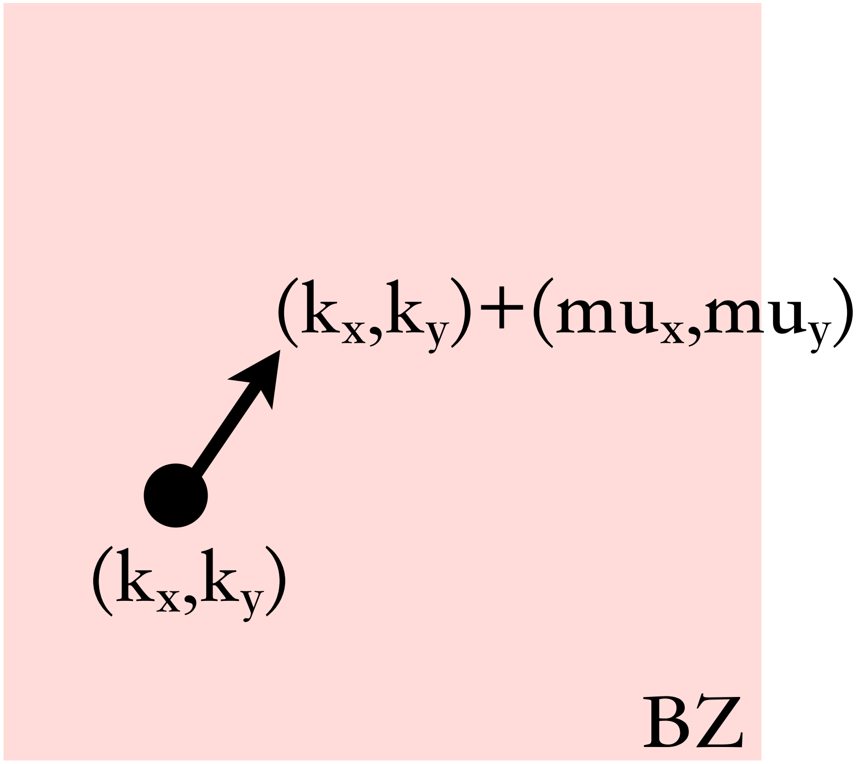

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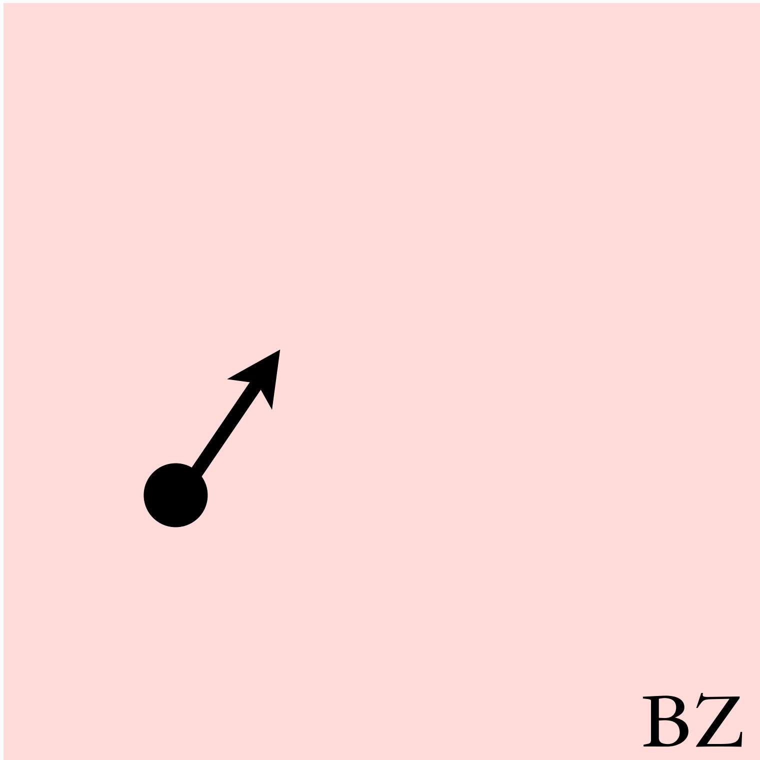

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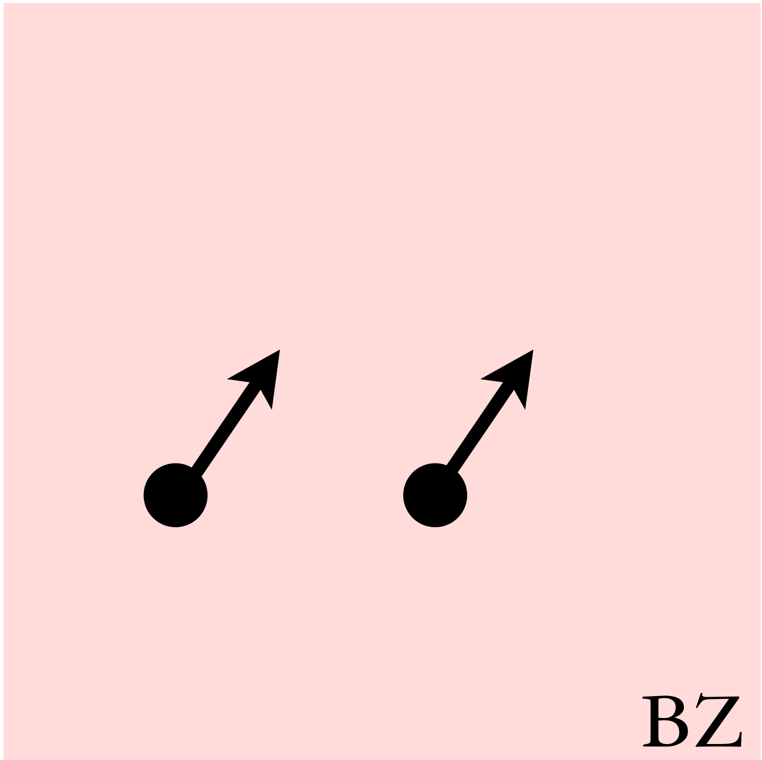

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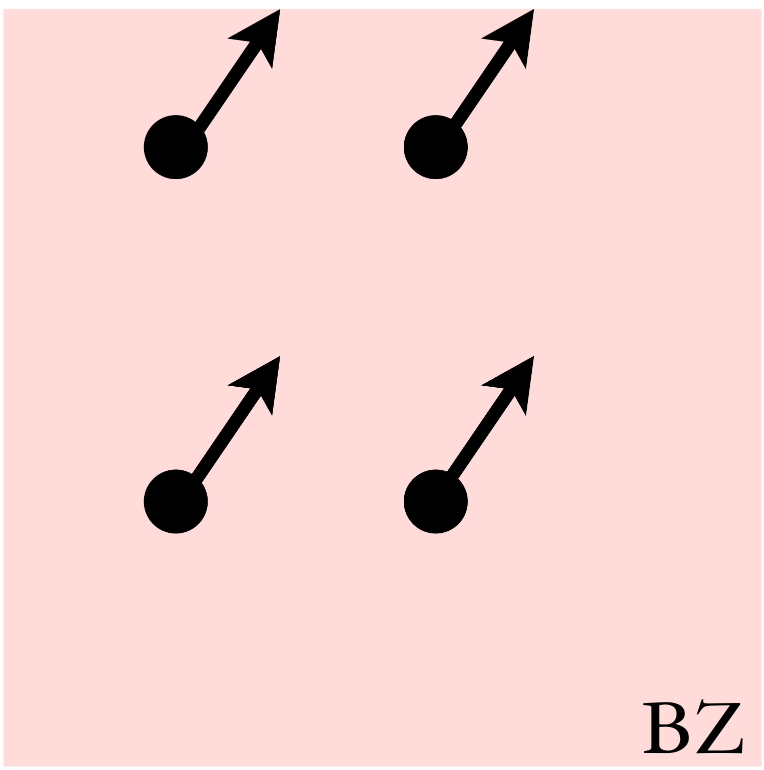

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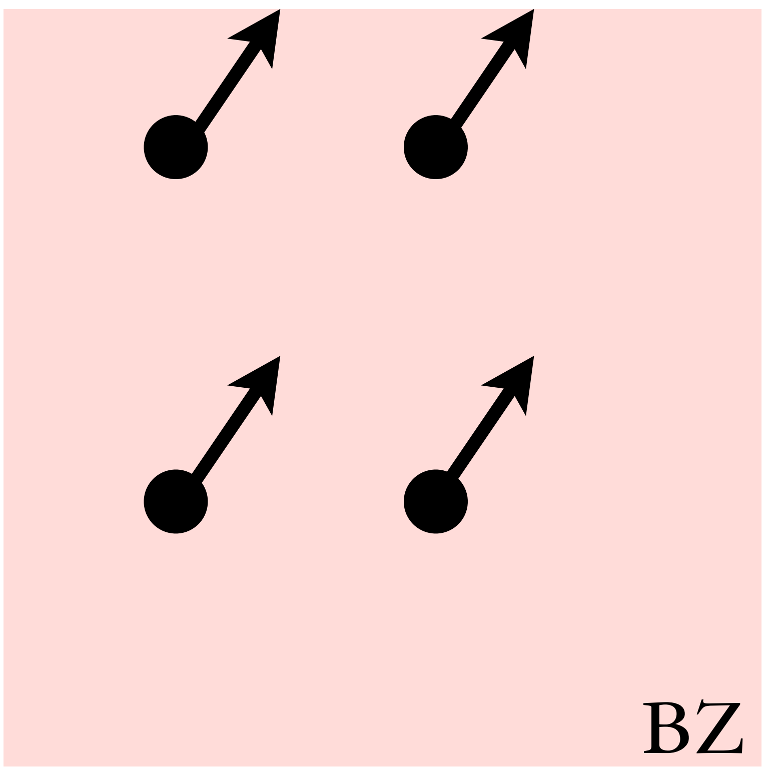

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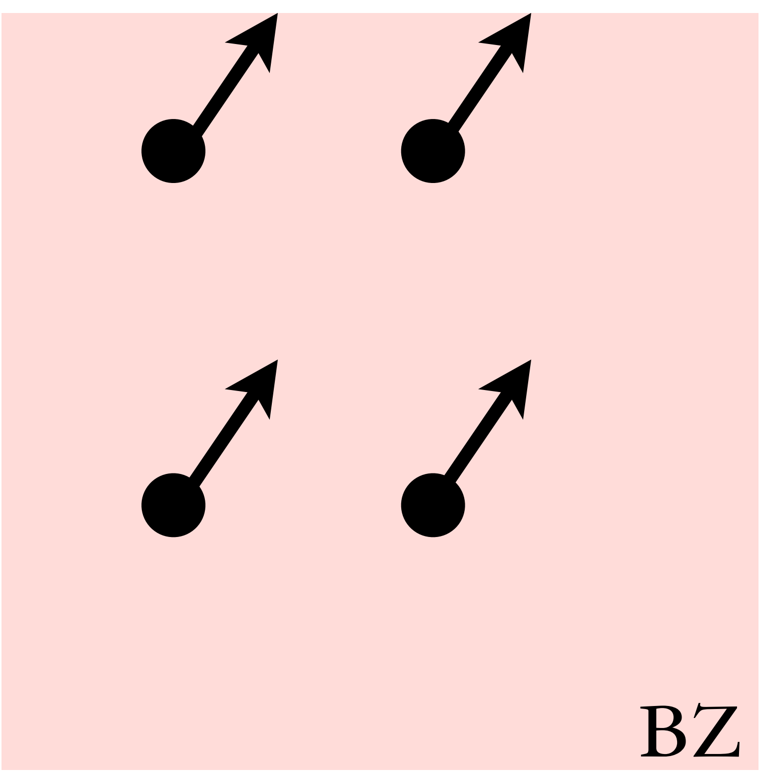

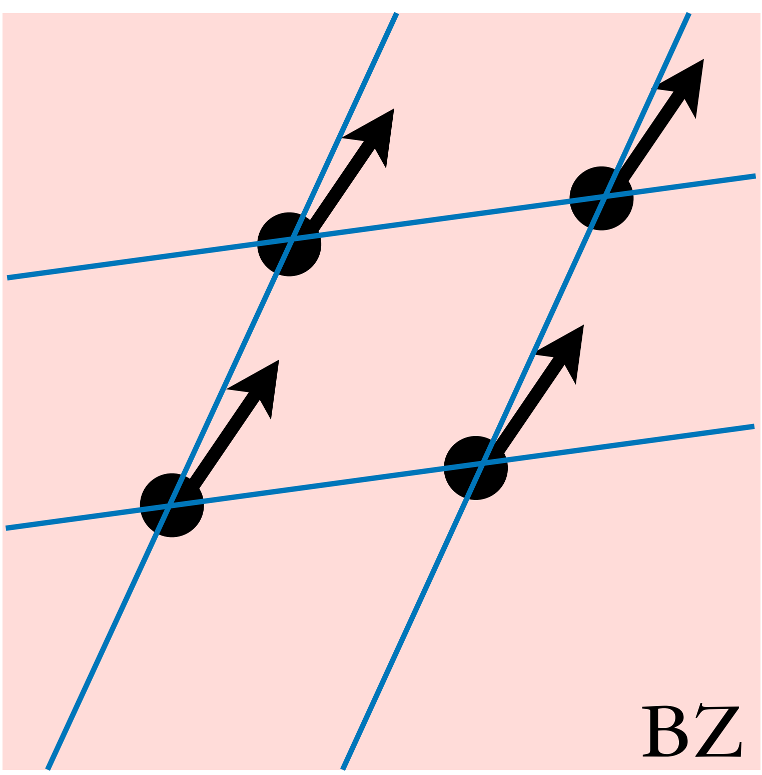

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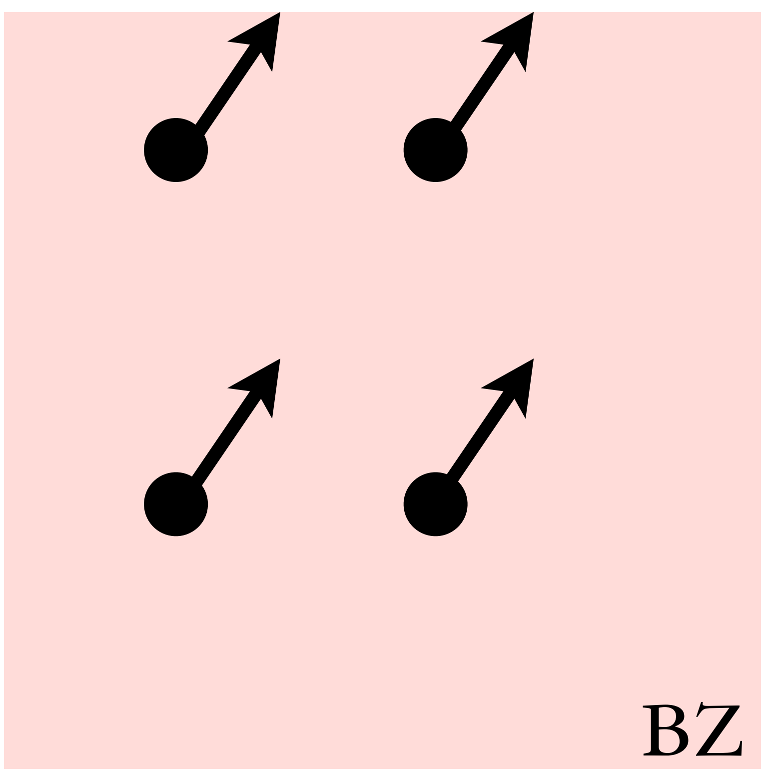
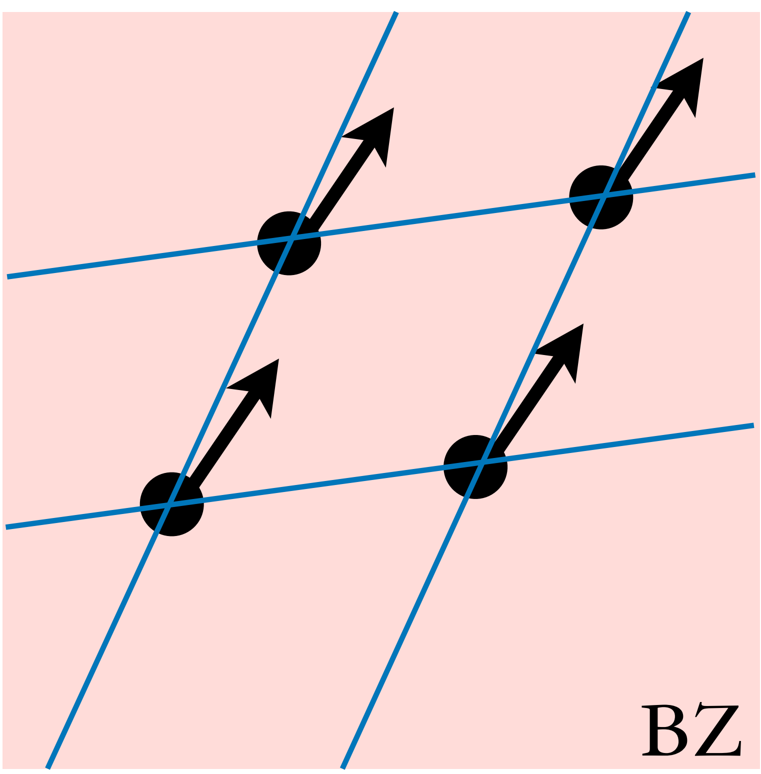




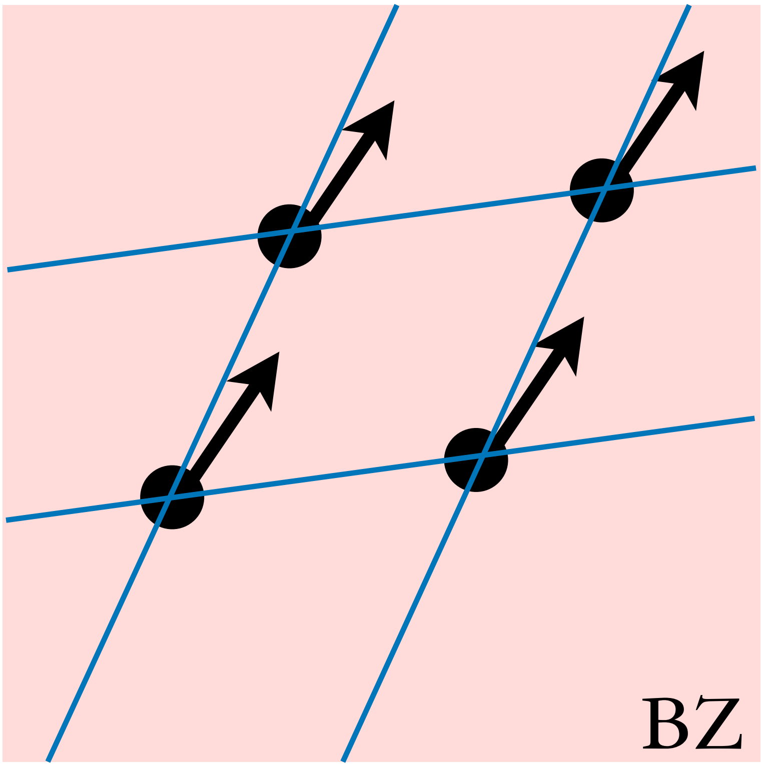
$$f(E_0, \beta_1, \beta_2) = f(E_0, \beta_1 e^{i\theta_1}, \beta_2) = f(E_0, \beta_1, \beta_2 e^{i\theta_2}) = 0$$

