# "Entanglement Entropy" and phase transition in classical statistical models

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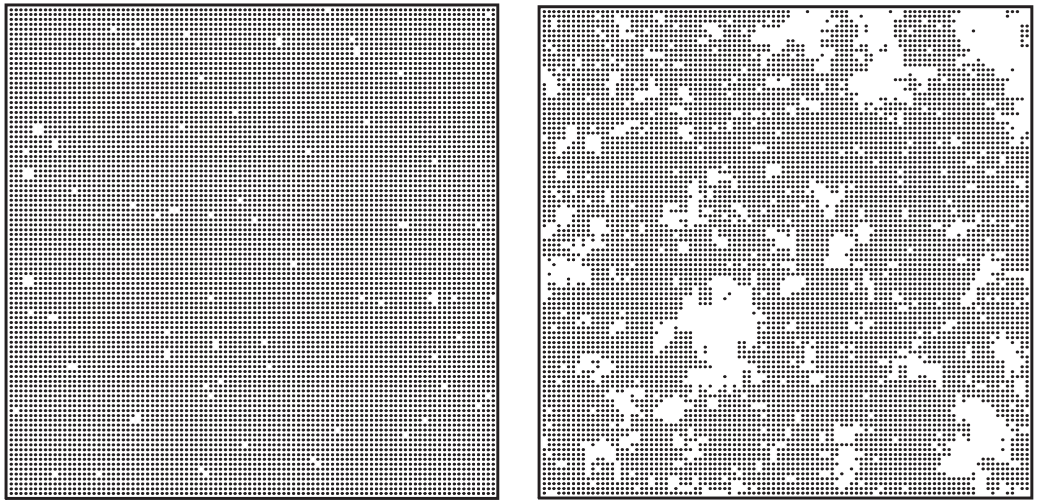
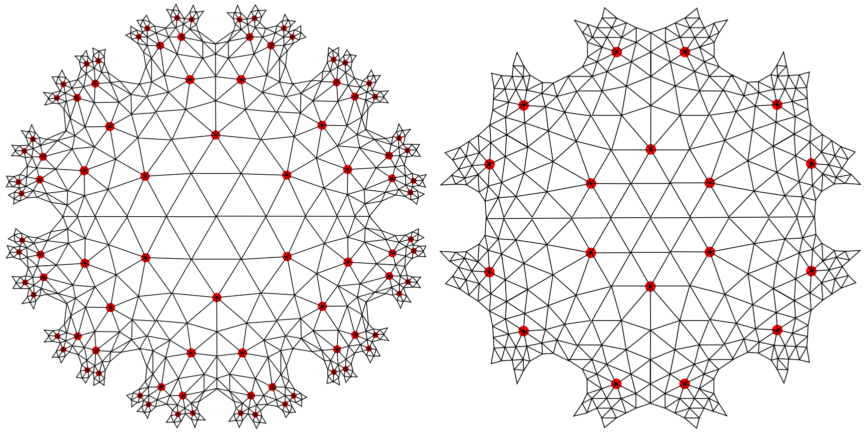
Ising model (simulated by DMRG)

