Monogamy of entanglement and mean-field ansatz for spin lattices

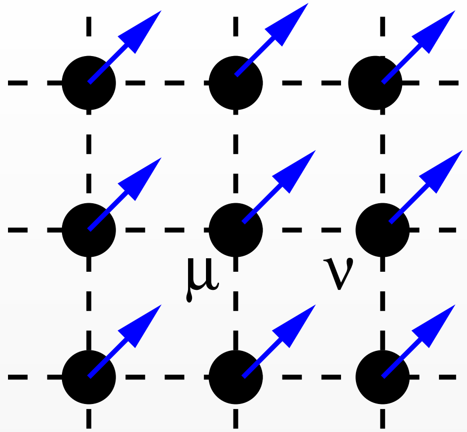
Ralf Schützhold

Fakultät für Physik

Universität Duisburg-Essen

UNIVERSITÄT DUISBURG ESSEN

Spin Lattice



Regular lattice of $1/2$-spins (qubits)
Pauli matrices $\hat{\boldsymbol{\sigma}}_\mu = (\hat{\sigma}^x_\mu, \hat{\sigma}^y_\mu, \hat{\sigma}^z_\mu)$
Coordination number Z (neighbours)

$$\hat{H} = \frac{1}{Z} \sum_{<\mu,\nu>} \hat{\boldsymbol{\sigma}}_\mu \cdot \boldsymbol{J} \cdot \hat{\boldsymbol{\sigma}}_\nu + \sum_\mu \boldsymbol{B} \cdot \hat{\boldsymbol{\sigma}}_\mu$$

In general very complicated $\longrightarrow$ mean-field ansatz

$$|\Psi_{\mathrm{mf}}\rangle = \bigotimes_\mu |\psi_\mu\rangle \ , \quad \text{e.g.,} \quad |\Psi_{\mathrm{mf}}\rangle = |\uparrow\rangle\,|\uparrow\rangle\,|\uparrow\rangle\,|\uparrow\rangle \ldots$$

Variational mean-field energy per lattice site

$$\frac{\langle \hat{H} \rangle_{\mathrm{mf}}}{N} = \frac{1}{2} \langle \hat{\boldsymbol{\sigma}}_\mu \rangle \cdot \boldsymbol{J} \cdot \langle \hat{\boldsymbol{\sigma}}_\nu \rangle + \boldsymbol{B} \cdot \langle \hat{\boldsymbol{\sigma}}_\mu \rangle$$

Neglect of entanglement!?

Entanglement for Pure States

Consider two spins (qubits) μ and ν: not entangled iff

$$|\Psi_{\mu\nu}\rangle = |\psi_\mu\rangle\,|\psi_\nu\rangle \;\;, \text{ e.g.,} \;\frac{|\!\uparrow\rangle_\mu + |\!\downarrow\rangle_\mu}{\sqrt{2}}\,\frac{|\!\uparrow\rangle_\nu + |\!\downarrow\rangle_\nu}{\sqrt{2}}$$

Maximum entanglement (Bell state)

$$|\Psi_{\mu\nu}\rangle = \frac{|\!\uparrow\rangle_\mu\,|\!\uparrow\rangle_\mu + |\!\downarrow\rangle_\nu\,|\!\downarrow\rangle_\nu}{\sqrt{2}} = |\mathrm{Bell}\rangle_{\mu\nu}$$

General state with concurrence C with $0 \leq C \leq 1$

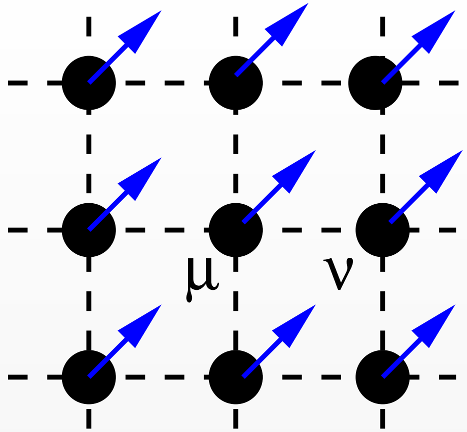
$$|\Psi_{\mu\nu}\rangle = \sqrt{1-C}\;|\psi_\mu\rangle\;|\psi_\nu\rangle + \sqrt{C}\,\hat{U}_\mu\hat{U}_\nu\,|\mathrm{Bell}\rangle_{\mu\nu}$$

W.K. Wooters, Phys. Rev. Lett. **80**, 2245 (1998).