2D GBZ theory: developments

- **Winding number condition**

Amoeba Formulation of Non-Bloch Band Theory in Arbitrary Dimensions

Hong-Yi Wang, Fei Song, and Zhong Wang
Phys. Rev. X **14**, 021011 – Published 16 April 2024



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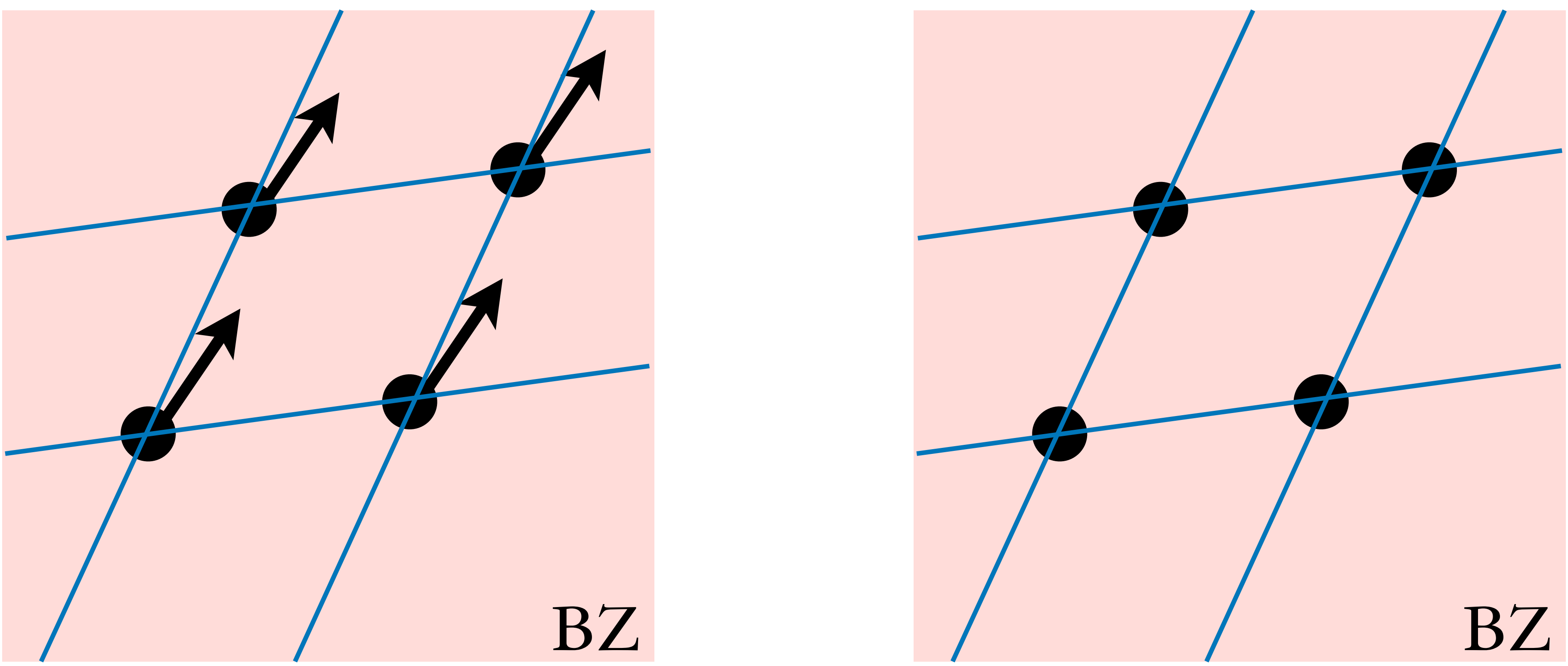

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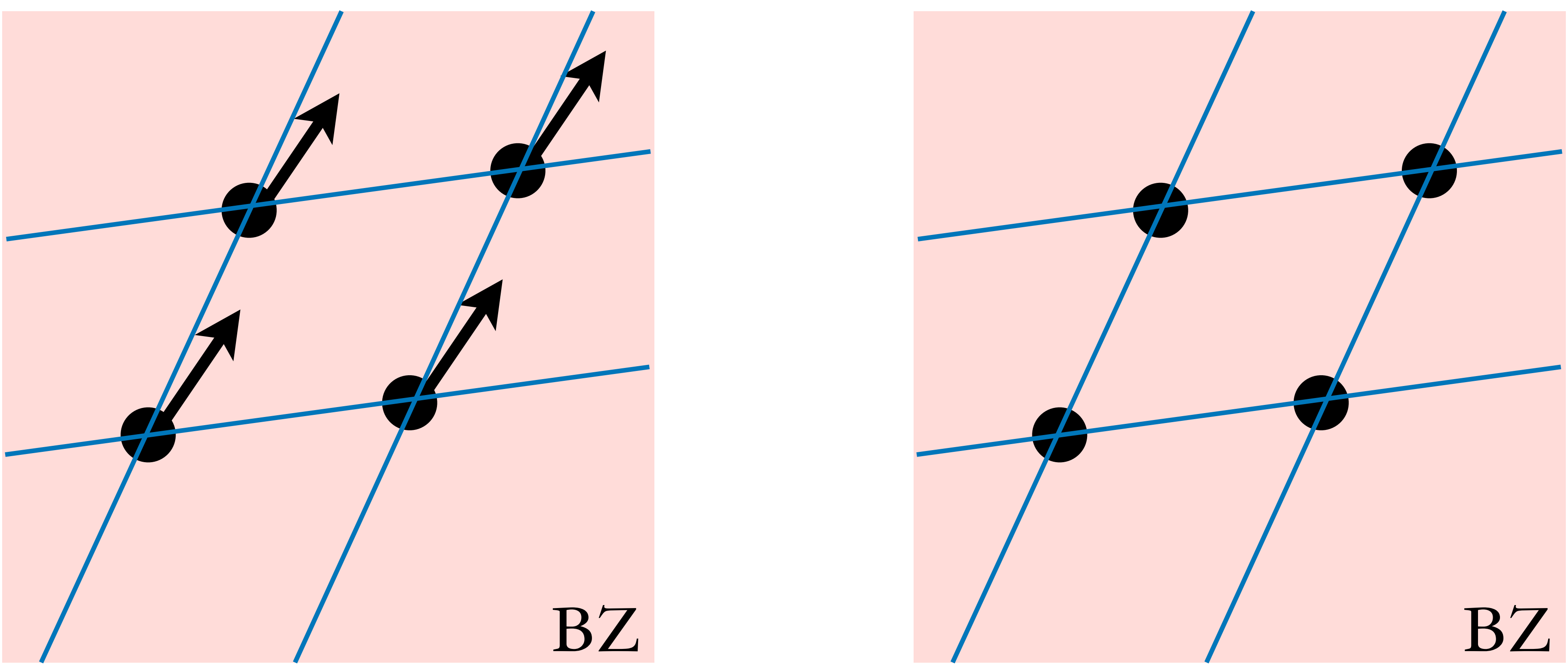

(ν_x, ν_y)

2D GBZ theory: developments

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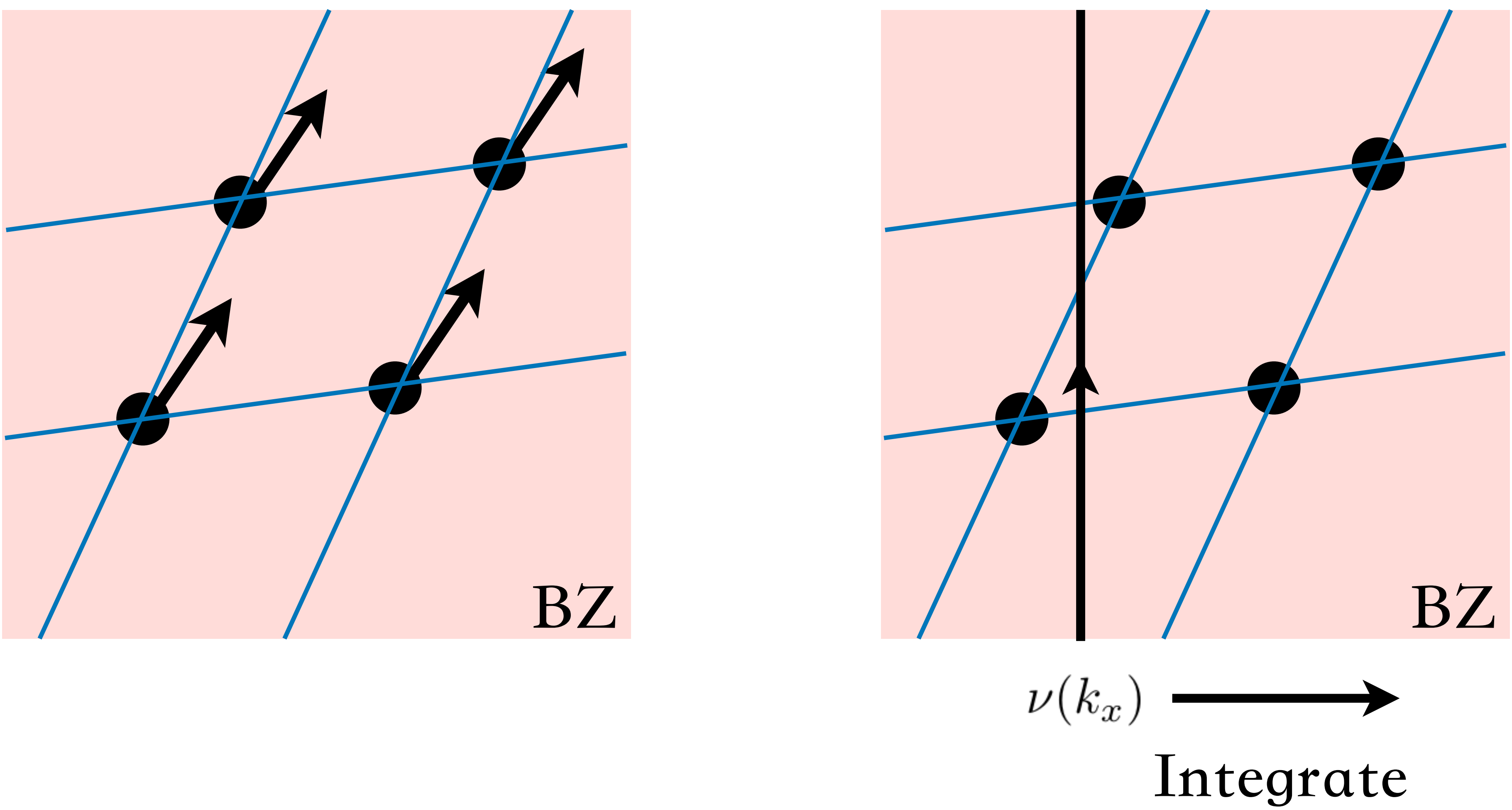

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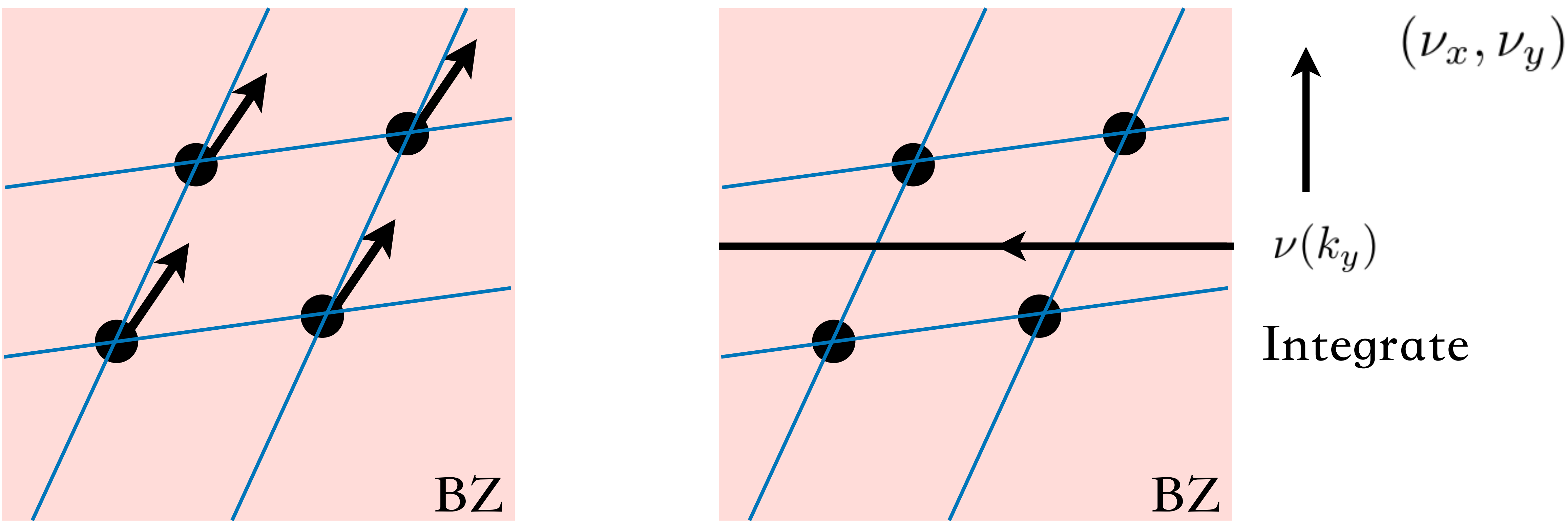

2D GBZ theory: developments

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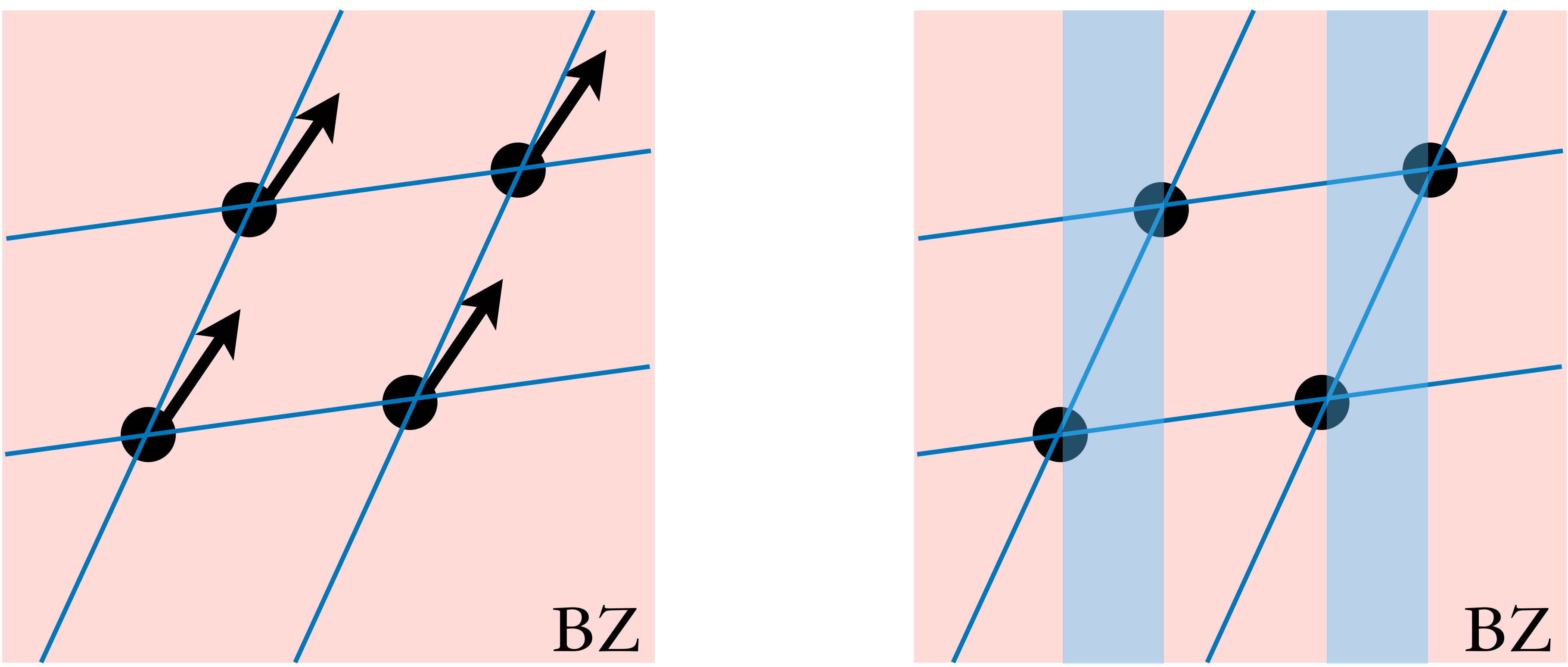


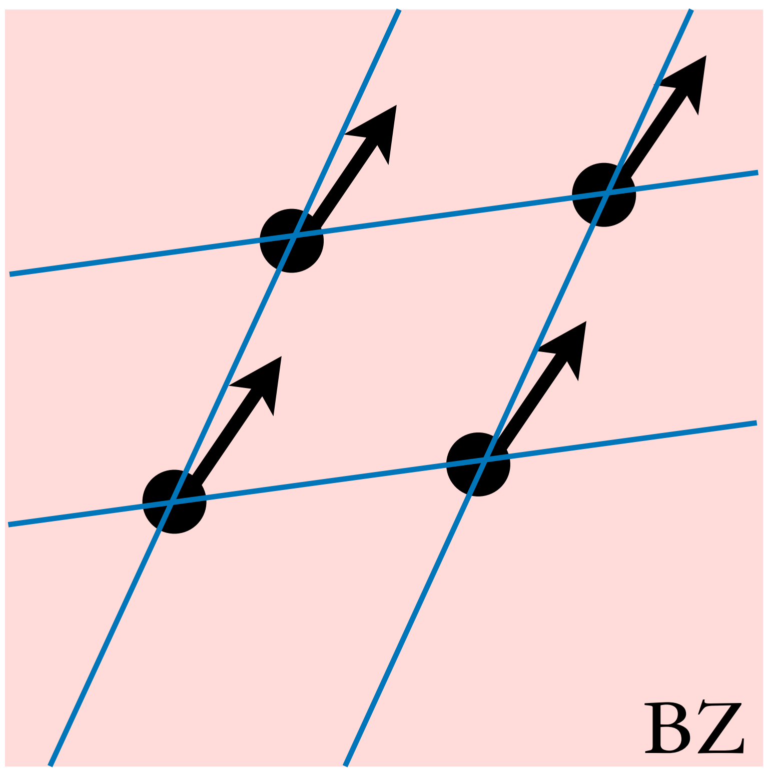
(ν_x, ν_y)

2D GBZ theory: developments

- Puzzles

2D GBZ theory: developments

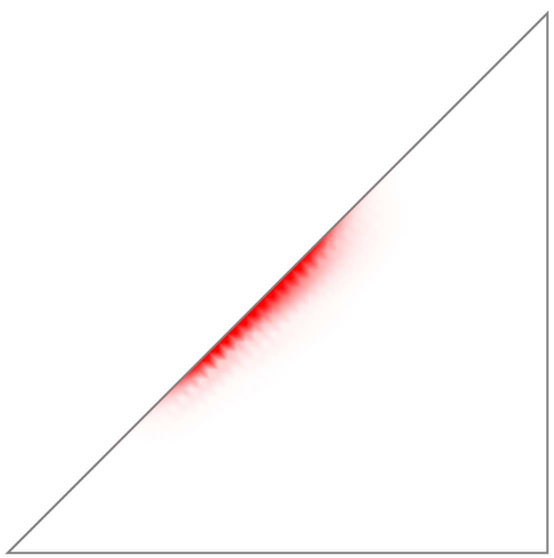
- **Puzzles**
  - → Puzzle 1: Scattering channel and standing wave