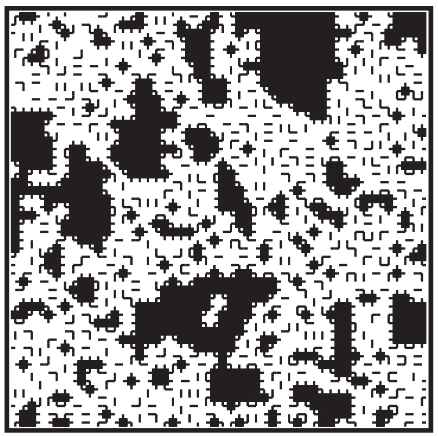


Low Temperature nearly Critical

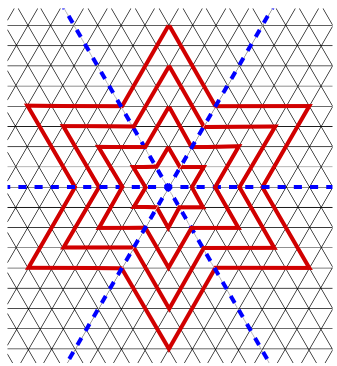


something nearly flat

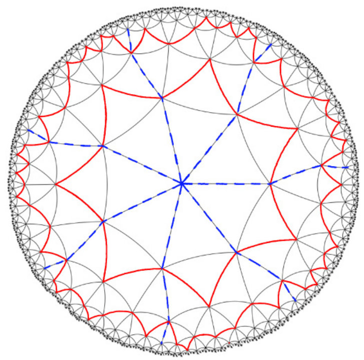
vertex model



Planer lattice



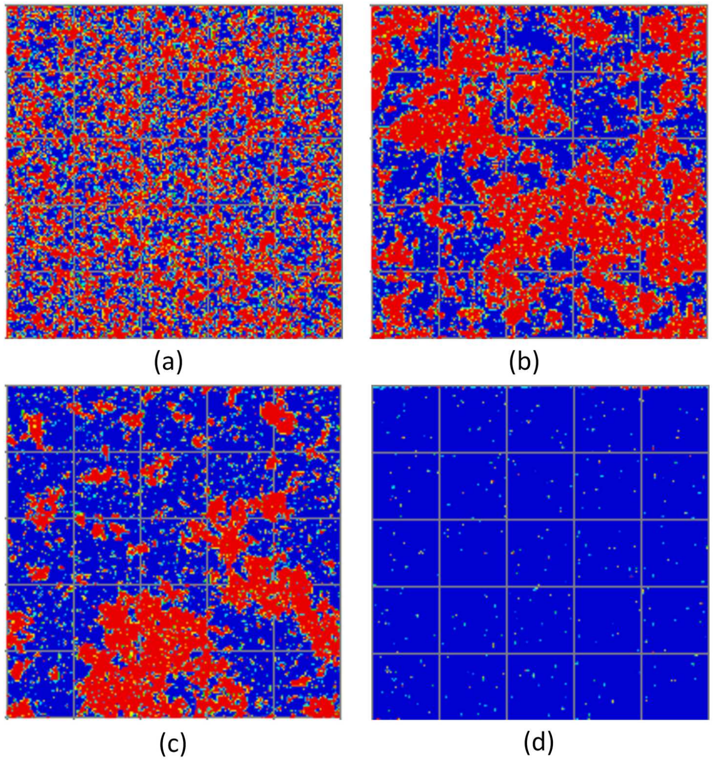
Hyperbolic

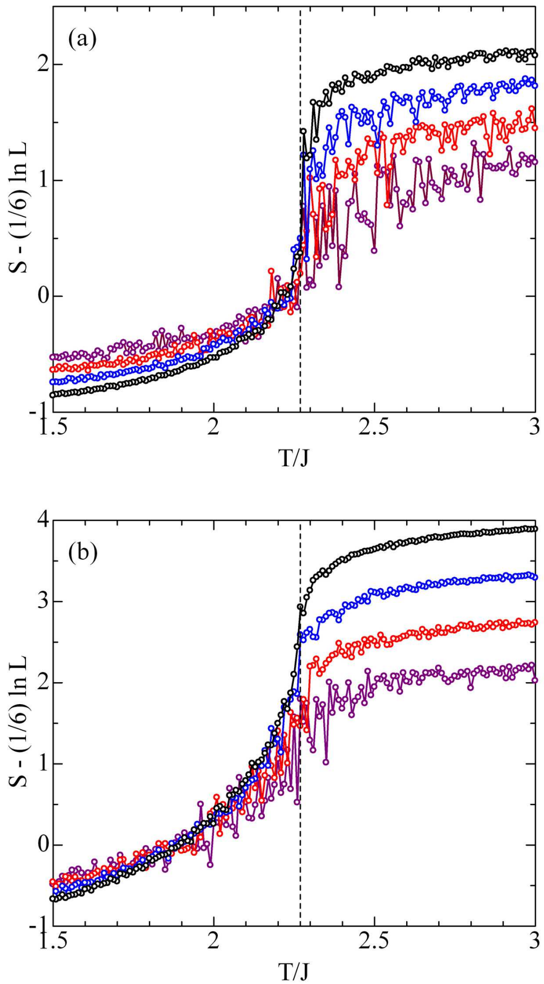


# There are a number of Entropies: an example

## Entropy of “quenched” thermal configuration

**Matsueda showed that there is an universality in the singular value distribution and corresponding “entropy” of a thermal snapshot of 2D spin system.**




**+ analytic approach by Okunishi et al.**

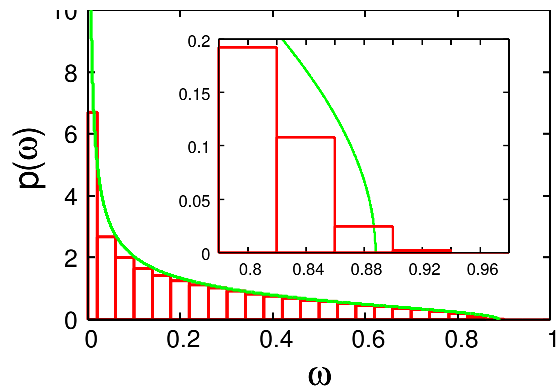
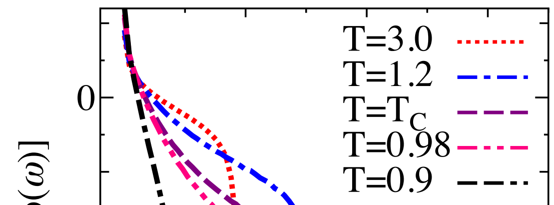
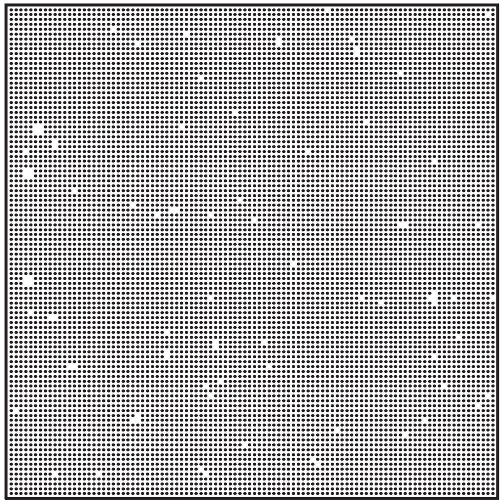



**Fig. 10.** The eigenvalue distribution $p(\omega)$ for the 3-states Potts model of the system size $N = 100$ at $T = 100.0$, where $\Delta\omega = 0.04$. We also plot the RMT curve for $N \to \infty$ with $\sigma^2 = 2/9$ and $Q = 1$ as a solid line. The inset is the enlarged view around the upper bound $\lambda_+ = 8/9$ of the RMT curve.

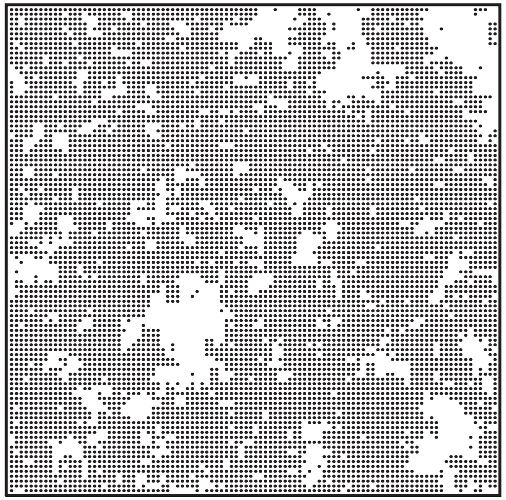


# "Entanglement Entropy" in statistical models

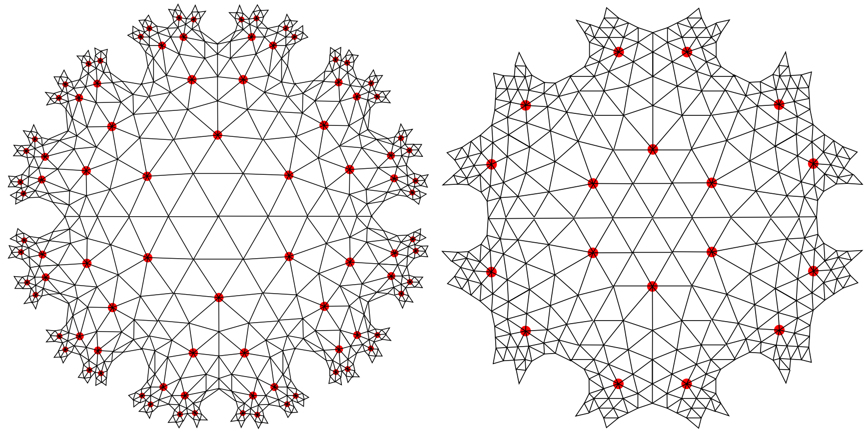
Ising model (simulated by DMRG)



Low Temperature

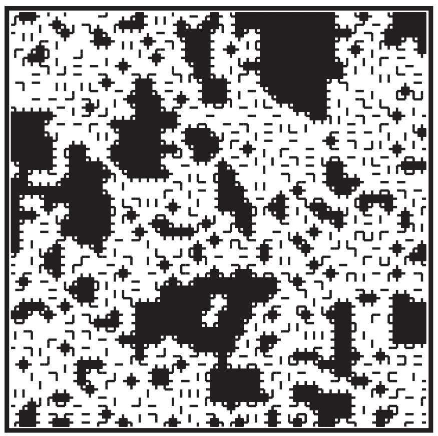


nearly Critical

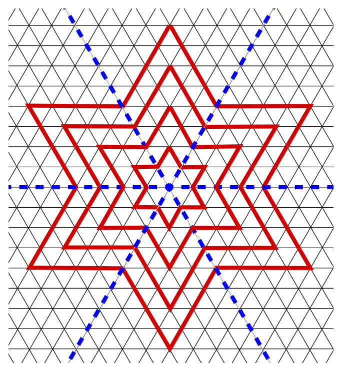


something nearly flat

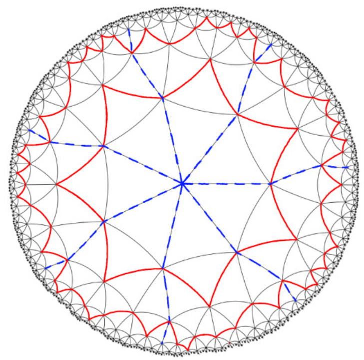
vertex model



Planer lattice



Hyperbolic



# Quantum - Classical Correspondence

**Path integral representation (?) for 1D Quantum System, such as Spin Chain, corresponds to 2D, or 1+1 D, Classical System.**

**>> Mathematical structure around Entanglement can be "exported" to 2D Classical lattice models, such as 2D classical Ising Model and its Critical Phenomena.**

**That's all. Any Questions?**

Your major is Quantum Information….
OK, then you are either Schroedinger's CAT or DOG.