Note: qutrits or three qubits are more complicated

$$|\Psi_{\mathrm{GHZ}}\rangle = \frac{|\!\uparrow\rangle\,|\!\uparrow\rangle\,|\!\uparrow\rangle + |\!\downarrow\rangle\,|\!\downarrow\rangle\,|\!\downarrow\rangle}{\sqrt{2}}$$

Mixed State of Two Spins



General decomposition (not unique)

$$\hat{\rho}_{<\mu\nu>} = \sum_I p_I \left|\Psi^I_{\mu\nu}\right\rangle \left\langle\Psi^I_{\mu\nu}\right|$$

Problem: consider all possible decompositions

$$\mathrm{ent}(\hat{\rho}_{<\mu\nu>}) = \min_{p_I, \left|\Psi^I_{\mu\nu}\right\rangle} \sum_I p_I \, \mathrm{ent}\left(\left|\Psi^I_{\mu\nu}\right\rangle \left\langle\Psi^I_{\mu\nu}\right|\right)$$

Problem solved for concurrence $C(\hat{\rho}_{<\mu\nu>})$ (2 qubits)

W.K. Wooters, Phys. Rev. Lett. **80**, 2245 (1998); A. Uhlmann, Phys. Rev. A **62**, 022307 (2000).

Symmetric decomposition for general mixed states

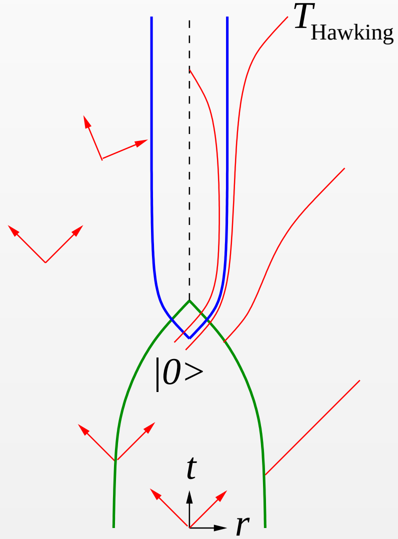
$$\hat{\rho}_{<\mu\nu>} = \sum_{I=1}^{4} p_I \left|\Psi^I_{\mu\nu}\right\rangle \left\langle\Psi^I_{\mu\nu}\right| \; : \; C(\hat{\rho}_{<\mu\nu>}) = C\left(\left|\Psi^I_{\mu\nu}\right\rangle \left\langle\Psi^I_{\mu\nu}\right|\right) \; \forall_I$$

Again: for 2 qubits only...

Monogamy of Entanglement

Upper bound for concurrence of qubit-pairs

$$\tau_1(\hat{\rho}_\mu) = 4\det(\hat{\rho}_\mu) \geq \sum_\nu C^2(\hat{\rho}_{<\mu\nu>})$$

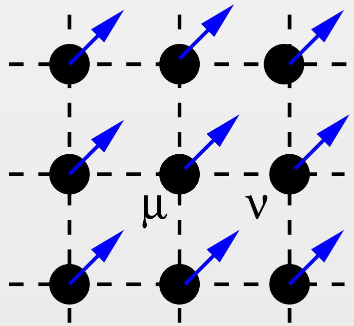


with one-tangle $\tau_1(\hat{\rho}_\mu) \leq 1$

V. Coffman, J. Kundu, W.K. Wootters, Phys. Rev. A **61**, 052306 (2000);

T.J. Osborne, F. Verstraete, Phys. Rev. Lett. **96**, 220503 (2006).

Lattice isotropy

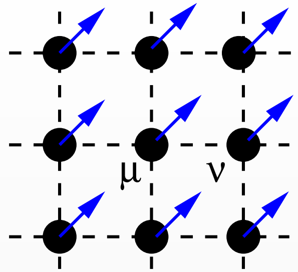


$$C(\hat{\rho}_{<\mu\nu>}) \leq \sqrt{\frac{\tau_1}{Z}} \leq \sqrt{\frac{1}{Z}}$$

Entanglement decreases for large Z

Expectation: mean-field ansatz becomes better

Ground State Energy



$$\hat{H} = \frac{1}{Z} \sum_{<\mu,\nu>} \hat{\boldsymbol{\sigma}}_\mu \cdot \boldsymbol{J} \cdot \hat{\boldsymbol{\sigma}}_\nu + \sum_\mu \boldsymbol{B} \cdot \hat{\boldsymbol{\sigma}}_\mu$$