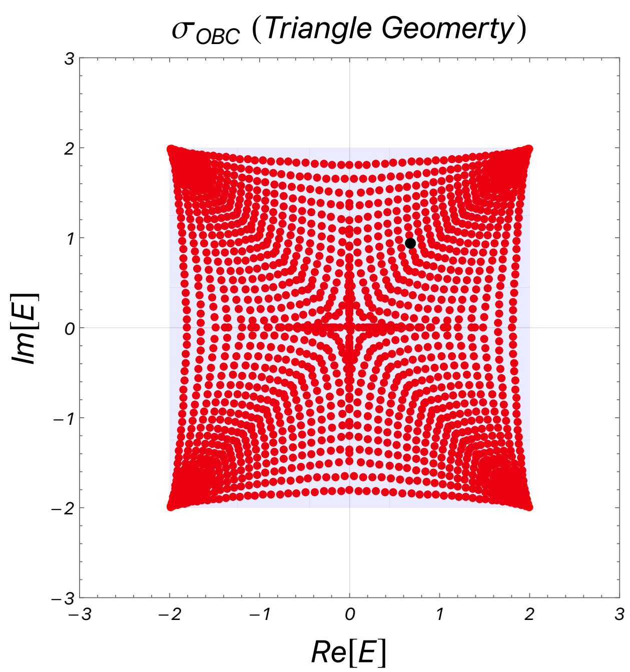


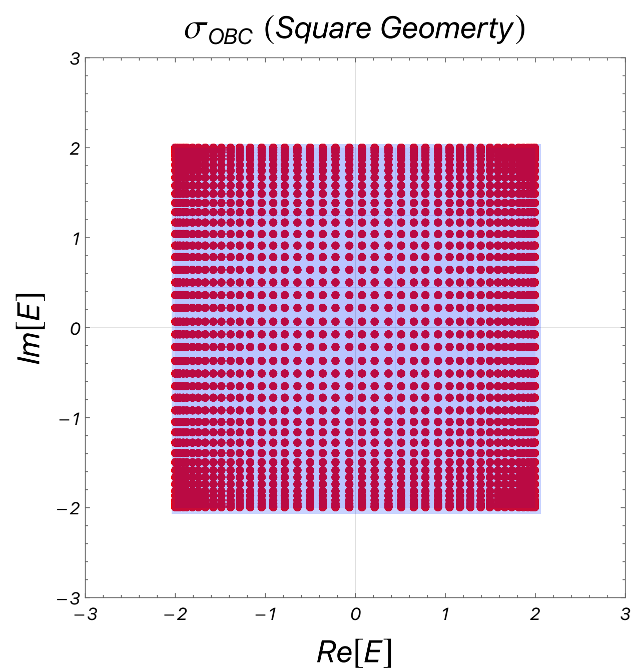
2D GBZ theory: developments

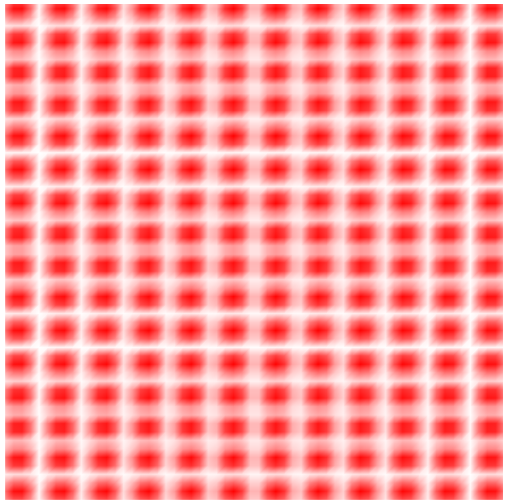
☐ Puzzles

→ Puzzle 2: Geometry dependent skin effect









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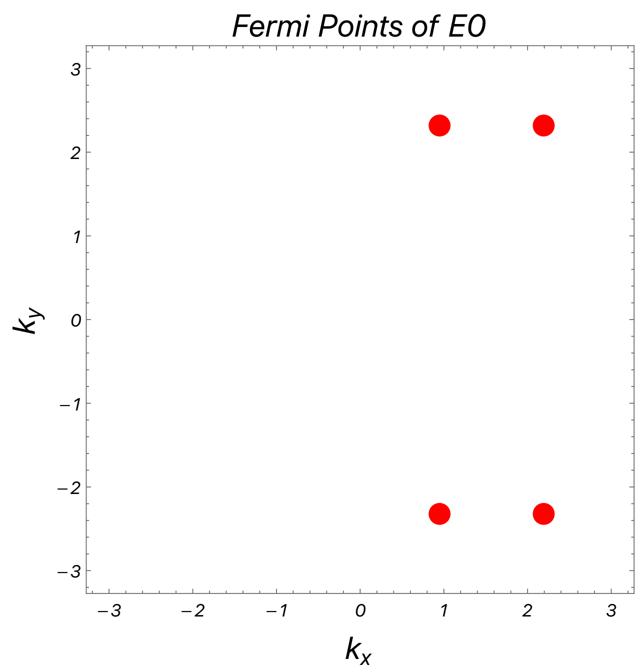


2D GBZ theory: developments

- **Puzzles**
  - → Puzzle 3: OBC spectral coverage

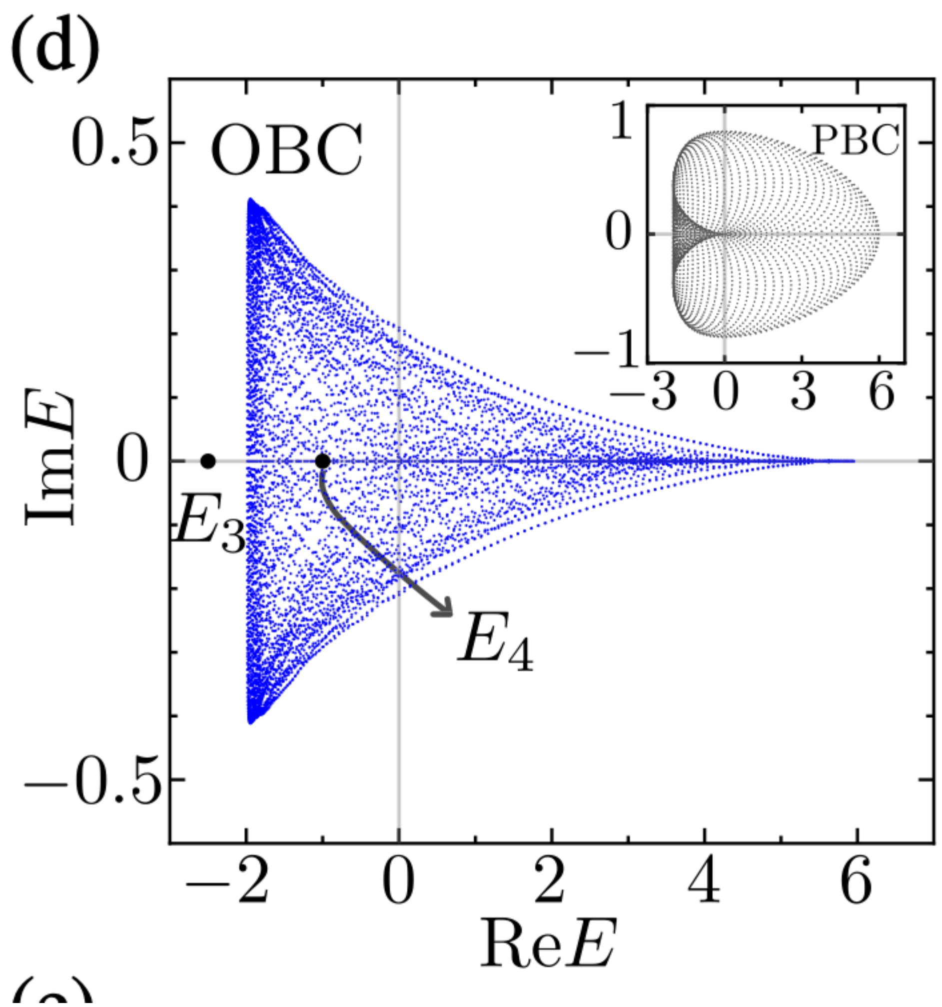


Figure from Amoeba theory

Outline

- Introduction
- 1D GBZ theory: review
- 2D NHSE: numerical summary
- 2D GBZ theory: recent developments
- 2D GBZ theory: wave function approach

2D GBZ theory: WF approach

- **Wave function approach**
  - → We find such a minimal model that we want to solve

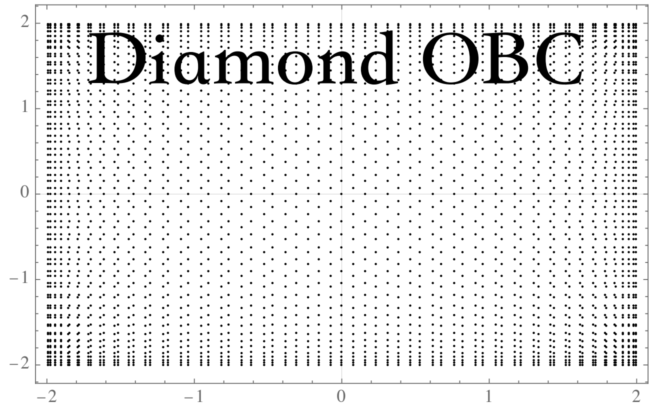
$$H_1(z_x, z_y) = -z_x z_y + z_x/z_y + z_y/z_x + 1/(z_x z_y).$$

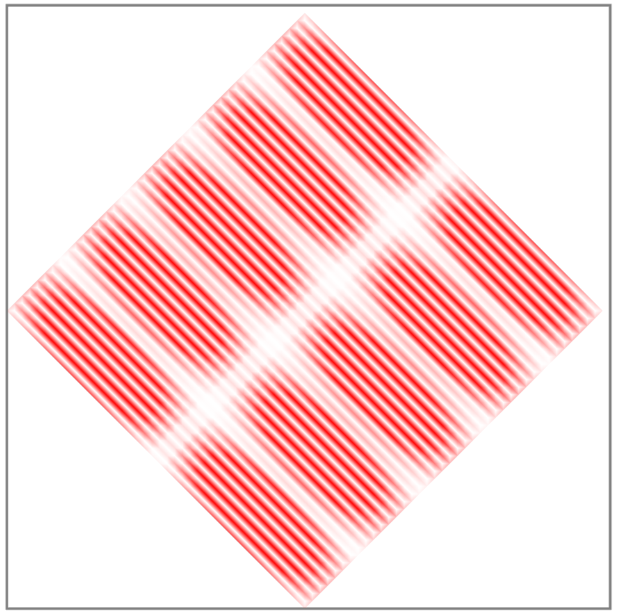
2D GBZ theory: WF approach

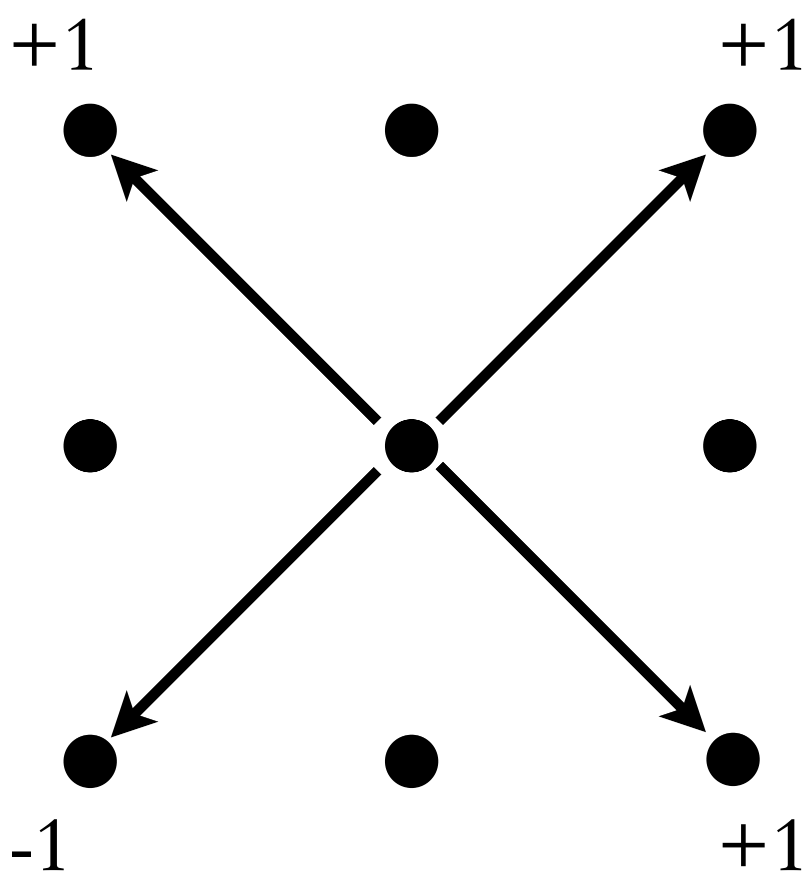
- **Wave function approach**
  - → We find such a minimal model that we want to solve

$$H_1(z_x, z_y) = -z_x z_y + z_x/z_y + z_y/z_x + 1/(z_x z_y).$$

  - → Typical wave function





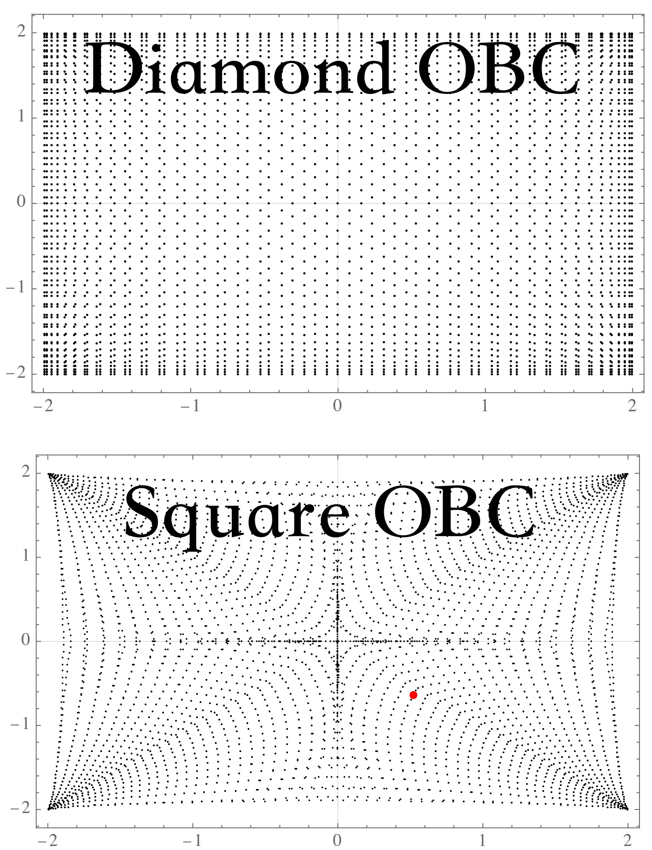


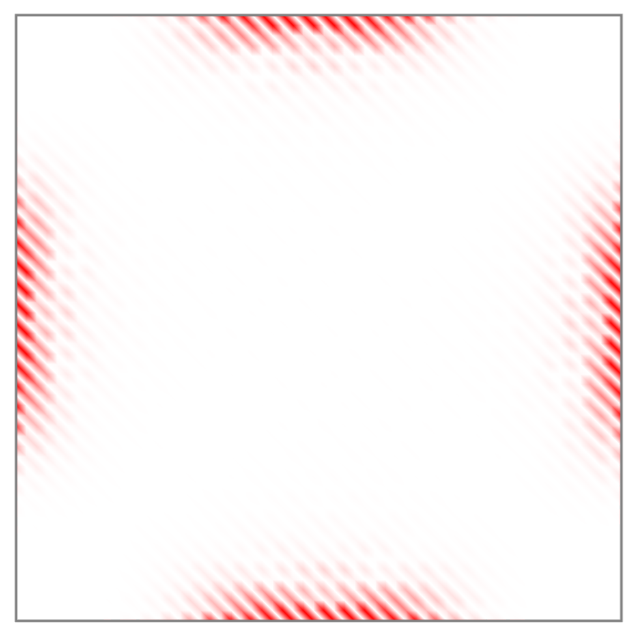
2D GBZ theory: WF approach

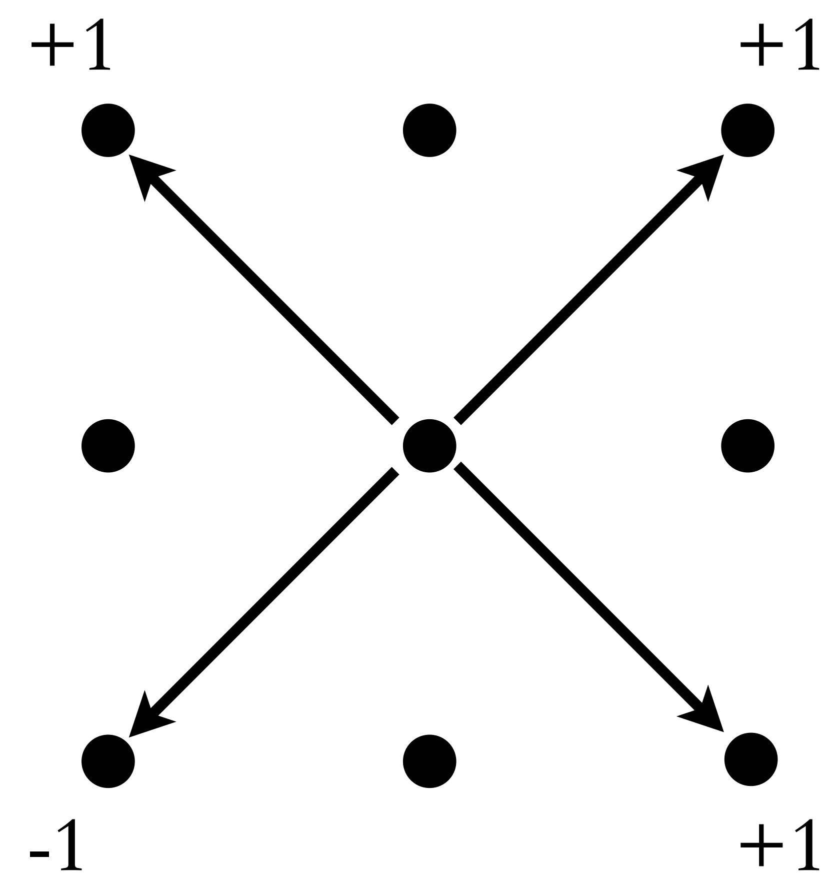
- ☐ **Wave function approach**
  - ➜ We find such a minimal model that we want to solve

$$H_1(z_x, z_y) = -z_x z_y + z_x/z_y + z_y/z_x + 1/(z_x z_y).$$

  - ➜ Typical wave function







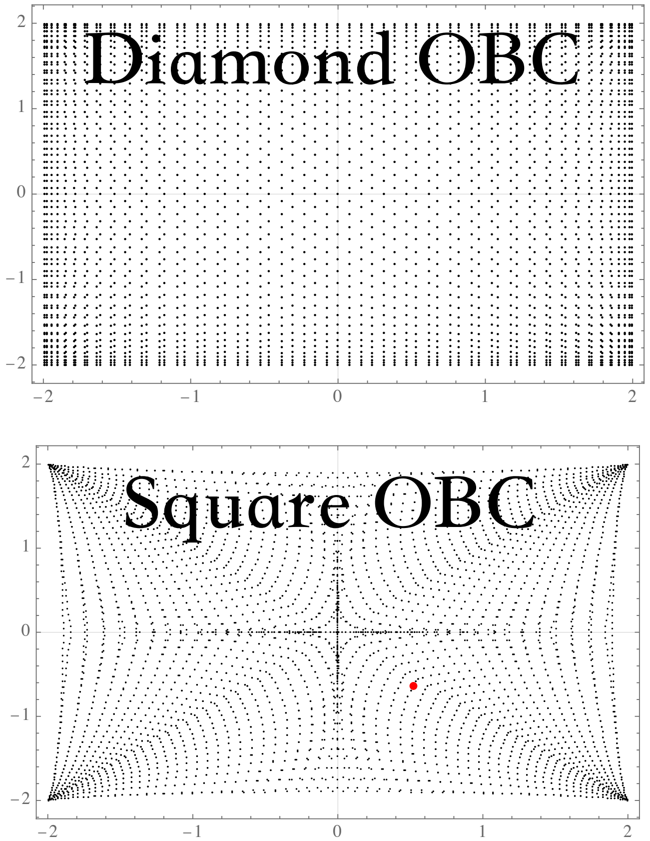
- ☐ **Question: what is the corresponding GBZ of this wave function?**