# Mapping: from 1D Quantum to 2D Classical

## 1D Transverse Field Ising Model (1D Quantum)

$$\hat{H}_{\rm TFI} = -J\sum_i \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z - \Gamma \sum_i \hat{\sigma}_i^x = \sum_i \hat{h}_i$$

$$\hat{h}_i = -J\hat{\sigma}_i^z \hat{\sigma}_{i+1}^z - \frac{\Gamma}{2}\left(\hat{\sigma}_i^x + \hat{\sigma}_{i+1}^x\right)$$

<<< Bond Operator

Jargon TM

**Trotter-Suzuki Decomposition**

**Partition function** at finite temperature

$$Z = {\rm Tr}\, e^{-\beta \hat{H}_{\rm TFI}} = {\rm Tr}\left(e^{-\beta \hat{H}_{\rm TFI}/N}\right)^N$$

$$\sim {\rm Tr}\left(e^{-\beta \hat{H}_{\rm e}/N}\, e^{-\beta \hat{H}_{\rm o}/N}\right)^N = {\rm Tr}\,(T_{\rm e} T_{\rm o})^N$$

Transfer Matrix = imaginary time evolution, which is written in the form of the product of Local operators, i.e. local weights)

$$T_{\rm e} = e^{-(\beta/N)(\hat{h}_0 + \hat{h}_2 + \hat{h}_4 + \cdots)} = \prod_{i={\rm even}} e^{-(\beta/N)\,\hat{h}_i}$$

$$T_{\rm o} = e^{-(\beta/N)(\hat{h}_1 + \hat{h}_3 + \hat{h}_5 + \cdots)} = \prod_{i={\rm odd}} e^{-(\beta/N)\,\hat{h}_i}$$

## >> Square Lattice Ising Model (2D Classical)

Matrix elements of the Local Weight (= the Local imaginary Time Evolution)

(This is a kind of 16-vertex weight)

Lowest order approx.

$$e^{-(\beta/N)\,\hat{h}_i} \sim 1 - \frac{\beta}{N}\hat{h}_i = \begin{bmatrix} 1+\frac{\beta J}{N} & \frac{\beta\Gamma}{2N} & \frac{\beta\Gamma}{2N} & 0 \\ \frac{\beta\Gamma}{2N} & 1-\frac{\beta J}{N} & 0 & \frac{\beta\Gamma}{2N} \\ \frac{\beta\Gamma}{2N} & 0 & 1-\frac{\beta J}{N} & \frac{\beta\Gamma}{2N} \\ 0 & \frac{\beta\Gamma}{2N} & \frac{\beta\Gamma}{2N} & 1+\frac{\beta J}{N} \end{bmatrix} \begin{matrix} \text{uu} \\ \text{ud} \\ \text{du} \\ \text{dd} \end{matrix}$$

Partition Function, as a trace of the product of transfer matrices

SKIP

$$Z = \mathrm{Tr}\,(T_\mathrm{e} T_\mathrm{o})^N$$

is equivalent to the spin configuration sum on the product of local weights "W" over the **2D chess board lattice**.

$$\begin{aligned} T_\mathrm{e} &= W_0 W_2 W_4 \cdots W_{M-2} \\ T_\mathrm{o} &= W_1 W_3 W_5 \cdots W_{M-1} \end{aligned}$$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| W | | W | | W | | W | |
| | W | | W | | W | | W |
| W | | W | | W | | W | |
| | W | | W | | W | | W |
| W | | W | | W | | W | |
| | W | | W | | W | | W |

Already we have reached a classical (=statistical) model defined on the square lattice, although it is chess board like.

## Quantum

$$e^{-(\beta/N)\,\hat{h}_i} \sim 1 - \frac{\beta}{N}\hat{h}_i = \begin{bmatrix} 1+\frac{\beta J}{N} & \frac{\beta\Gamma}{2N} & \frac{\beta\Gamma}{2N} & 0 \\ \frac{\beta\Gamma}{2N} & 1-\frac{\beta J}{N} & 0 & \frac{\beta\Gamma}{2N} \\ \frac{\beta\Gamma}{2N} & 0 & 1-\frac{\beta J}{N} & \frac{\beta\Gamma}{2N} \\ 0 & \frac{\beta\Gamma}{2N} & \frac{\beta\Gamma}{2N} & 1+\frac{\beta J}{N} \end{bmatrix}$$

comparison

## Classical

$$W \sim e^{2K_1} \begin{bmatrix} 1+2K_2 & e^{-2K_2} & e^{-2K_2} & 0 \\ e^{-2K_2} & 1-2K_2 & 0 & e^{-2K_2} \\ e^{-2K_2} & 0 & 1-2K_2 & e^{-2K_2} \\ 0 & e^{-2K_2} & e^{-2K_2} & 1+2K_2 \end{bmatrix}$$

**SKIP**

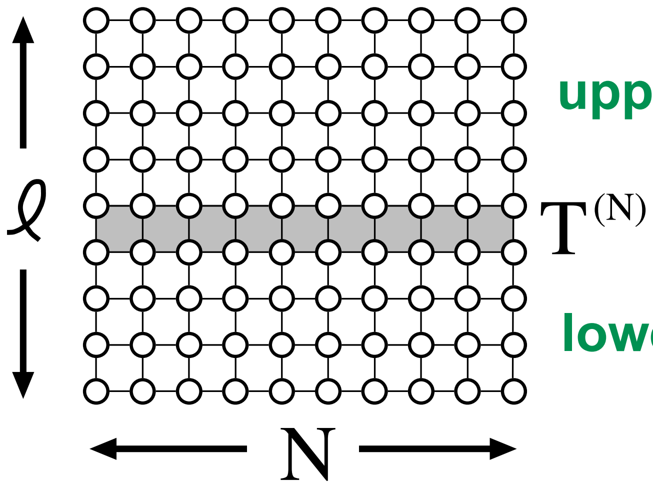
Horizontal $\quad 2K_2 = \frac{\beta J}{N}$