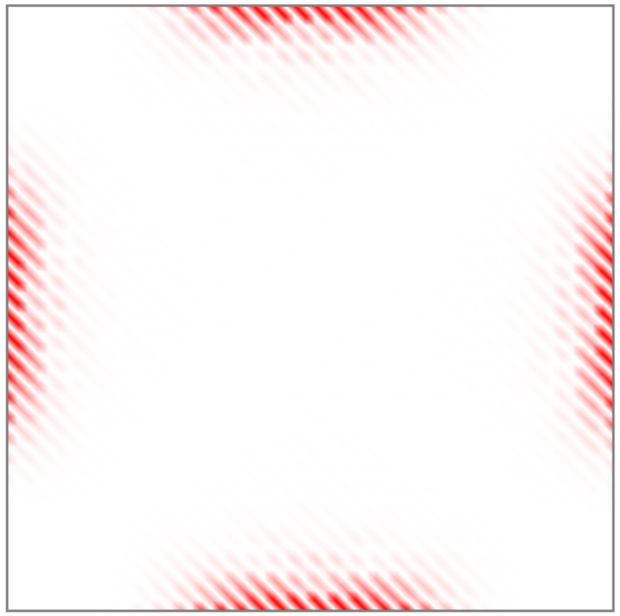
2D GBZ theory: WF approach

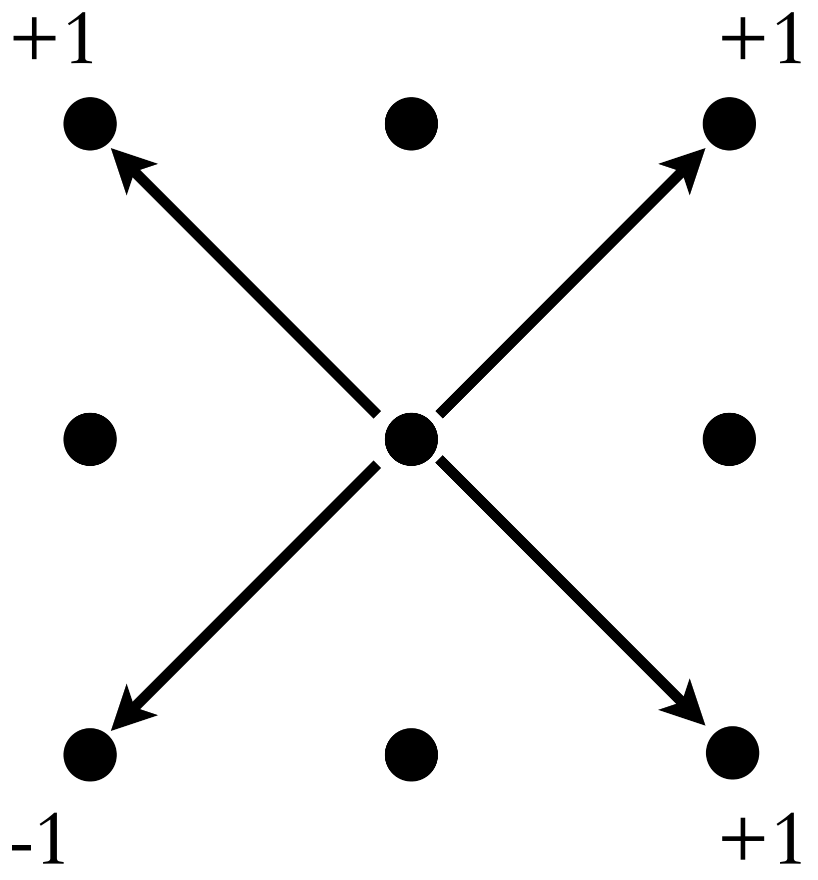
- **Wave function approach**
  - → We find such a minimal model that we want to solve

$$H_1(z_x, z_y) = -z_x z_y + z_x/z_y + z_y/z_x + 1/(z_x z_y).$$

  - → Typical wave function







- **Question: what is the corresponding GBZ of this wave function?**

2D GBZ theory: WF approach

□ Advantage

Notably, our approach offers two significant advantages that distinguish it from previous studies:

(i) **Direct GBZ Calculation:**

We can calculate the GBZ of $E_0 \in \sigma_G^{\mathrm{OBC}}$ diretly from the perspective of eigenstate wavefunction.

2D GBZ theory: WF approach

☐ Advantage

Notably, our approach offers two significant advantages that distinguish it from previous studies:

(i) **Direct GBZ Calculation:**

We can calculate the GBZ of $E_0 \in \sigma_G^{\mathrm{OBC}}$ diretly from the perspective of eigenstate wavefunction.

(ii) **Geometry Considerations:**

Our approach allows us to explicitly discuss the role of geometry.

2D GBZ theory: developments

- **Main difficulty**

  → Analytic diffuclty

  Now suppose that the Hamiltonian is $H(z_x, z_y)$ and the OBC geometry is G_0.

$$\det[H(z_x, z_y) - E_0] = 0. \tag{1}$$

  Here E_0 is an arbitrary compelx energy (not necessaey to be the OBC eigenvalues).

2D GBZ theory: developments

- **Main difficulty**

  → Analytic diffuclty

Now suppose that the Hamiltonian is $H(z_x, z_y)$ and the OBC geometry is G_0.

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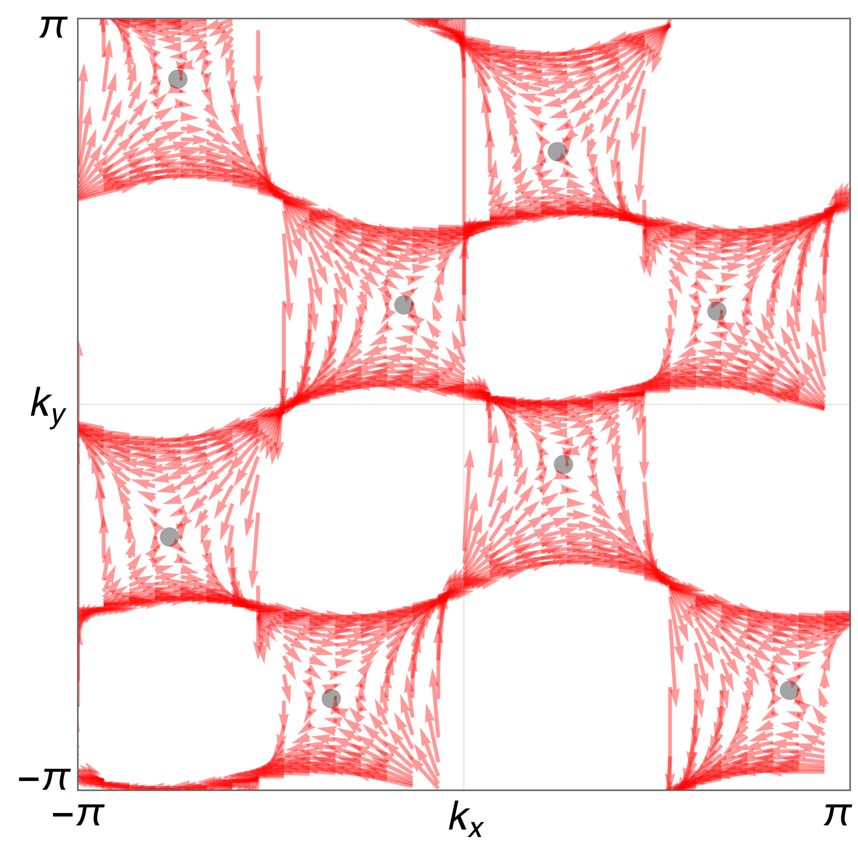
Here E_0 is an arbitrary compelx energy (not necessaey to be the OBC eigenvalues).

$$\left(\frac{2224}{4279} - \frac{1792\, i}{2797}\right) - \frac{1}{zx\, zy} - \frac{zx}{zy} - \frac{zy}{zx} + zx\, zy$$

$$z_x = e^{ik_x + \mu_x}, \qquad z_y = e^{ik_x + \mu_y}$$

$$(\mu_x(k_x, k_y), \mu_y(k_x, k_y))$$



2D GBZ theory: developments

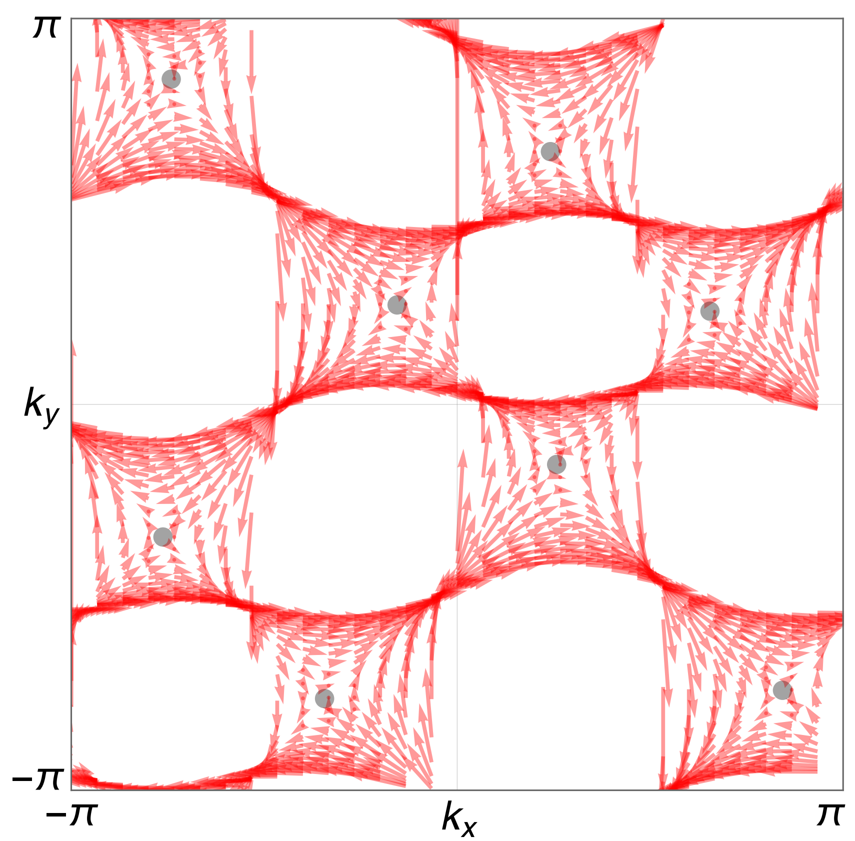
□ Main difficulty

➜ Analytic diffuclty

Defining the solution space of these characteristic equations as

$$F_H(E_0) := \{(z_x, z_y) \in C^2 | \det[E_0 - H(z_x, z_y)] = 0\}. \quad (2)$$

Then, each point in this solution space corresponds to a non-Bloch wave (or bulk solution), i.e. $|z_x, z_y\rangle$.



2D GBZ theory: developments

- **Main difficulty**
  - ➜ Analytic diffuclty

Second Step: The second step is to write down a linear superposition wavefunction, i.e., combining each non-Bloch wave linearly,

$$|\psi_{E_0,G}\rangle = \sum_{(z_x,z_y)\in F_H(E_0)} A_{E_0,G_0}(z_x, z_y)|z_x, z_y\rangle. \qquad (3)$$

Here $A_{E_0,G_0}(z_x, z_y)$ is the linear superposition coefficient.

2D GBZ theory: developments

- **Main difficulty**

  → Analytic diffuclty

It is crucial to recognize that, for a given E_0, there are infinity solutions for the following bulk equation

$$\det[H(z_x, z_y) - E_0] = 0.$$

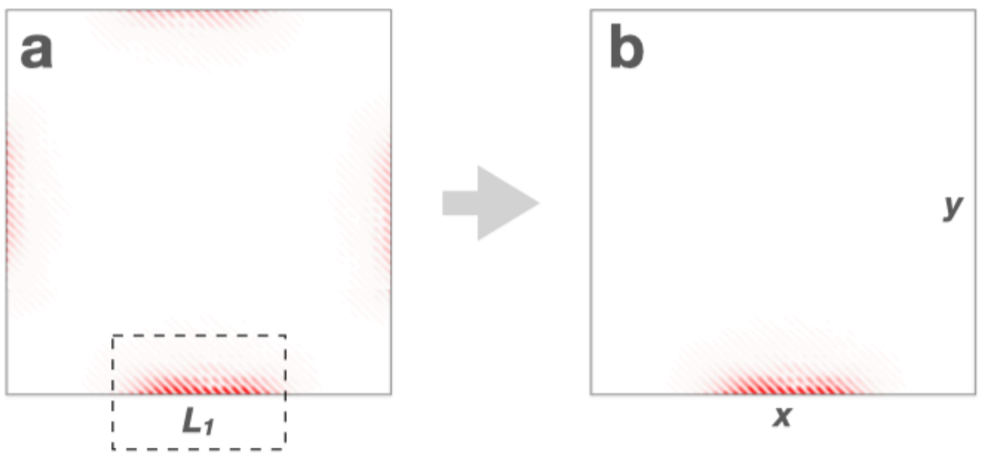
Consequently, in principle, there exists an infinite number of linear superposition wavefunctions in the following equation

$$|\psi_{E_0,G}\rangle = \sum_{(z_x,z_y)\in F_H(E_0)} A_{E_0,G_0}(z_x, z_y)|z_x, z_y\rangle.$$

However, for a system of finite size, we only have a finite number of boundary conditions.

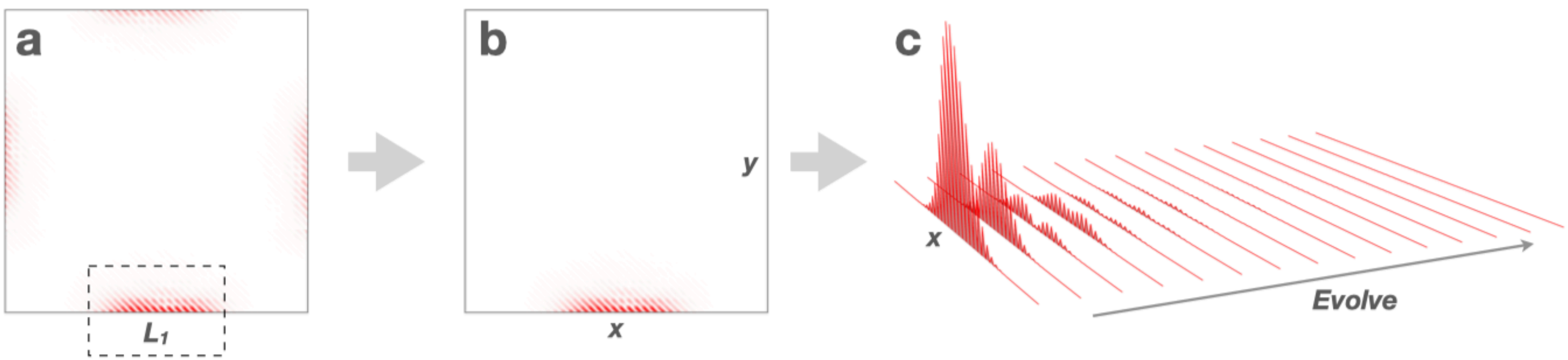
2D GBZ theory: WF approach

- **Dynamical duality method**
  - → Basic idea



2D GBZ theory: WF approach

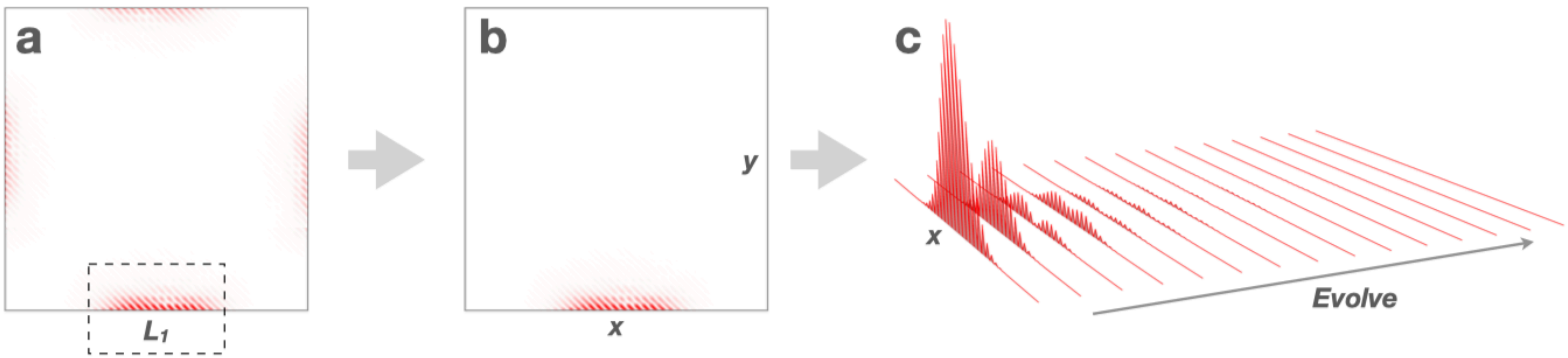
- **Dynamical duality method**
  - → Basic idea



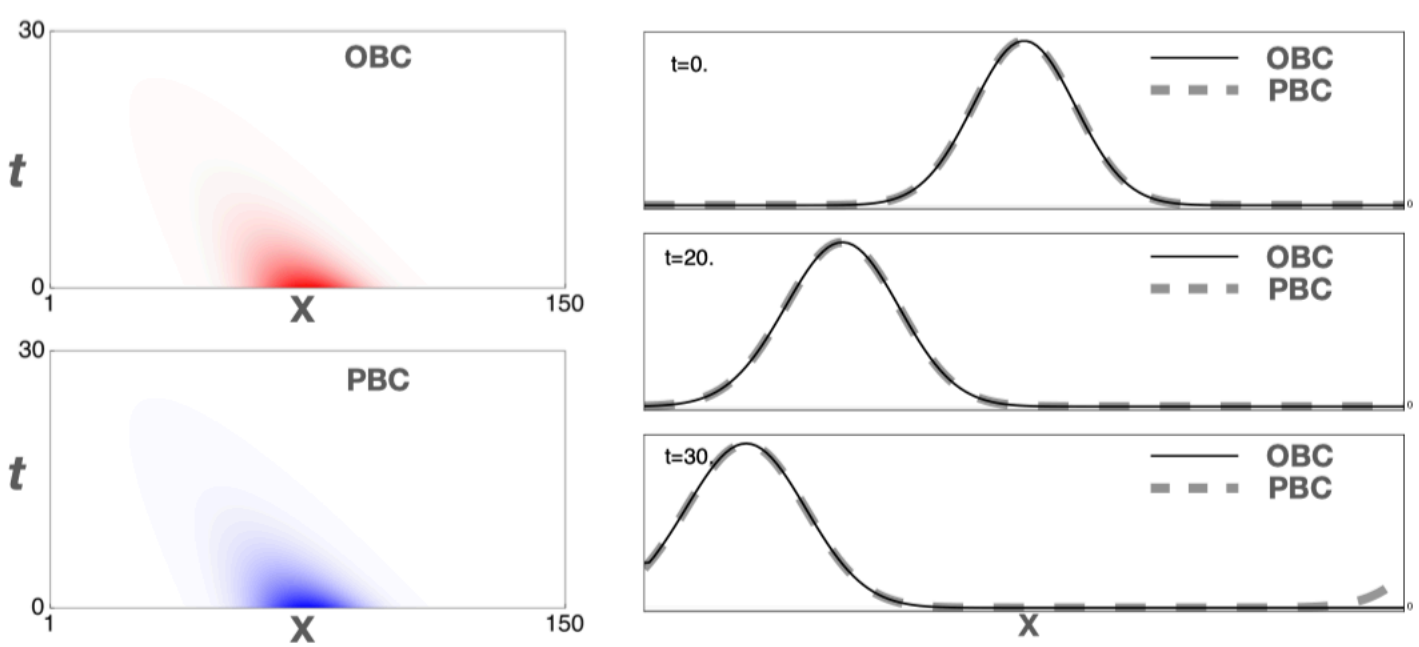
2D GBZ theory: WF approach

☐ Dynamical duality method

→ Basic idea



→ Observation



2D GBZ theory: WF approach

- **Dynamical duality method**
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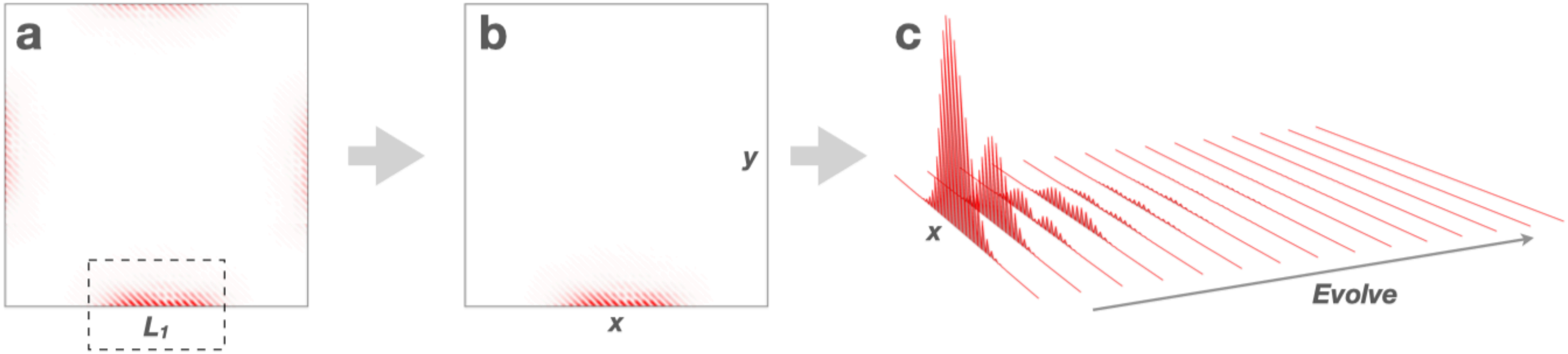
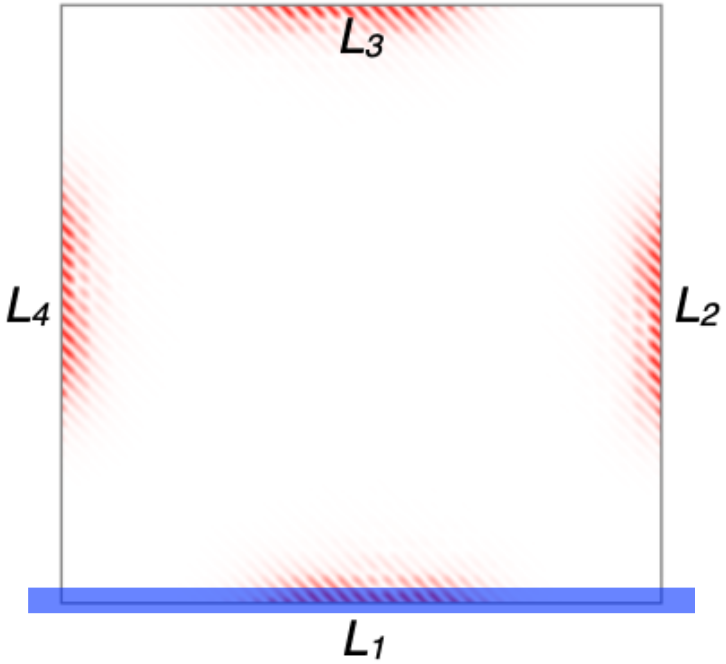


  - → Central question



2D GBZ theory: WF approach

- **Dynamical duality method**
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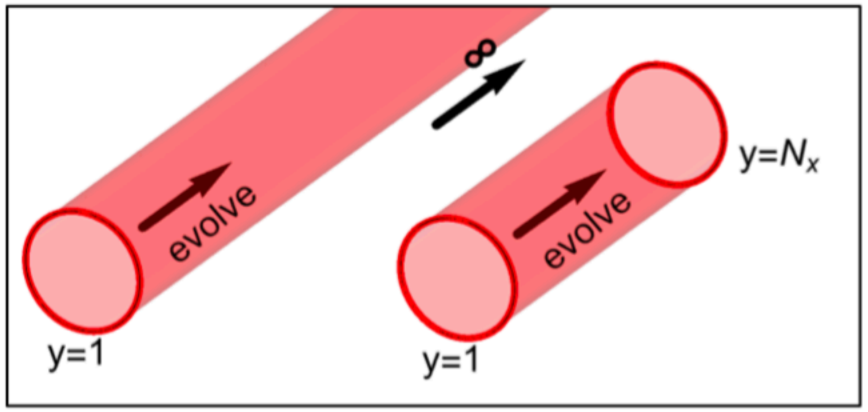
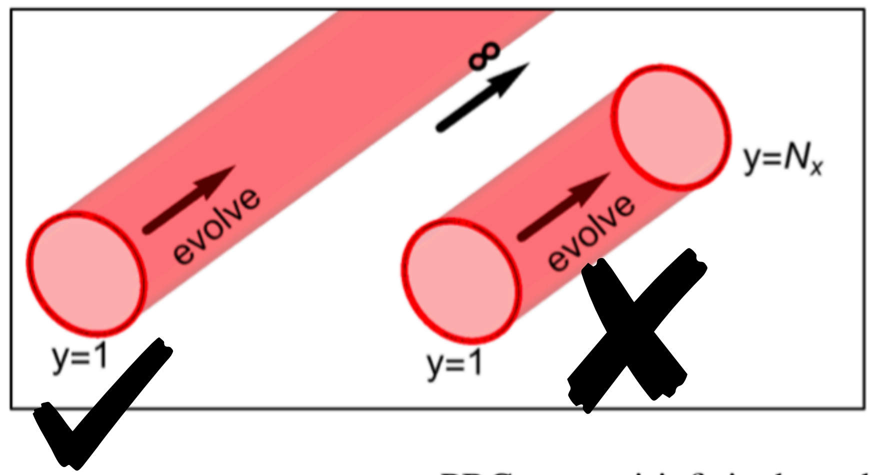
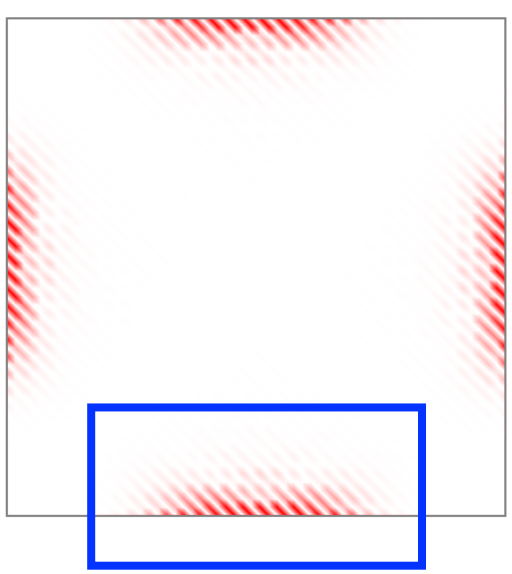


$$U_{k_x}^{\mathrm{I}//\mathrm{II}}(y) = \frac{1}{2\pi i} \oint_{\Gamma_{\mathrm{I}/\mathrm{II}}} \frac{dz_y}{z_y} \frac{z_y^y}{E_0 - H(z_x = e^{ik_x}, z_y) + i0^+}$$