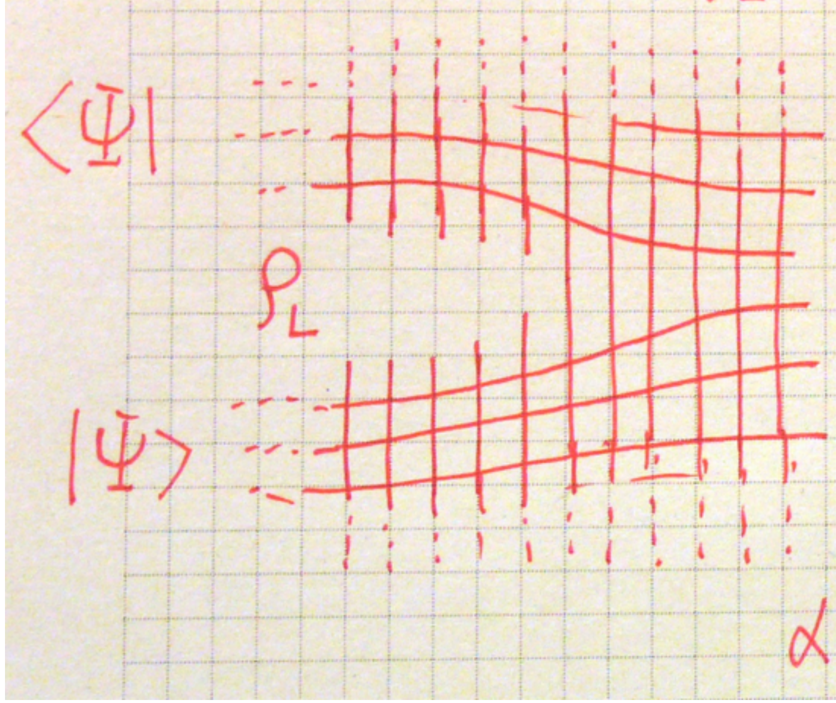
Vertical $\quad e^{-2K_1} = \frac{\beta\Gamma}{2N}$

Critical Condition $K_2\, e^{2K_1} \sim 1$ draws the Quantum Criticality $J = \Gamma$.

**Both 1D quantum and 2D classical Ising models have been represented by a subset of the 16 vertex model. Thus it is natural that the ground-state phase transition of the former shows the same universality with the order-disorder transition of the latter at the finite temperature.**

**Consider the Square Lattice Ising Model, or Vertex Model.**



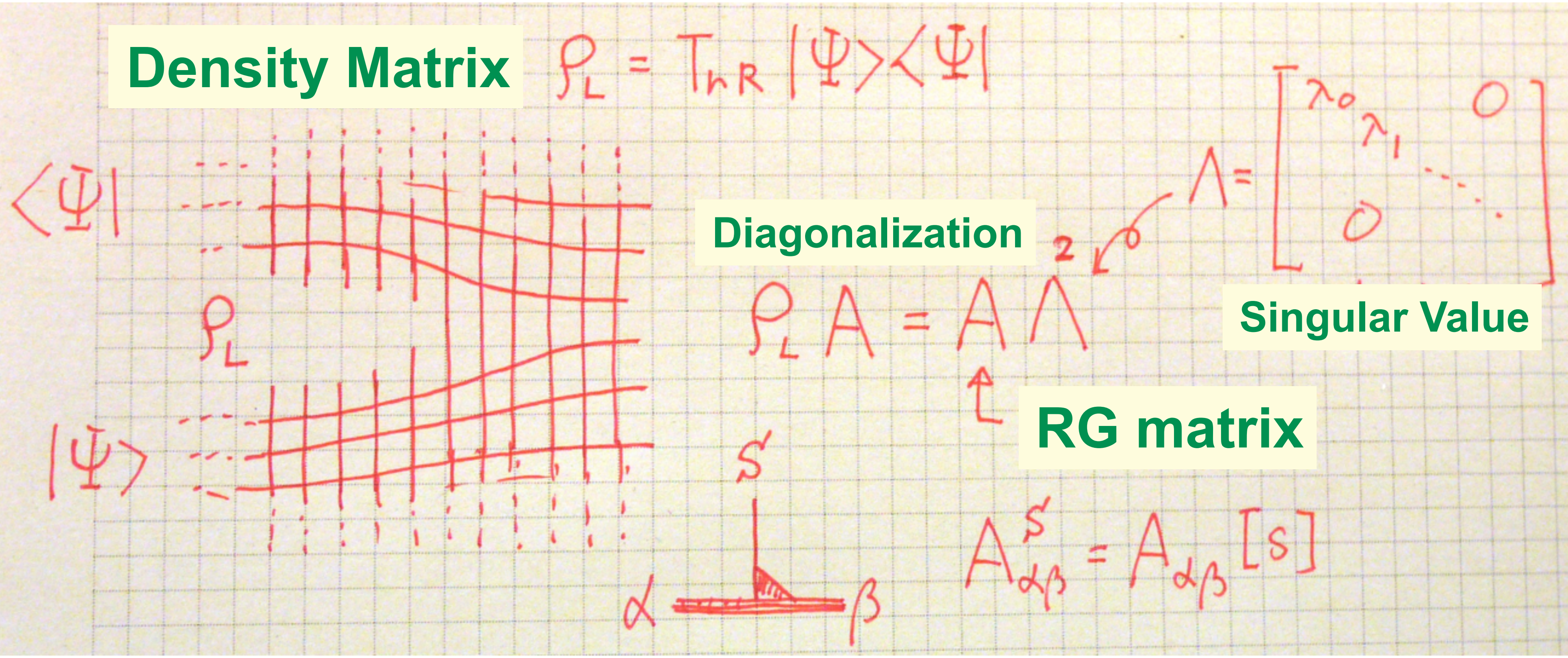




Let us divide the whole lattice into lower half, say the past, and the upper half, the future. (In the picture a Transfer Matrix joints these two.)

Taking the configuration sum for all the spins other than those on the T.M., one gets the eigenvectors of the Transfer Matrix.

Identifying these eigenvectors as Quantum Bra and Ket states, one can construct Density Matrix, by partially tracing out those spins on the right side of the horizontal spin row.

Density Matrix appears on any cut (or any boundary) of a given 2D (or even in any dimensional) classical statistical model.



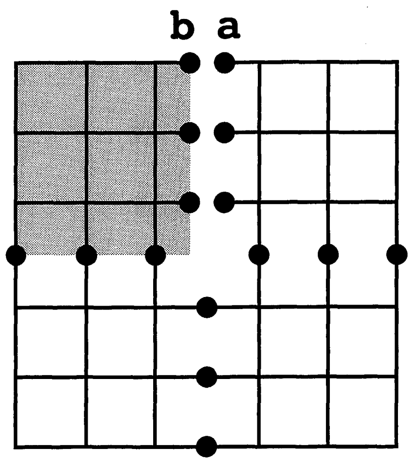

Identifying the eigenvalues of the density matrix (= Square of Singular Values) as probability, one can define an Entropy, which is nothing but the Entanglement Entropy as long as one speaks about its mathematical structure.

(***) Note that we are considering thermal statistical average. I don't know any experimental procedure to measure (or at least observe the effect of) the E.E. thus defined.

>> What I speak about, is, some profit in numerical analyses.

# Besides, Baxter reached the Density Matrix in 1968!!!

## (He used IBM system360!)




[*] R.J.Baxter J. Math. Phys. 9, 650 (1968)

[*] R.J.Baxter: "Exactly Solved Models in Statistical Mechanics", http://physics.anu.edu.au/theophys/baxter_book.php

**The Corner Transfer formalism by Baxter is essentially the same as the Density Matrix Renormalization Group method.**

**Analytic Formulation for E.E. can be obtained from those singular values obtained within the CTM formulation.**

**Numerically: to obtain Correlation Length is rather hard.**

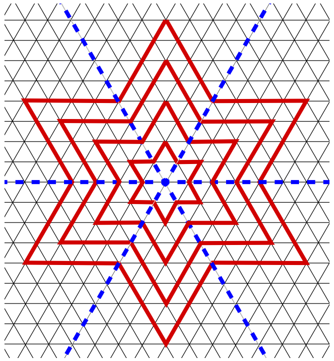
**to obtain E.E is rather easy, if one uses CTM formulation.**

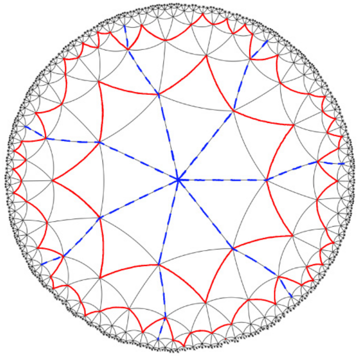
**from the area low, one can obtain the correlation length.**

**Jargon TM**

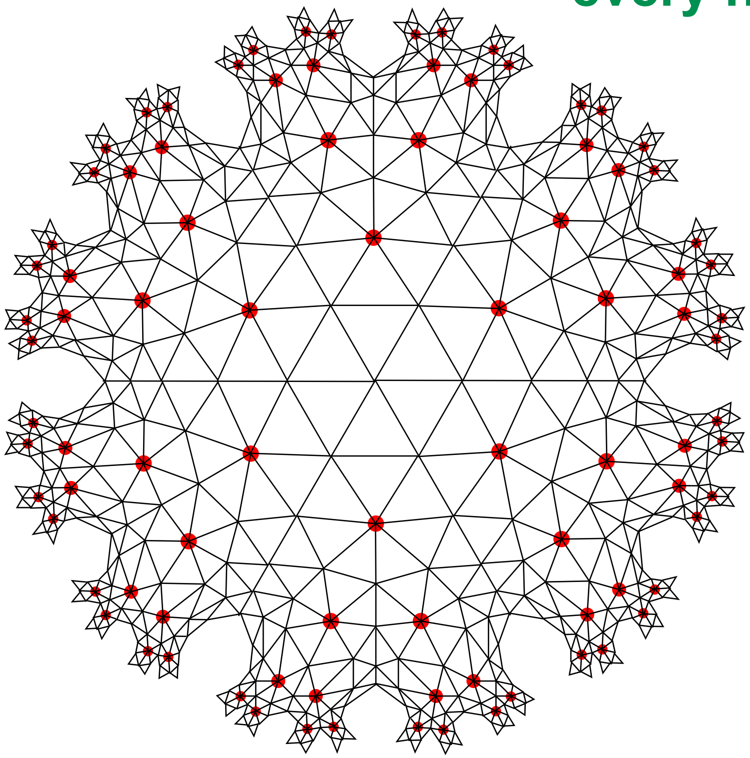
# Ising model on Hyperbolic Lattice

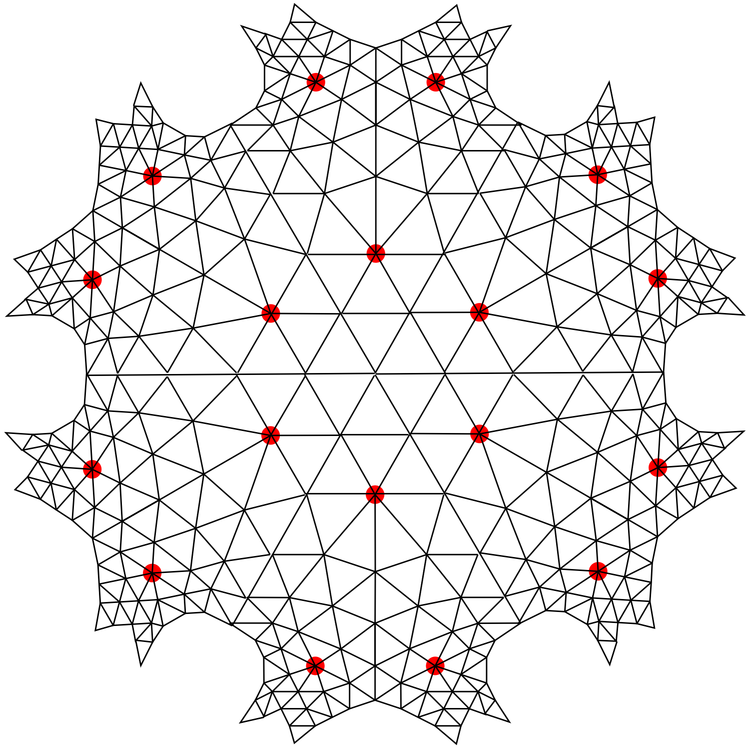
**flat lattice   Hyperbolic Lattice <<< These two lattices are too different.**