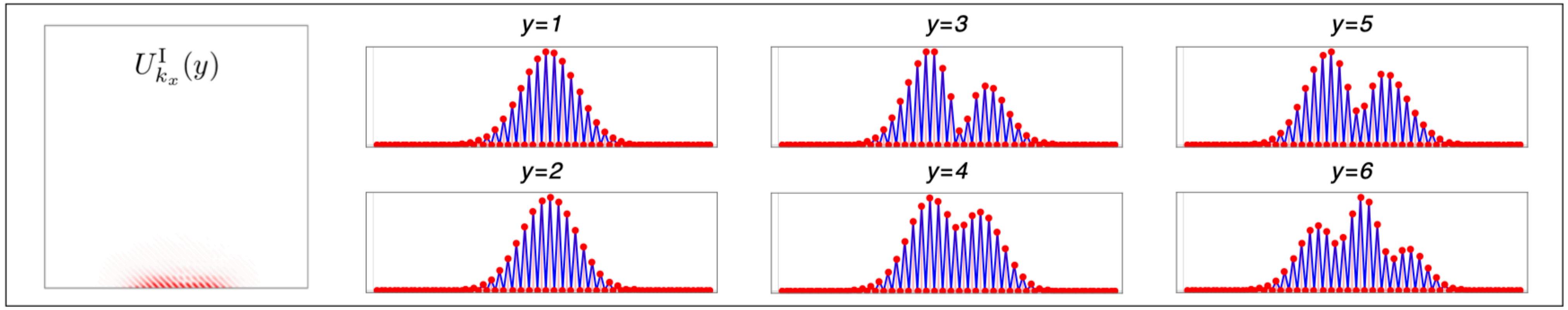
2D GBZ theory: WF approach

- **Dynamical duality method**
  - → Basic idea

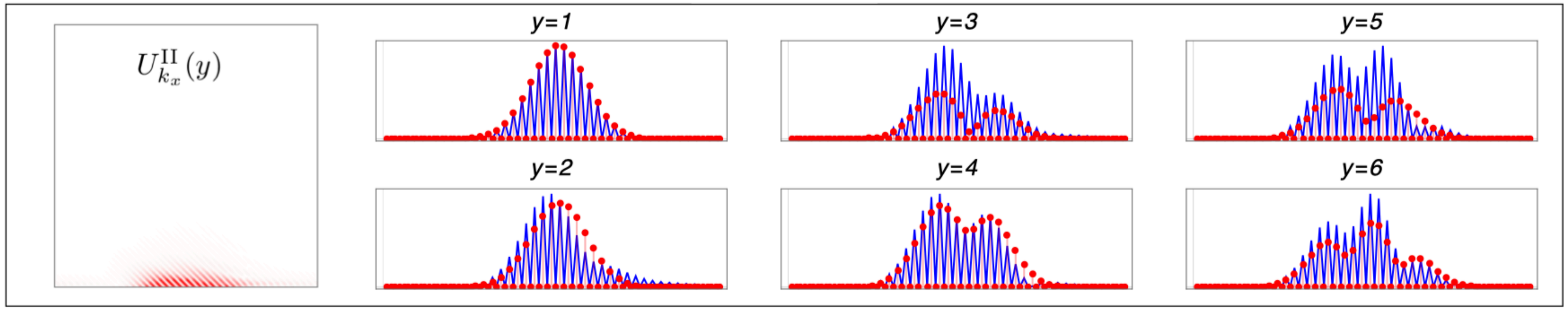




PBC$_x$ + semi-infinite boundary condition$_y$



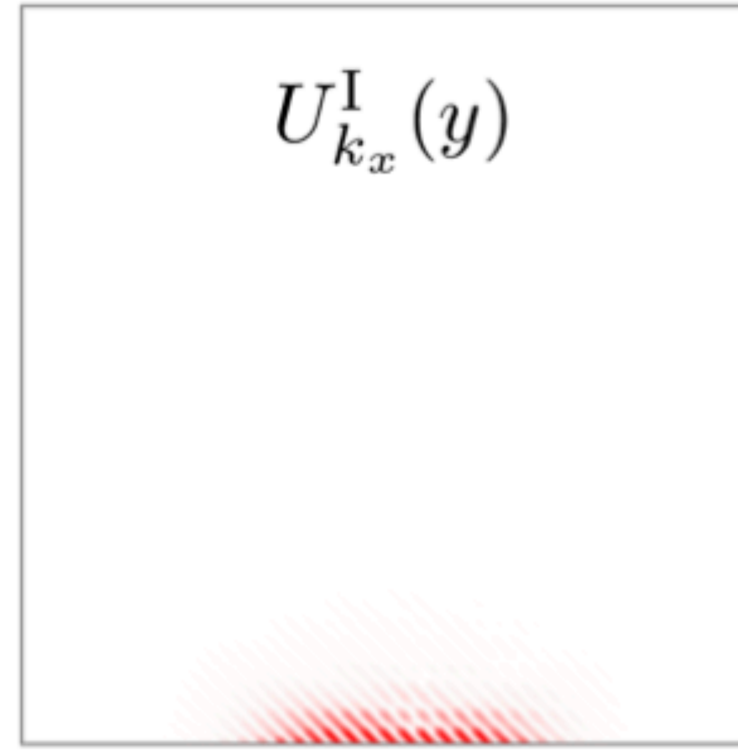
PBC$_x$ + OBC$_y$



2D GBZ theory: WF approach

- **Sub-GBZ**

  → Pick GBZ



$$
\begin{aligned}
&(-3.69767\times10^{-6} - 0.0000245306\,\mathrm{i})\,(-0.505522 + 0.791911\,\mathrm{i})^{y}\,\mathrm{e}^{\frac{4\mathrm{i}\pi x}{15}} + (3.69767\times10^{-6} + 0.0000245306\,\mathrm{i})\,(0.807816 - 0.515675\,\mathrm{i})^{y}\,\mathrm{e}^{\frac{4\mathrm{i}\pi x}{15}} +\\
&(6.68379\times10^{-7} + 4.69063\times10^{-6}\,\mathrm{i})\,(0.479609 + 0.489001\,\mathrm{i})^{y}\,\mathrm{e}^{-\frac{3}{10}\mathrm{i}\pi x} + (0.0000721745 + 0.000216555\,\mathrm{i})\,(-0.450439 + 0.713447\,\mathrm{i})^{y}\,\mathrm{e}^{\frac{3\mathrm{i}\pi x}{10}} -\\
&(0.0000721745 + 0.000216555\,\mathrm{i})\,(0.728119 - 0.459703\,\mathrm{i})^{y}\,\mathrm{e}^{\frac{3\mathrm{i}\pi x}{10}} - (2.91465\times10^{-6} - 0.0000192649\,\mathrm{i})\,(-0.682365 - 0.668668\,\mathrm{i})^{y}\,\mathrm{e}^{-\frac{1}{3}\mathrm{i}\pi x} +\\
&(2.91465\times10^{-6} - 0.0000192649\,\mathrm{i})\,(0.422964 + 0.431628\,\mathrm{i})^{y}\,\mathrm{e}^{-\frac{1}{3}\mathrm{i}\pi x} - (0.000568431 + 0.00118188\,\mathrm{i})\,(-0.394412 + 0.639979\,\mathrm{i})^{y}\,\mathrm{e}^{\frac{\mathrm{i}\pi x}{3}} +\\
&(0.000568431 + 0.00118188\,\mathrm{i})\,(0.653813 - 0.402938\,\mathrm{i})^{y}\,\mathrm{e}^{\frac{\mathrm{i}\pi x}{3}} + (0.0000506092 - 0.000154525\,\mathrm{i})\,(-0.610477 - 0.597314\,\mathrm{i})^{y}\,\mathrm{e}^{-\frac{11}{30}\mathrm{i}\pi x} -\\
&(0.0000506092 - 0.000154525\,\mathrm{i})\,(0.36457 + 0.372604\,\mathrm{i})^{y}\,\mathrm{e}^{-\frac{11}{30}\mathrm{i}\pi x} + (0.00260979 + 0.00423381\,\mathrm{i})\,(-0.335359 + 0.567939\,\mathrm{i})^{y}\,\mathrm{e}^{\frac{11\mathrm{i}\pi x}{30}} -\\
&(0.00260979 + 0.00423381\,\mathrm{i})\,(0.581267 - 0.343229\,\mathrm{i})^{y}\,\mathrm{e}^{\frac{11\mathrm{i}\pi x}{30}} - (0.000412168 - 0.000890787\,\mathrm{i})\,(-0.538057 - 0.525128\,\mathrm{i})^{y}\,\mathrm{e}^{-\frac{2}{5}\mathrm{i}\pi x} +\\
&(0.000412168 - 0.000890787\,\mathrm{i})\,(0.301849 + 0.30928\,\mathrm{i})^{y}\,\mathrm{e}^{-\frac{2}{5}\mathrm{i}\pi x} - (0.00789507 + 0.01051\,\mathrm{i})\,(-0.270314 + 0.493368\,\mathrm{i})^{y}\,\mathrm{e}^{\frac{2\mathrm{i}\pi x}{5}} + (0.00789507 + 0.01051\,\mathrm{i})\,(0.506522 - 0.277521\,\mathrm{i})^{y}\,\mathrm{e}^{\frac{2\mathrm{i}\pi x}{5}} +\\
&(0.00217967 - 0.0037283\,\mathrm{i})\,(-0.460407 - 0.447346\,\mathrm{i})^{y}\,\mathrm{e}^{-\frac{13}{30}\mathrm{i}\pi x} - (0.00217967 - 0.0037283\,\mathrm{i})\,(0.23074 + 0.237477\,\mathrm{i})^{y}\,\mathrm{e}^{-\frac{13}{30}\mathrm{i}\pi x} + (0.0171641 + 0.0191827\,\mathrm{i})\,(-0.19391 + 0.410134\,\mathrm{i})^{y}\,\mathrm{e}^{\frac{13\mathrm{i}\pi x}{30}} -\\
&(0.0171641 + 0.0191827\,\mathrm{i})\,(0.423576 - 0.200265\,\mathrm{i})^{y}\,\mathrm{e}^{\frac{13\mathrm{i}\pi x}{30}} - (0.00860175 - 0.0121296\,\mathrm{i})\,(-0.368421 - 0.354534\,\mathrm{i})^{y}\,\mathrm{e}^{-\frac{7}{15}\mathrm{i}\pi x} + (0.00860175 - 0.0121296\,\mathrm{i})\,(0.142536 + 0.14812\,\mathrm{i})^{y}\,\mathrm{e}^{-\frac{7}{15}\mathrm{i}\pi x} -\\
&(0.0312435 + 0.0293389\,\mathrm{i})\,(-0.0919021 + 0.302849\,\mathrm{i})^{y}\,\mathrm{e}^{\frac{7\mathrm{i}\pi x}{15}} + (0.0312435 + 0.0293389\,\mathrm{i})\,(0.317787 - 0.0964352\,\mathrm{i})^{y}\,\mathrm{e}^{\frac{7\mathrm{i}\pi x}{15}} + (0.0421258 - 0.046756\,\mathrm{i})\,(-0.224648 - 0.205283\,\mathrm{i})^{y}\,\mathrm{e}^{-\frac{1}{2}\mathrm{i}\pi x} -\\
&(0.0421258 - 0.0467562\,\mathrm{i})\,(0.224648 + 0.205283\,\mathrm{i})^{y}\,\mathrm{e}^{\frac{\mathrm{i}\pi x}{2}} - (0.0312434 + 0.0293389\,\mathrm{i})\,(-0.317787 + 0.0964352\,\mathrm{i})^{y}\,\mathrm{e}^{-\frac{8}{15}\mathrm{i}\pi x} + (0.0312434 + 0.0293389\,\mathrm{i})\,(0.0919021 - 0.302849\,\mathrm{i})^{y}\,\mathrm{e}^{-\frac{8}{15}\mathrm{i}\pi x} -\\
&(0.00860174 - 0.0121297\,\mathrm{i})\,(-0.142536 - 0.14812\,\mathrm{i})^{y}\,\mathrm{e}^{\frac{8\mathrm{i}\pi x}{15}} + (0.00860174 - 0.0121297\,\mathrm{i})\,(0.368421 + 0.354534\,\mathrm{i})^{y}\,\mathrm{e}^{\frac{8\mathrm{i}\pi x}{15}} + (0.017164 + 0.0191827\,\mathrm{i})\,(-0.423576 + 0.200265\,\mathrm{i})^{y}\,\mathrm{e}^{-\frac{17}{30}\mathrm{i}\pi x} -\\
&(0.017164 + 0.0191827\,\mathrm{i})\,(0.19391 - 0.410134\,\mathrm{i})^{y}\,\mathrm{e}^{-\frac{17}{30}\mathrm{i}\pi x} + (0.00217966 - 0.00372832\,\mathrm{i})\,(-0.23074 - 0.237477\,\mathrm{i})^{y}\,\mathrm{e}^{\frac{17\mathrm{i}\pi x}{30}} - (0.00217966 - 0.00372832\,\mathrm{i})\,(0.460407 + 0.447346\,\mathrm{i})^{y}\,\mathrm{e}^{\frac{17\mathrm{i}\pi x}{30}} -\\
&(0.00789503 + 0.0105099\,\mathrm{i})\,(-0.506522 + 0.277521\,\mathrm{i})^{y}\,\mathrm{e}^{-\frac{3}{5}\mathrm{i}\pi x} + (0.00789503 + 0.0105099\,\mathrm{i})\,(0.270314 - 0.493368\,\mathrm{i})^{y}\,\mathrm{e}^{-\frac{3}{5}\mathrm{i}\pi x} - (0.000412167 - 0.000890789\,\mathrm{i})\,(-0.301849 - 0.30928\,\mathrm{i})^{y}\,\mathrm{e}^{\frac{3\mathrm{i}\pi x}{5}} +\\
&(0.000412167 - 0.000890789\,\mathrm{i})\,(0.538057 + 0.525128\,\mathrm{i})^{y}\,\mathrm{e}^{\frac{3\mathrm{i}\pi x}{5}} + (0.00260977 + 0.00423381\,\mathrm{i})\,(-0.581267 + 0.343229\,\mathrm{i})^{y}\,\mathrm{e}^{-\frac{19}{30}\mathrm{i}\pi x} -\\
&(0.00260977 + 0.00423381\,\mathrm{i})\,(0.335359 - 0.567939\,\mathrm{i})^{y}\,\mathrm{e}^{-\frac{19}{30}\mathrm{i}\pi x} + (0.0000506089 - 0.000154526\,\mathrm{i})\,(-0.36457 - 0.372604\,\mathrm{i})^{y}\,\mathrm{e}^{\frac{19\mathrm{i}\pi x}{30}} -\\
&(0.0000506089 - 0.000154526\,\mathrm{i})\,(0.610477 + 0.597314\,\mathrm{i})^{y}\,\mathrm{e}^{\frac{19\mathrm{i}\pi x}{30}} - (0.000568427 + 0.00118188\,\mathrm{i})\,(-0.653813 + 0.402938\,\mathrm{i})^{y}\,\mathrm{e}^{-\frac{2}{3}\mathrm{i}\pi x} +\\
&(0.000568427 + 0.00118188\,\mathrm{i})\,(0.394412 - 0.639979\,\mathrm{i})^{y}\,\mathrm{e}^{-\frac{2}{3}\mathrm{i}\pi x} - (2.91461\times10^{-6} - 0.0000192649\,\mathrm{i})\,(-0.422964 - 0.431628\,\mathrm{i})^{y}\,\mathrm{e}^{\frac{2\mathrm{i}\pi x}{3}} +\\
&(2.91461\times10^{-6} - 0.0000192649\,\mathrm{i})\,(0.682365 + 0.668668\,\mathrm{i})^{y}\,\mathrm{e}^{\frac{2\mathrm{i}\pi x}{3}} + (0.0000721738 + 0.000216555\,\mathrm{i})\,(-0.728119 + 0.459703\,\mathrm{i})^{y}\,\mathrm{e}^{-\frac{7}{10}\mathrm{i}\pi x} -\\
&(0.0000721738 + 0.000216555\,\mathrm{i})\,(0.450439 - 0.713447\,\mathrm{i})^{y}\,\mathrm{e}^{-\frac{7}{10}\mathrm{i}\pi x} - (6.68391\times10^{-7} + 4.69064\times10^{-6}\,\mathrm{i})\,(-0.479609 - 0.489001\,\mathrm{i})^{y}\,\mathrm{e}^{\frac{7\mathrm{i}\pi x}{10}} -\\
&(3.69761\times10^{-6} + 0.0000245305\,\mathrm{i})\,(-0.807816 + 0.515675\,\mathrm{i})^{y}\,\mathrm{e}^{-\frac{11}{15}\mathrm{i}\pi x}\ \boxed{(3.69761\times10^{-6} + 0.0000245305\,\mathrm{i})\,(0.505522 - 0.791911\,\mathrm{i})^{y}\,\mathrm{e}^{-\frac{11}{15}\mathrm{i}\pi x}}
\end{aligned}
$$

$$\boxed{(3.69761\times10^{-6} + 0.0000245305\,\mathrm{i})\,(0.505522 - 0.791911\,\mathrm{i})^{y}\,\mathrm{e}^{-\frac{11}{15}\mathrm{i}\pi x}}$$

→ Subset of $F_H(E_0) := \{(z_x, z_y) \in C^2 | \det[E_0 - H(z_x, z_y)] = 0\}. \quad (2)$

2D GBZ theory: WF approach

- **Sub-GBZ**

→ Pick GBZ

$$\left\{\left\{\frac{4\pi}{15}, 2.13894-0.0623983\,i\right\}, \left\{\frac{4\pi}{15}, -0.568147-0.0425134\,i\right\}, \left\{-\frac{3\pi}{10}, 0.795094-0.37842\,i\right\}, \left\{\frac{3\pi}{10}, 2.13395-0.169908\,i\right\}, \left\{\frac{3\pi}{10}, -0.563157-0.149551\,i\right\}, \left\{-\frac{\pi}{3}, -2.36633-0.0456525\,i\right\},\right.$$

$$\left\{-\frac{\pi}{3}, 0.795536-0.503654\,i\right\}, \left\{\frac{\pi}{3}, 2.12311-0.285346\,i\right\}, \left\{\frac{\pi}{3}, -0.552311-0.26396\,i\right\}, \left\{-\frac{11\pi}{30}, -2.36709-0.157721\,i\right\}, \left\{-\frac{11\pi}{30}, 0.796296-0.651446\,i\right\},$$

$$\left\{\frac{11\pi}{30}, 2.10419-0.416181\,i\right\}, \left\{\frac{11\pi}{30}, -0.533393-0.392986\,i\right\}, \left\{-\frac{2\pi}{5}, -2.36835-0.28523\,i\right\}, \left\{-\frac{2\pi}{5}, 0.797558-0.838947\,i\right\}, \left\{\frac{2\pi}{5}, 2.07202-0.575245\,i\right\},$$

$$\left\{\frac{2\pi}{5}, -0.501225-0.548932\,i\right\}, \left\{-\frac{13\pi}{30}, -2.37058-0.443253\,i\right\}, \left\{-\frac{13\pi}{30}, 0.799785-1.10529\,i\right\}, \left\{\frac{13\pi}{30}, 2.01244-0.790398\,i\right\}, \left\{\frac{13\pi}{30}, -0.441648-0.758149\,i\right\},$$

$$\left\{-\frac{7\pi}{15}, -2.3754-0.670798\,i\right\}, \left\{-\frac{7\pi}{15}, 0.804606-1.58201\,i\right\}, \left\{\frac{7\pi}{15}, 1.86542-1.15047\,i\right\}, \left\{\frac{7\pi}{15}, -0.294626-1.10233\,i\right\}, \left\{-\frac{\pi}{2}, -2.40121-1.18969\,i\right\}, \left\{\frac{\pi}{2}, 0.740388-1.18969\,i\right\},$$

$$\left\{-\frac{8\pi}{15}, 2.84697-1.10233\,i\right\}, \left\{-\frac{8\pi}{15}, -1.27617-1.15047\,i\right\}, \left\{\frac{8\pi}{15}, -2.33699-1.58201\,i\right\}, \left\{\frac{8\pi}{15}, 0.766191-0.670798\,i\right\}, \left\{-\frac{17\pi}{30}, 2.69994-0.758149\,i\right\},$$

$$\left\{-\frac{17\pi}{30}, -1.12915-0.790398\,i\right\}, \left\{\frac{17\pi}{30}, -2.34181-1.10529\,i\right\}, \left\{\frac{17\pi}{30}, 0.771011-0.443253\,i\right\}, \left\{-\frac{3\pi}{5}, 2.64037-0.548932\,i\right\}, \left\{-\frac{3\pi}{5}, -1.06957-0.575245\,i\right\},$$

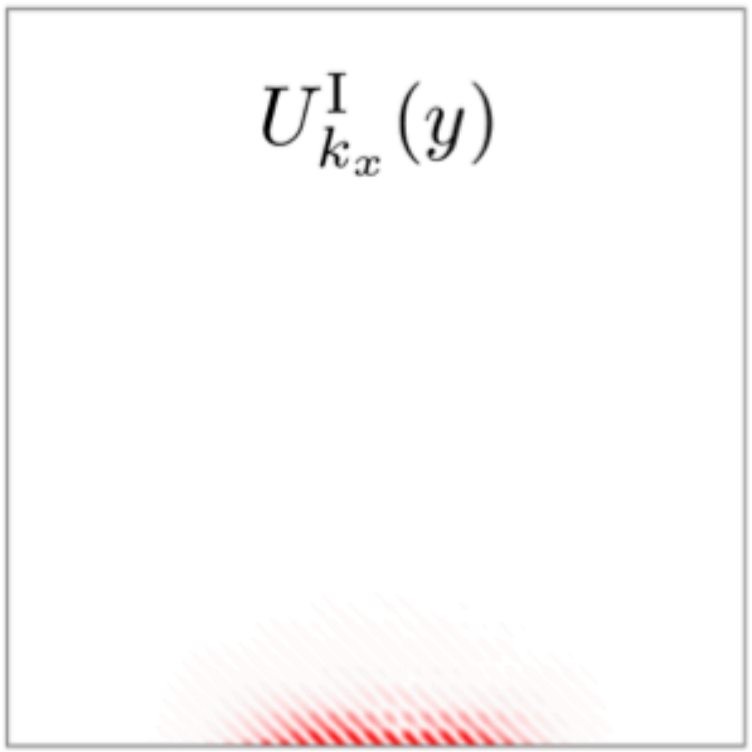
$$\left\{\frac{3\pi}{5}, -2.34403-0.838947\,i\right\}, \left\{\frac{3\pi}{5}, 0.773238-0.28523\,i\right\}, \left\{-\frac{19\pi}{30}, 2.6082-0.392986\,i\right\}, \left\{-\frac{19\pi}{30}, -1.0374-0.416181\,i\right\}, \left\{\frac{19\pi}{30}, -2.3453-0.651446\,i\right\},$$

$$\left\{\frac{19\pi}{30}, 0.7745-0.157721\,i\right\}, \left\{-\frac{2\pi}{3}, 2.58928-0.26396\,i\right\}, \left\{-\frac{2\pi}{3}, -1.01849-0.285346\,i\right\}, \left\{\frac{2\pi}{3}, -2.34606-0.503654\,i\right\}, \left\{\frac{2\pi}{3}, 0.775261-0.0456525\,i\right\},$$

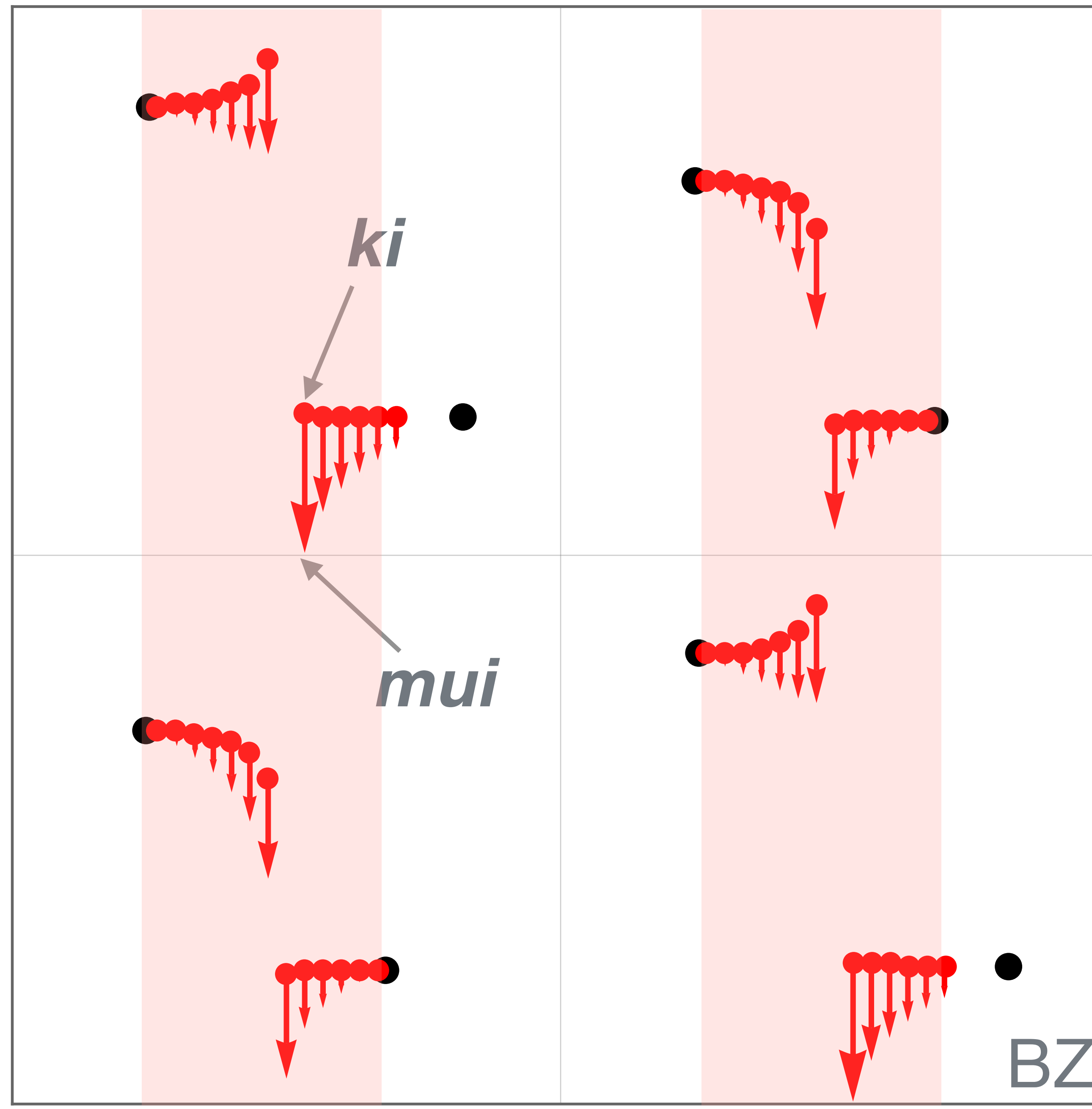
$$\left.\left\{-\frac{7\pi}{10}, 2.57844-0.149551\,i\right\}, \left\{-\frac{7\pi}{10}, -1.00764-0.169908\,i\right\}, \left\{\frac{7\pi}{10}, -2.3465-0.37842\,i\right\}, \left\{-\frac{11\pi}{15}, 2.57345-0.0425134\,i\right\}, \left\{-\frac{11\pi}{15}, -1.00265-0.0623983\,i\right\}\right\}$$

2D GBZ theory: WF approach

- **Sub-GBZ**
  - ➔ Pick GBZ

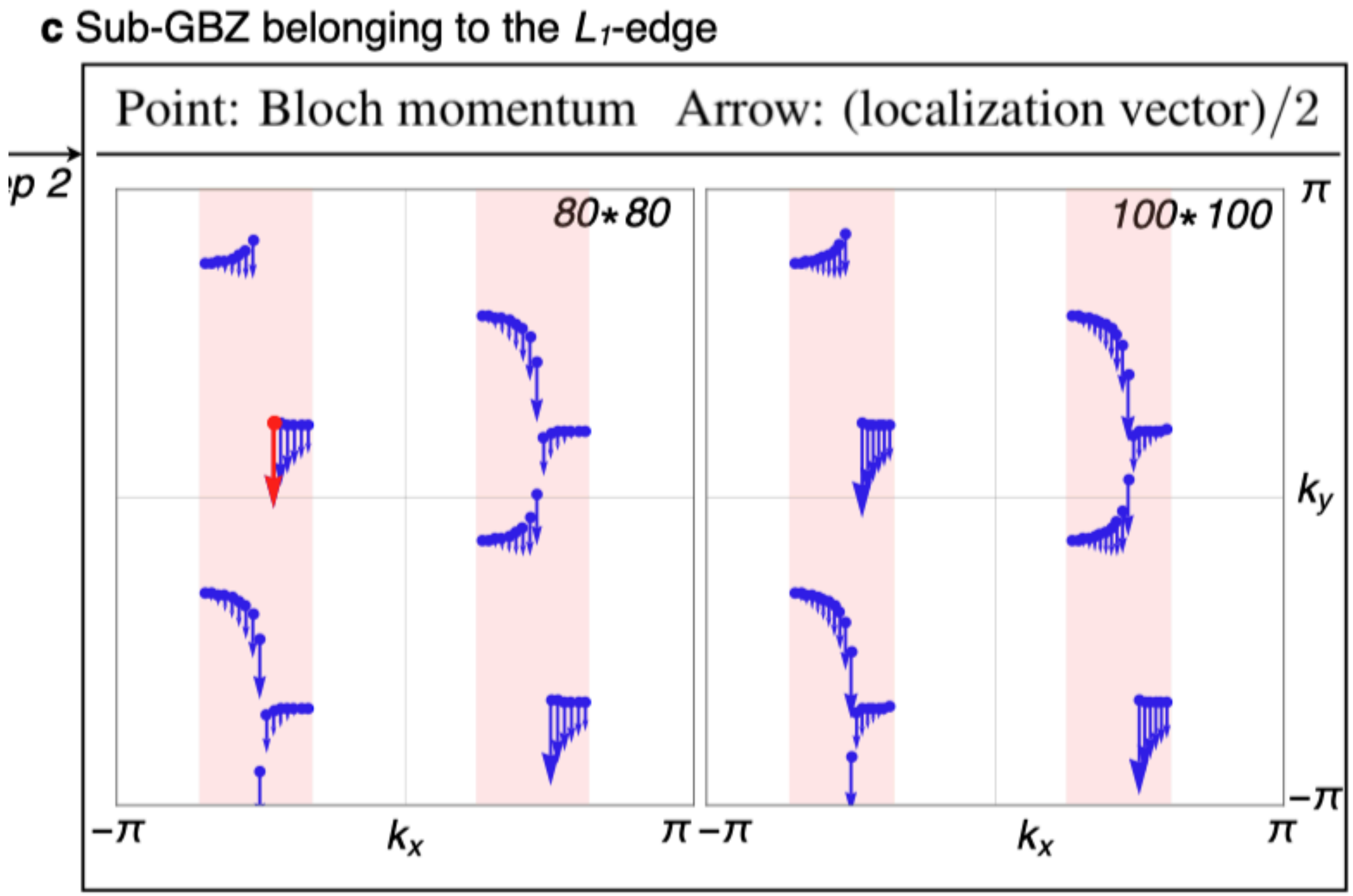


  - ➔ Sub-GBZ
  - ➔ mux=0



2D GBZ theory: WF approach

- **Sub-GBZ**
  - → Increase lattice size



How to calculate the result with N-> infinity??