**"special points" are distributed every n sites.**



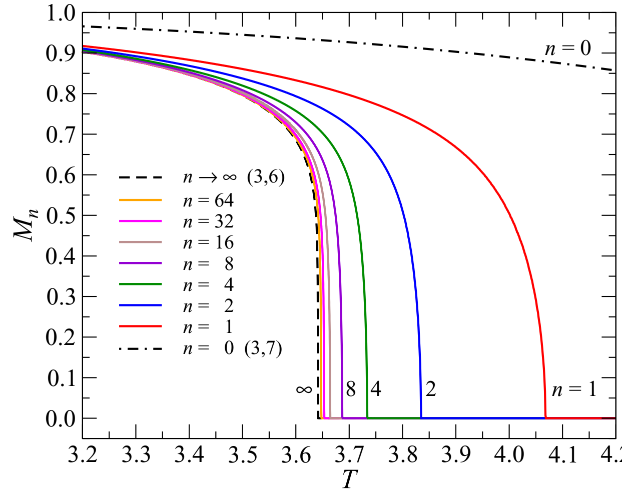


**Ising model on the Hyperbolic lattice shows 2nd order phase transition.**

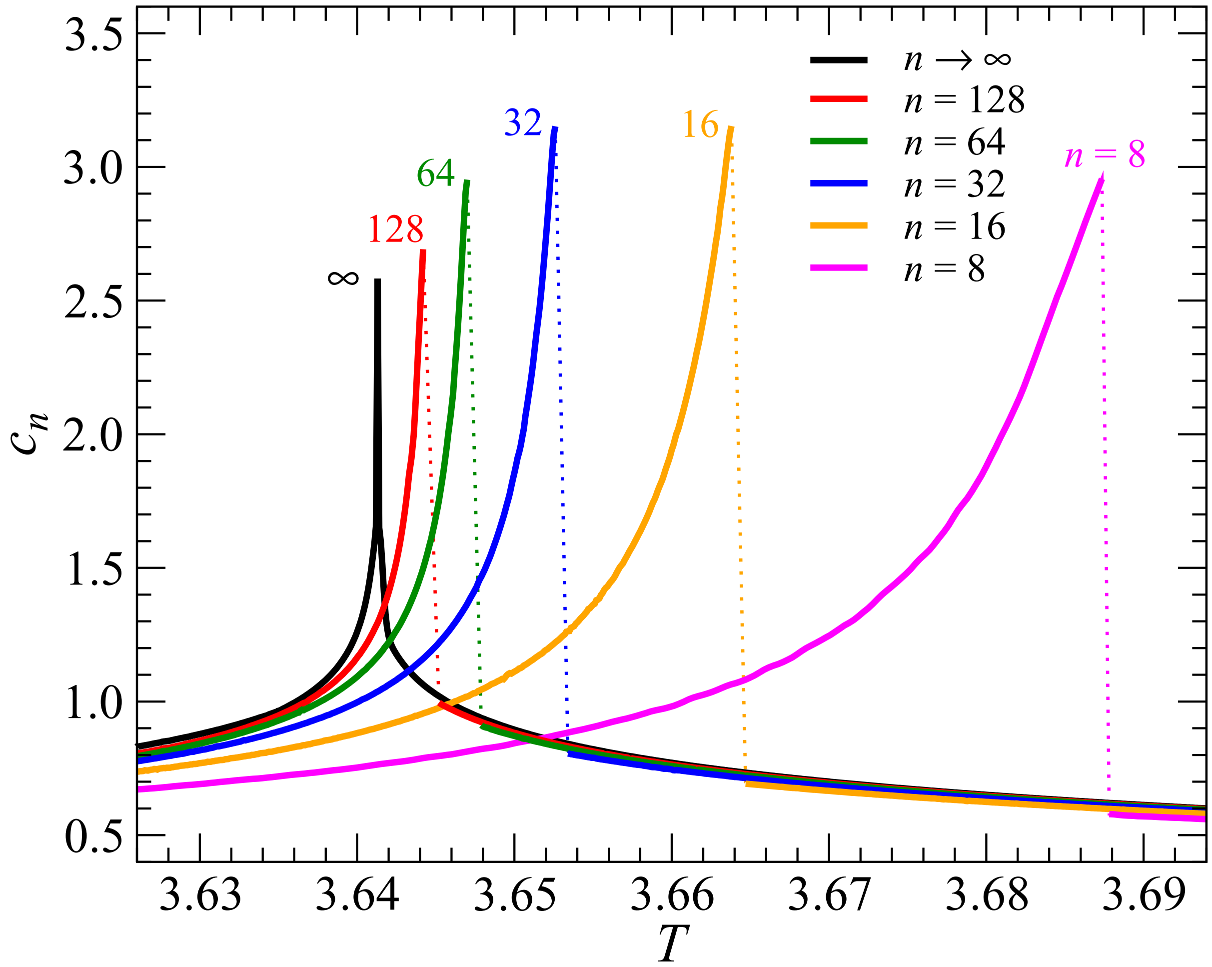
**….There are several thermodynamic quantities.**

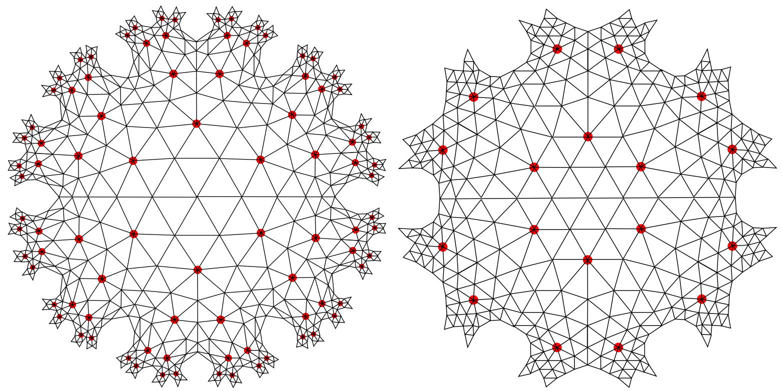
**….It is not easy to understand what is going on at Tc.**