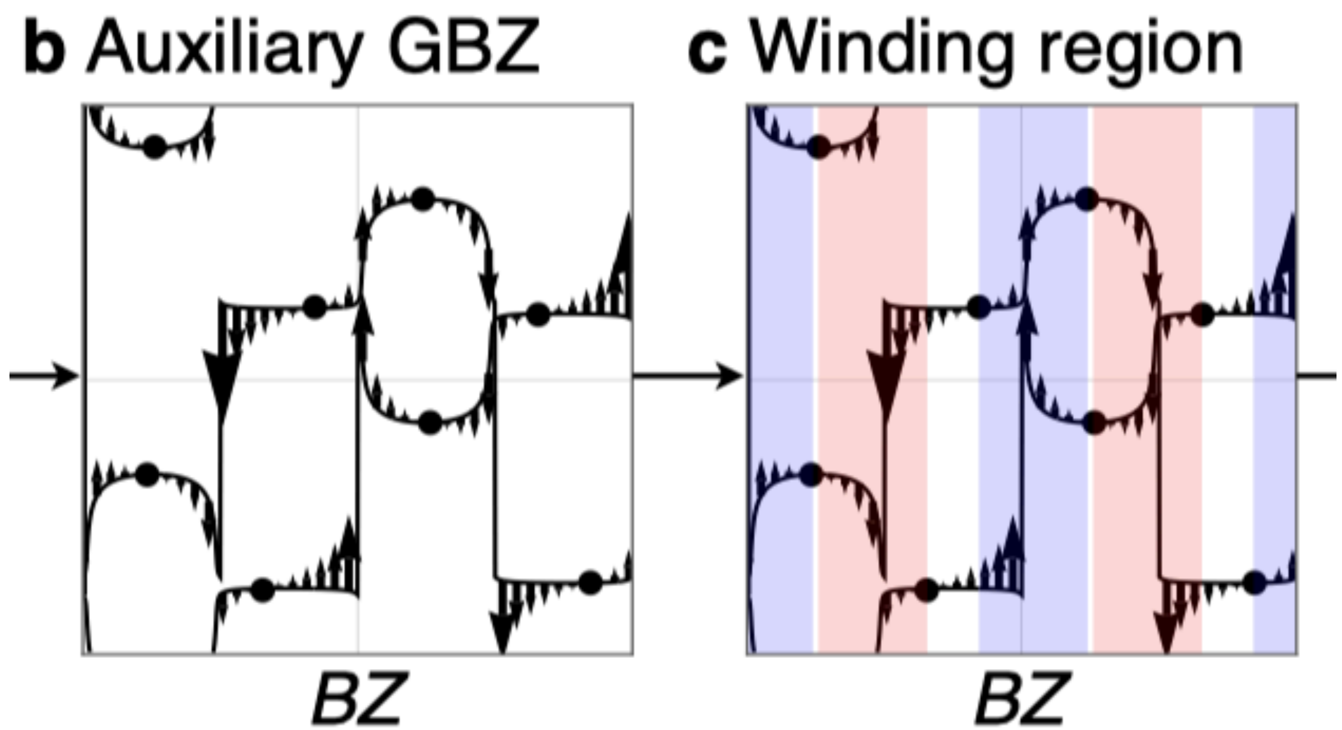
2D GBZ theory: WF approach

- **Sub-GBZ**
  - → Auxiliary GBZ

$$\det[E_0 - H(e^{ik_x}, e^{ik_y+\mu_y})] = 0,$$

2D GBZ theory: WF approach

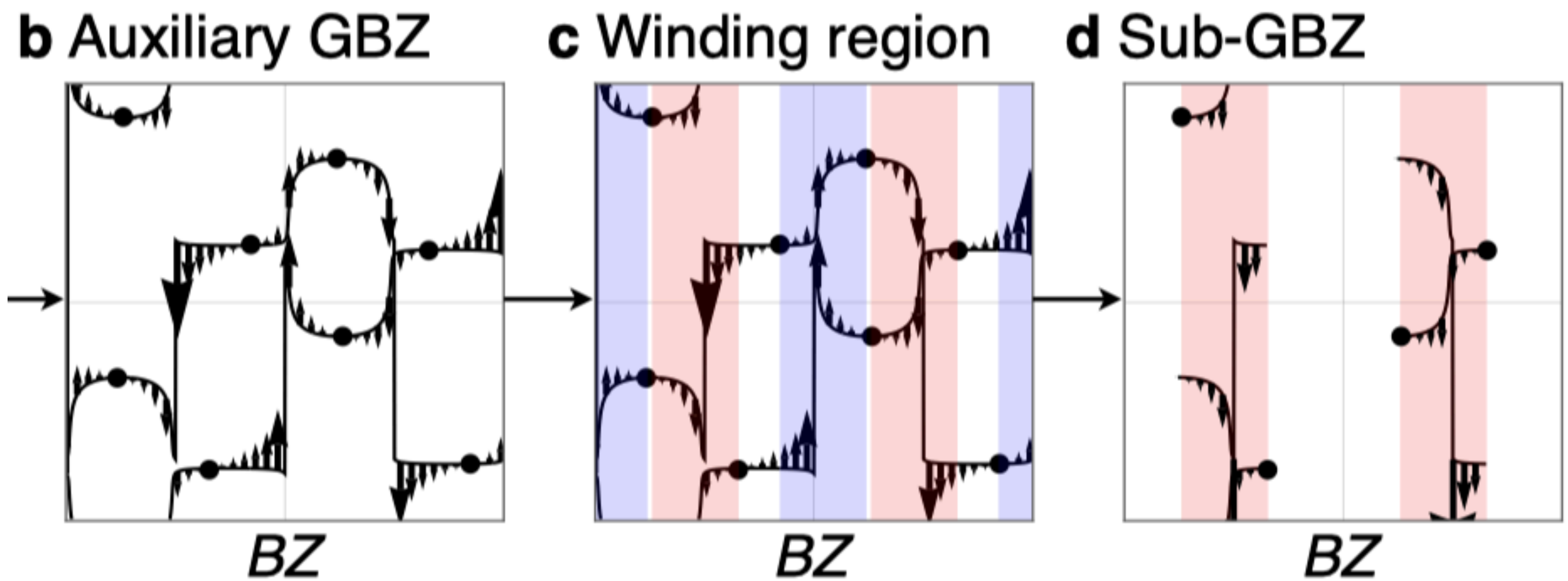
- **Sub-GBZ**
  - → Auxiliary GBZ



$$\nu_{L_1}(k_x) = \frac{1}{2\pi i}\int_{-\pi}^{\pi} \mathrm{d}k_y \partial_{k_y} \ln\det[E_0 - H(e^{ik_x}, e^{ik_y})].$$

2D GBZ theory: WF approach

- **Sub-GBZ**
  - → Auxiliary GBZ



  - → GBZ

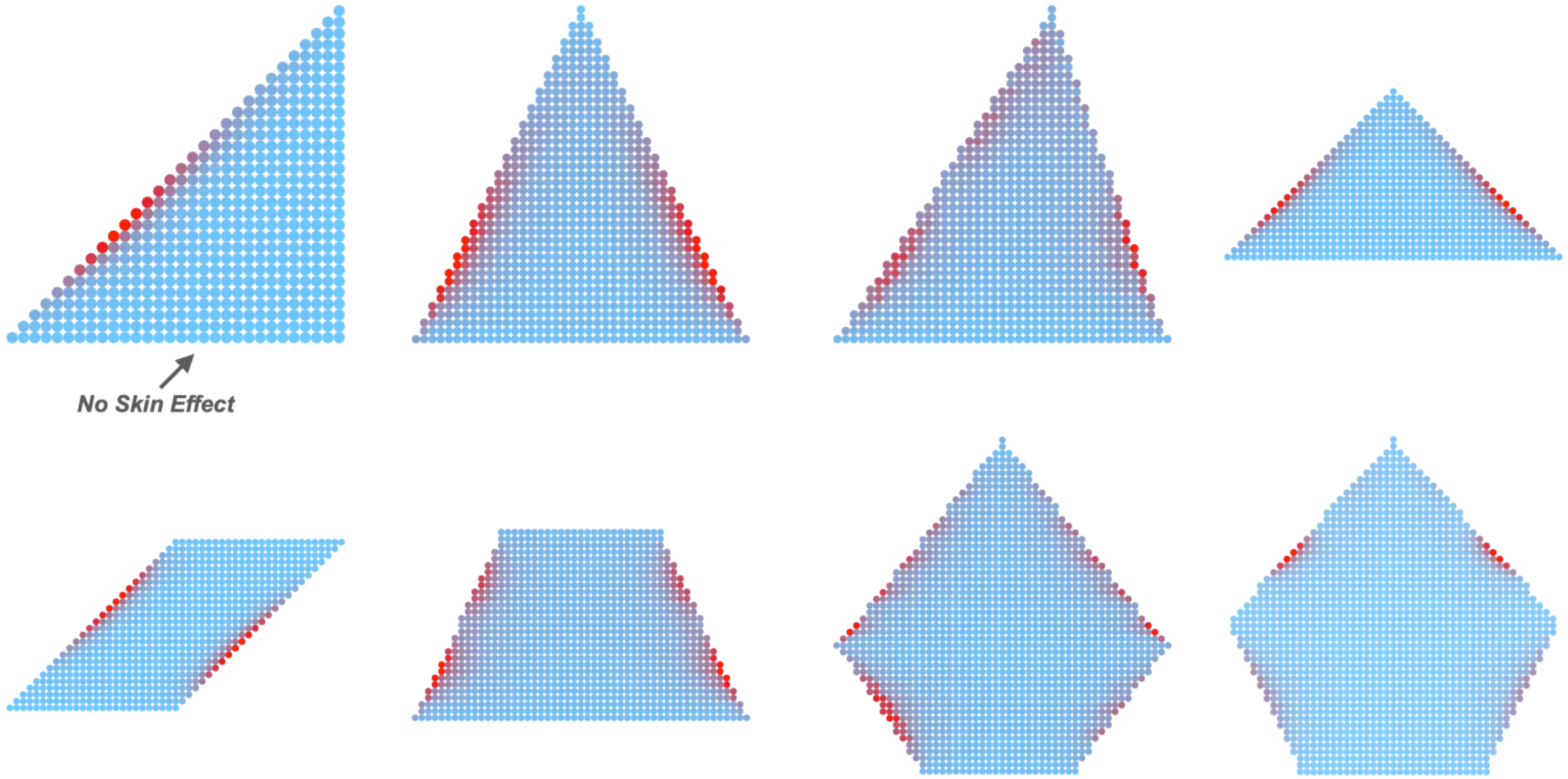
$$\beta_{E_0,G_{sq}} = \beta_{E_0,L_1} \cup \beta_{E_0,L_2} \cup \beta_{E_0,L_3} \cup \beta_{E_0,L_4},$$

  - → The role of OBC goeometry
  - → OBC spectrum

2D GBZ theory: WF approach

- **Geometry independent quantities** Zhang, ZY, and Fang, Hu NC 13, 2496 (2022).

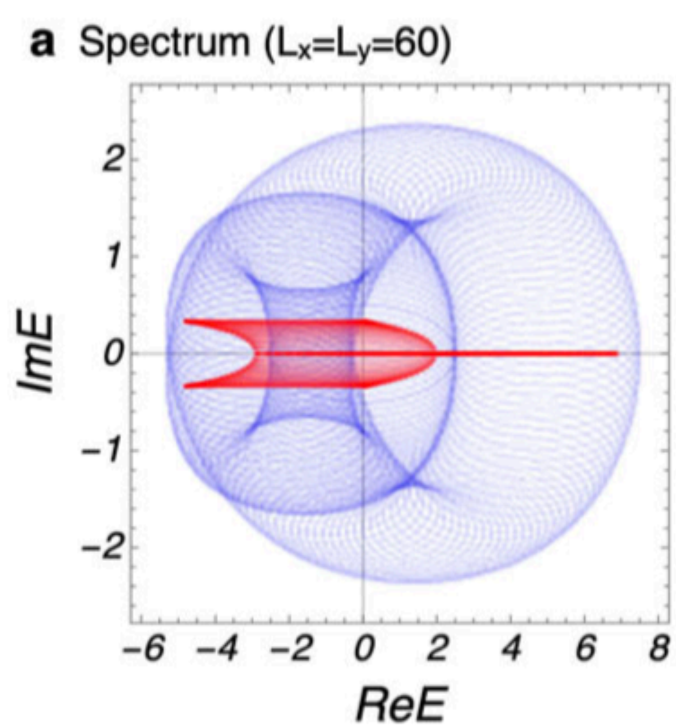
→ Hint 2: Particular edge



2D GBZ theory: numerical summary

- **GRSE v.s. NRSE** — Zhang, ZY, and Fang, Hu NC 13, 2496 (2022).

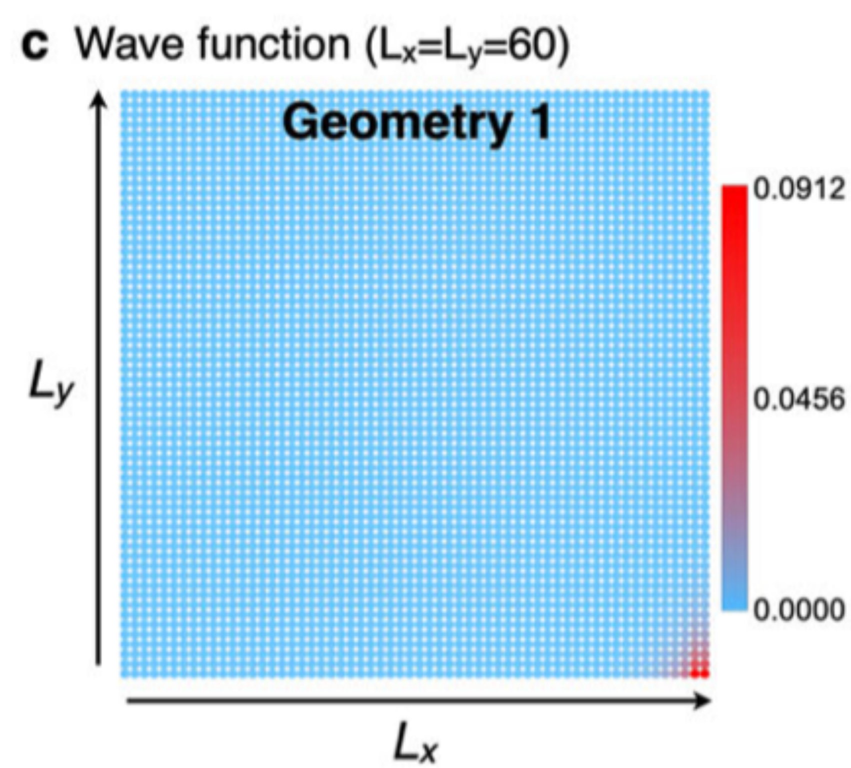
  → Two types of NHSE: non-reciprocal skin effect



→ Why coverage regions

$$\sigma^{\mathrm{PBC}} \neq \sigma^{\mathrm{OBC}}$$

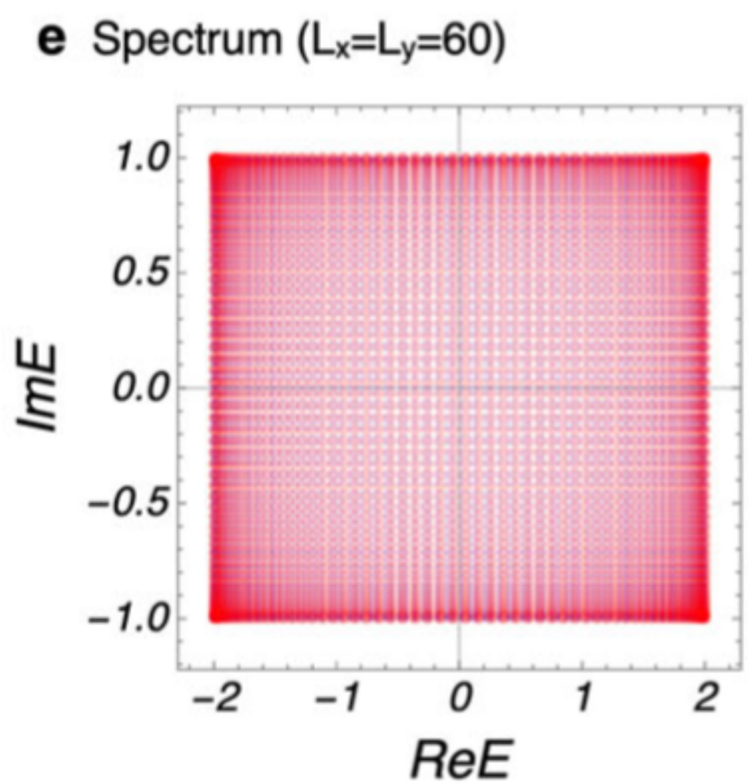
→ Why corner localization



2D GBZ theory: numerical summary

- **GRSE v.s. NRSE** Zhang, ZY, and Fang, Hu NC 13, 2496 (2022).