**Magnetization: easy to obtain.**



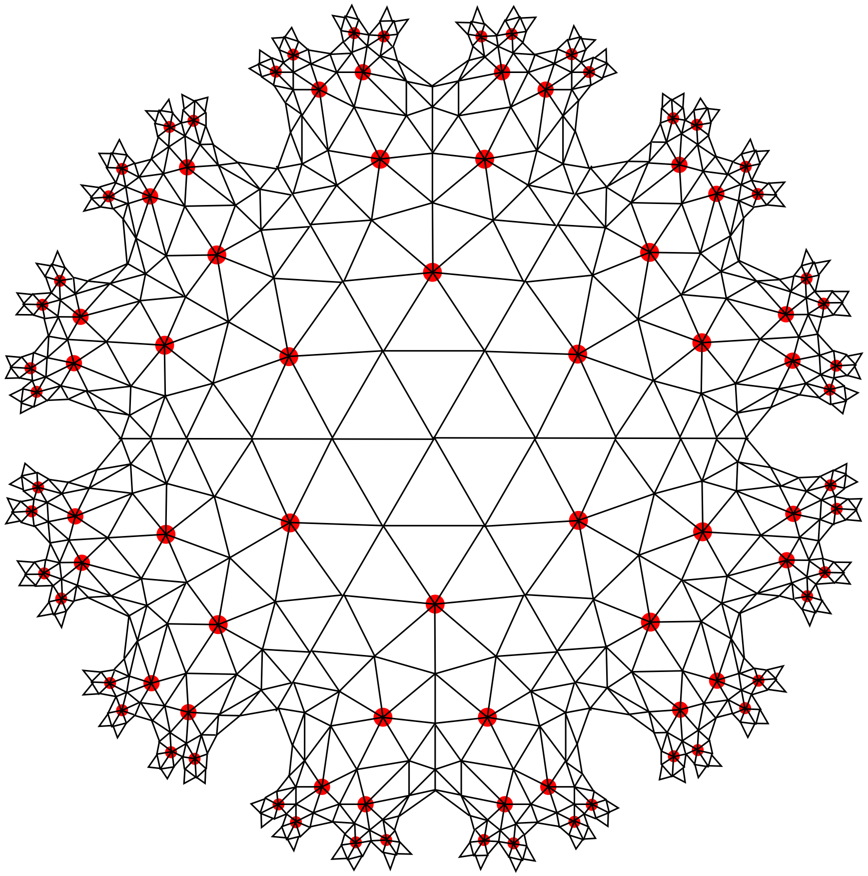

**Specific Heat: rather hard to obtain.**

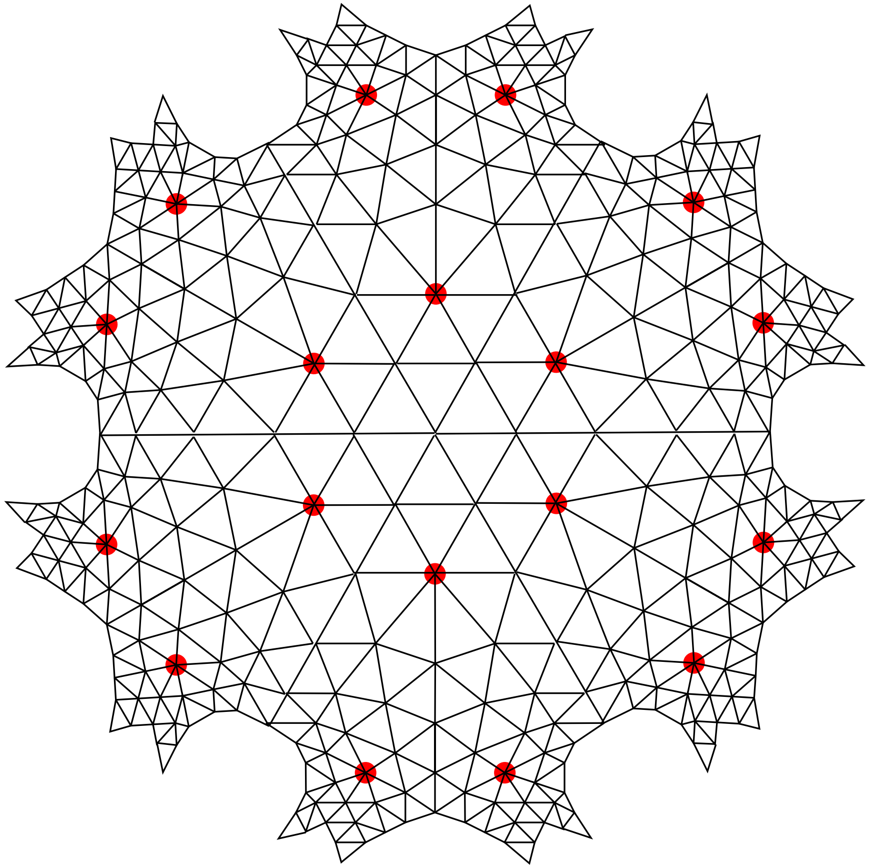


# Entanglement Entropy

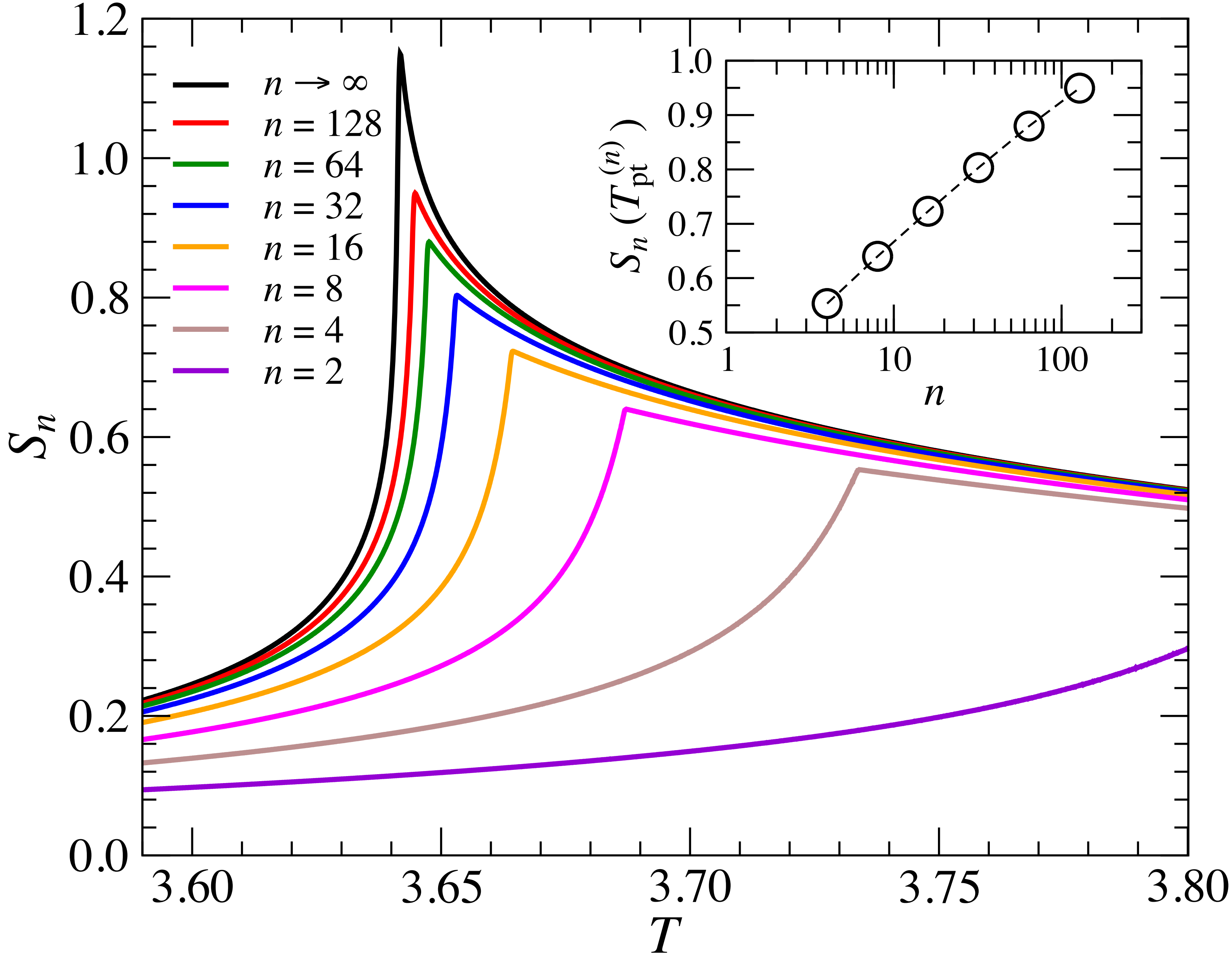
**Tattice consists of upper and lower halves.**

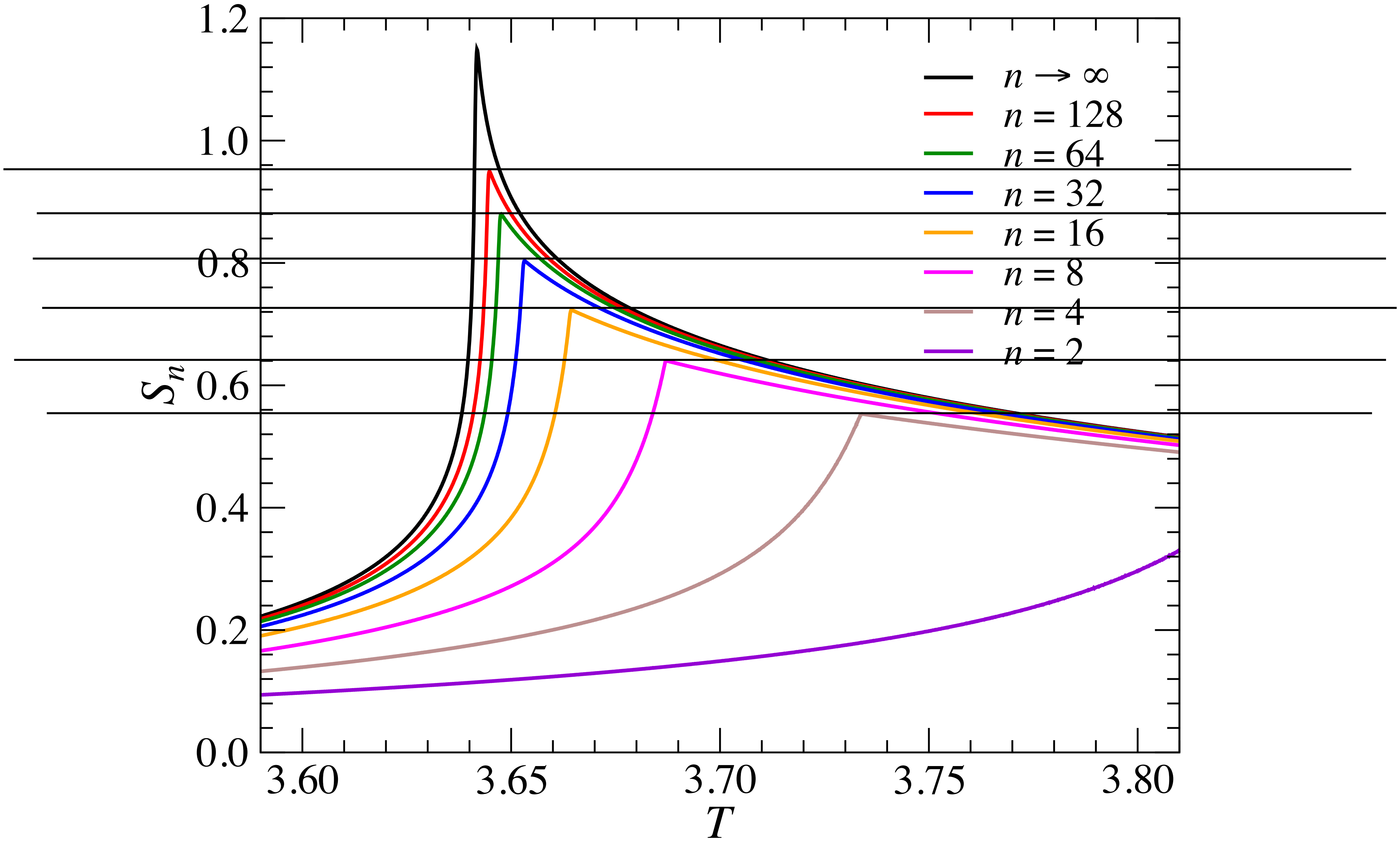




**Let us obtain the E.E, which corresponds to a straight cut from the center to any border of the system.**

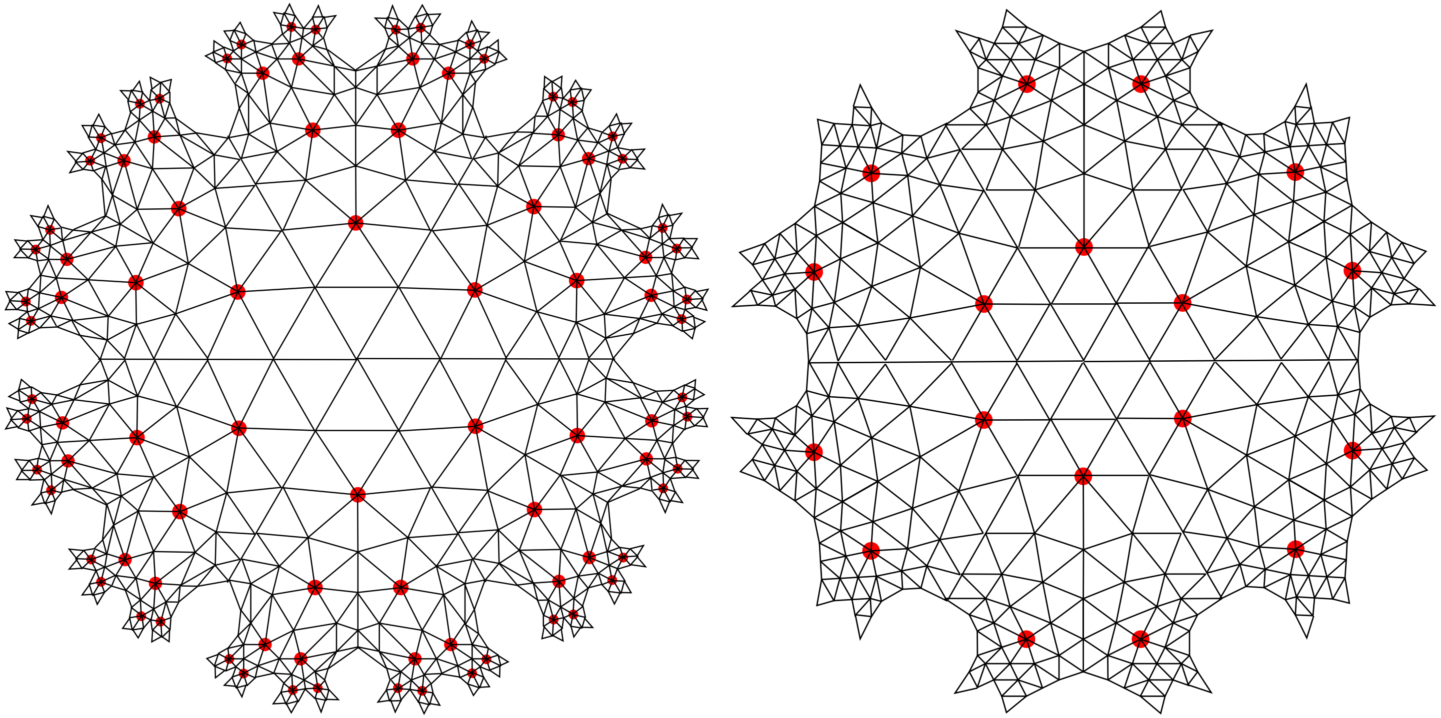
**n: average distance between "red points".**
**Note that the curvature radius of the lattice is proportional to -1/n.**





$n \to \infty$
$n = 128$
$n = 64$
$n = 32$
$n = 16$
$n = 8$
$n = 4$
$n = 2$
$S_n$
$T$
0.0
0.2
0.4
0.6
0.8
1.0
1.2
3.60
3.65
3.70
3.75
3.80

Due to the hyperbolic nature of the lattice, correlation length is limited by the average distance between red points.



The calculate E.E certainly captures the length.

**That's all. Any Questions?**

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**>>> Next Speaker >>>**