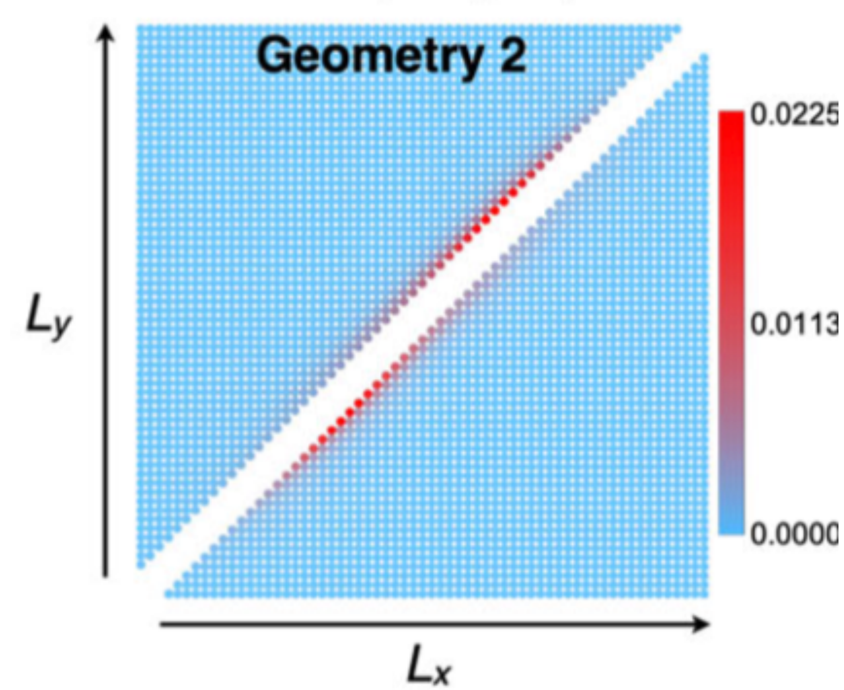
→ Two types of NHSE: generalized reciprocal skin effect



→ Why common coverage regions

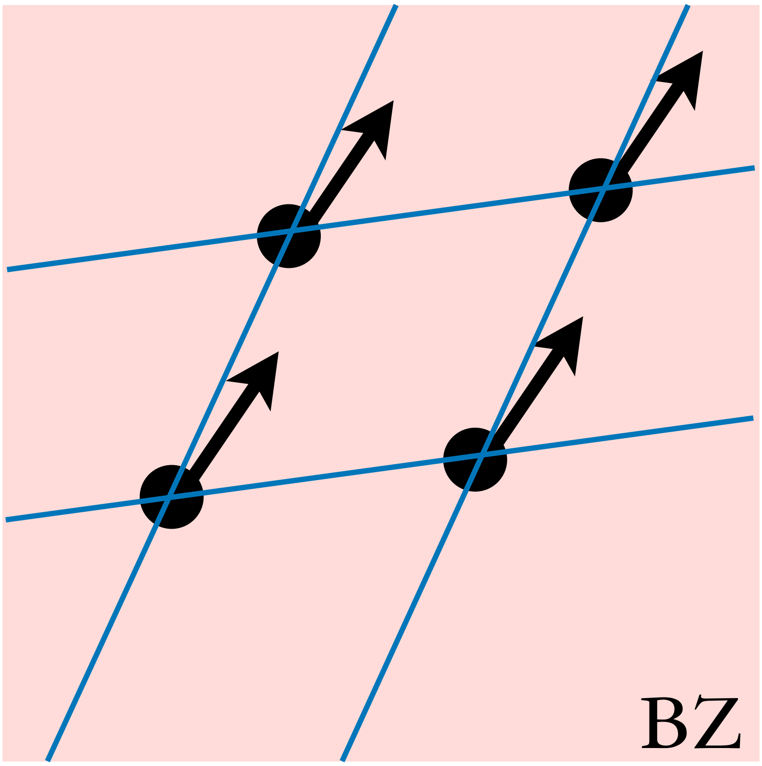
$$\text{GRSE}: \ \sigma^{\text{PBC}} = \sigma^{\text{OBC}}$$

→ Why edge localization



2D GBZ theory: WF approach

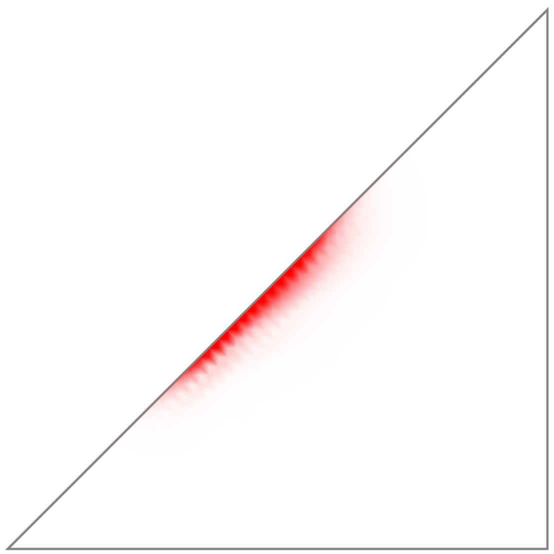
- **Puzzles**
  - → Puzzle 1: Scattering channel and standing wave

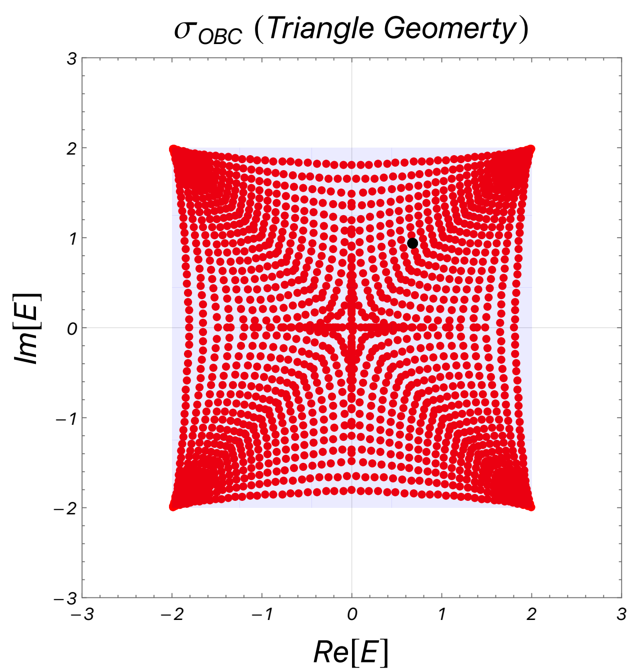


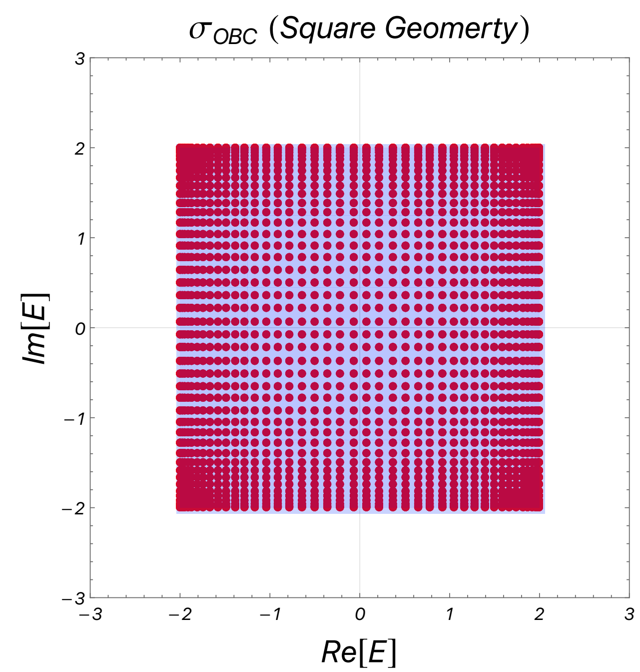
2D GBZ theory: WF approach

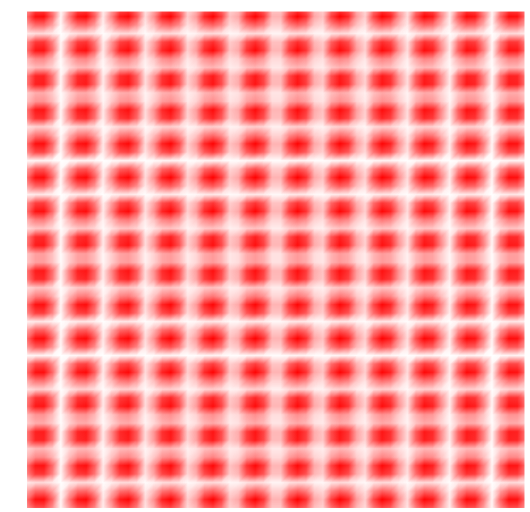
☐ Puzzles

→ Puzzle 2: Geometry dependent skin effect

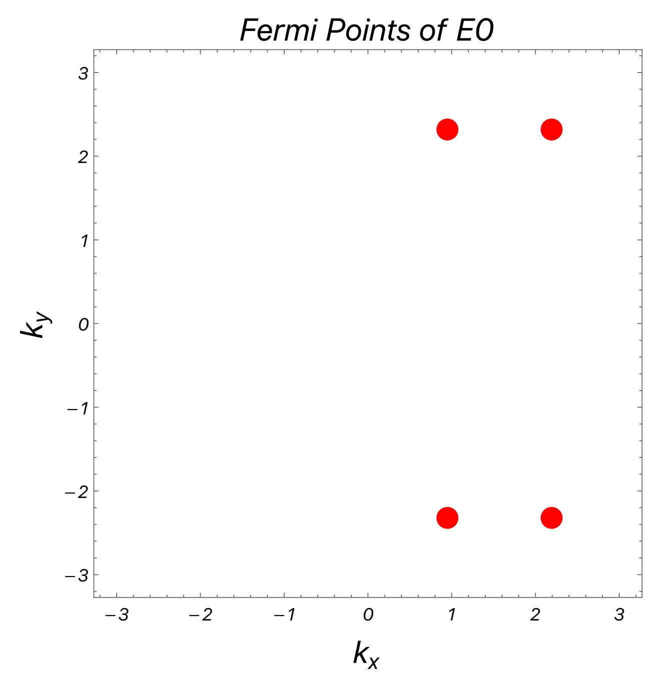








????



2D GBZ theory: WF approach

- **Puzzles**
  - → Puzzle 3: OBC spectral coverage

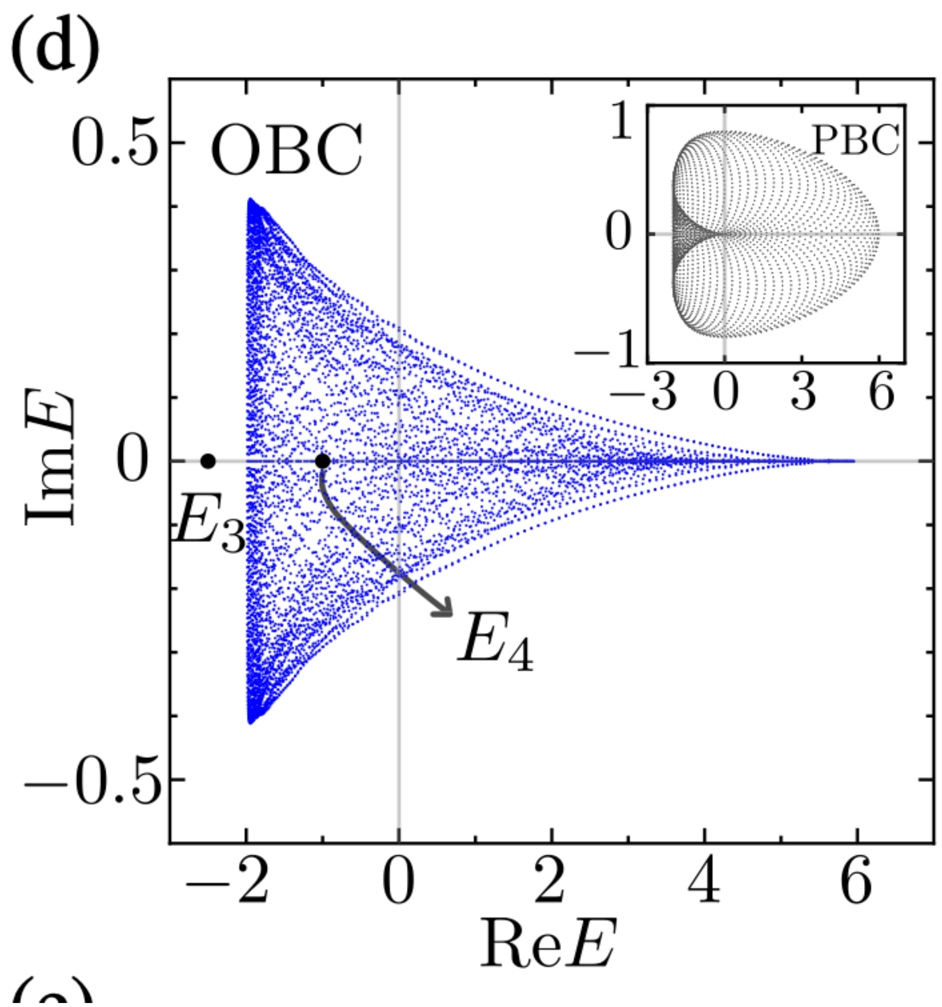


Figure from Amoeba theory

2D GBZ theory: WF approach

□ Summary

→ Our basic physical picture on the 2D GBZ

2D GBZ

Geometry part

Depend on the OBC geometry shapes

Determine: DoS of OBC spectrum,
Localization details of the OBC eigenstate

Intrinsic part

Determine: coverage region of the OBC spectrum,
Robust edge localizations

2D GBZ theory: WF approach

- **Questions related to the 2D GBZ**

For a given 2D non-Hermitian Hamiltonian within a given OBC geometry, denoted as G_0, the GBZ theory should answer:

1. what is the coverage region of the OBC spectrum, denoted by $\sigma_{G_0}^{\mathrm{OBC}}$?
2. what is the density of states on the OBC spectrum?
3. for a given $E_0 \in \sigma_{G_0}^{\mathrm{OBC}}$, what is the corresponding OBC eigenstate and GBZ?
4. when the OBC geometry G_0 undergoes changes, how do the above three quantities change accordingly, and is there a fundamental rule to identify the corresponding changes?

2D GBZ theory: WF approach

□ Summary

Our central conclusion is that the 2D GBZ comprises an intrinsic component dubbed as the **intrinsic GBZ**, alongside a geometry-dependent counterpart referred to as the **geometry-dependent GBZ**.

Firstly, for the intrinsic GBZ, we have identified these parts as the **Fermi points** for the Generalized Reciprocal Skin Effect (GRSE) and the **non-Bloch Fermi points** for the Non-Reciprocal Skin Effect (NRSE). Physically, the intrinsic GBZ plays a pivotal role in determining the coverage of the OBC spectrum σ_G^{OBC}, which remarkably remains invariant under variations of OBC geometry G.

Secondly, the geometry-dependent GBZ has been proved corresponding to the non-Bloch Equal Frequency Contours (**non-Bloch EFCs**), which can be effectively approximated using the **asymptotic GBZ theory**. Notably, as the OBC geometry evolves, the corresponding geometry-dependent GBZ undergoes changes, influencing not only the density of states within σ_G^{OBC} but also the localization properties of the associated eigenstate wavefunctions. This explains the geometrical dependent behaviours of the 2D NHSE.

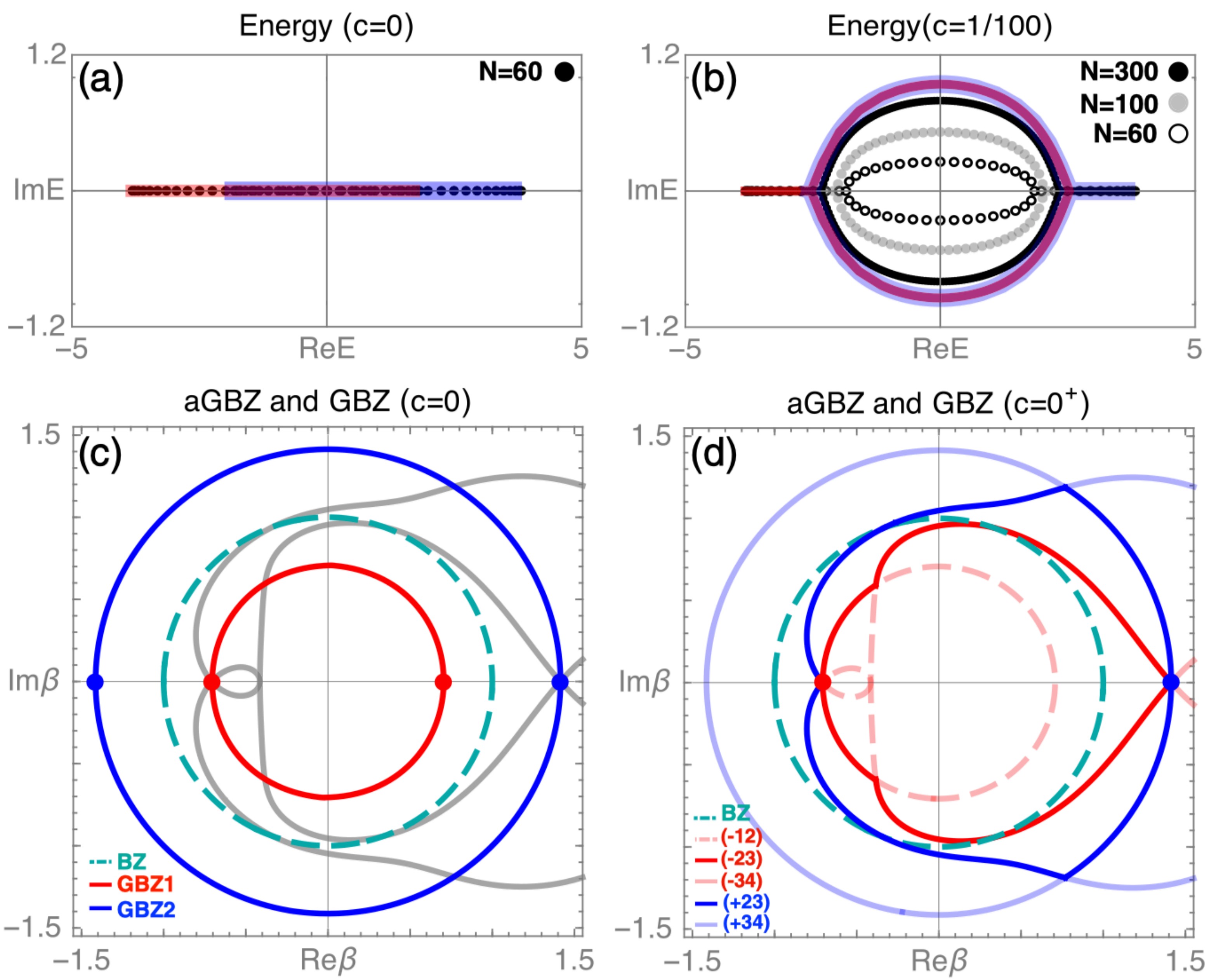
Thans for your attention

1D GBZ theory: review

- 1D GBZ calculation: numerical method
  - → Critical skin effect

$$\mathcal{H}(\beta) = \begin{pmatrix} t_0 + t_{-1}/\beta + t_1\beta & c \\ c & w_0 + w_{-1}/\beta + w_1\beta \end{pmatrix}, \quad (6)$$

with $t_0 = 4, t_1 = t_{-1} = 1, w_0 = -2, w_1 = 3, w_{-1} = 1, c = -1.$



Okuma, Sato, PRL
Li, Lee, Mu, Gong, NC