Insert $\hat{\rho}_{<\mu\nu>} = \sum_{I=1}^{4} p_I \left|\Psi^I_{\mu\nu}\right\rangle \left\langle\Psi^I_{\mu\nu}\right|$ with

$$\left|\Psi^I_{\mu\nu}\right\rangle = \sqrt{1-C}\, \left|\psi^I_\mu\right\rangle \left|\psi^I_\nu\right\rangle + \sqrt{C}\, \hat{U}^I_\mu \hat{U}^I_\nu \left|\text{Bell}\right\rangle_{\mu\nu}$$

$\rightarrow$ estimate for ground-state energy

$$\frac{\langle\hat{H}\rangle}{N} = \sum_{I=1}^{4} \frac{p_I}{2} \left[\langle\hat{\boldsymbol{\sigma}}^I_\mu\rangle \cdot \boldsymbol{J} \cdot \langle\hat{\boldsymbol{\sigma}}^I_\nu\rangle + \boldsymbol{B} \cdot \left(\langle\hat{\boldsymbol{\sigma}}^I_\mu\rangle + \langle\hat{\boldsymbol{\sigma}}^I_\nu\rangle\right)\right] + \mathcal{O}(\sqrt{C})$$

with $\langle\hat{\boldsymbol{\sigma}}^I_\mu\rangle = \left\langle\psi^I_\mu\right| \hat{\boldsymbol{\sigma}}_\mu \left|\psi^I_\mu\right\rangle$ $\rightarrow$ mean-field ansatz

Ergo: $Z \gg 1 \rightarrow C \ll 1$ $\rightarrow$ mean-field behaviour

Intermediate Summary

Concurrence C measures deviation from mean-field

$$\frac{\langle \hat{H} \rangle_{\mathrm{mf}}}{N} - \frac{\langle \hat{H} \rangle_{\mathrm{exact}}}{N} \leq (||\boldsymbol{J}|| + 2||\boldsymbol{B}||)\sqrt{C} + \mathcal{O}(C)$$

$\rightarrow$ $C = 0$ only if mean-field yields exact ground state

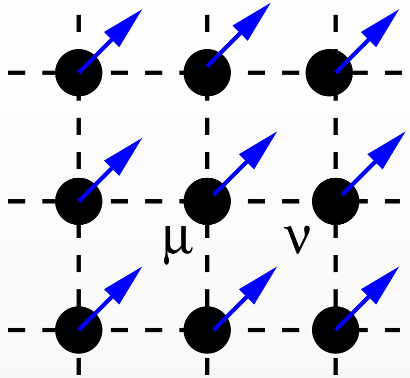
Large $Z \gg 1$ $\rightarrow$ small $C \leq 1/\sqrt{Z} \ll 1$
$\rightarrow$ mean-field becomes better for large $Z \gg 1$

Note: different from quantum de Finetti theorem
(full permutational invariance vs lattice symmetry)
E.g., Lipkin-Meshkov-Glick model

$$\hat{H} = \frac{1}{N} \sum_{\mu,\nu} \hat{\boldsymbol{\sigma}}_\mu \cdot \boldsymbol{J} \cdot \hat{\boldsymbol{\sigma}}_\nu + \sum_{\mu} \boldsymbol{B} \cdot \hat{\boldsymbol{\sigma}}_\mu$$

$\rightarrow$ large spin $\boldsymbol{\Sigma} = \sum_\mu \hat{\boldsymbol{\sigma}}_\mu / N$

Example: Ising Model



$$\hat{H} = -\frac{J}{Z}\sum_{<\mu,\nu>}\hat{\sigma}_\mu^x\hat{\sigma}_\nu^x - B\sum_\mu \hat{\sigma}_\mu^z$$

Mean-field ansatz: paramagnetic for $B > |J|$

$$|\Psi_{\rm mf}\rangle = |\uparrow\uparrow\uparrow\dots\rangle$$

Estimate for exact on-site density matrix

$$\hat{\rho}_\mu = |\uparrow\rangle\langle\uparrow| + \mathcal{O}(\sqrt{C}) = |\uparrow\rangle\langle\uparrow| + \mathcal{O}(1/Z^{1/4})$$

$\rightarrow$ iterate monogamy argument $C \le \sqrt{\tau_1/Z}$

$$C \le \mathcal{O}(Z^{-2/3})\,, \quad \tau_1 = 4\det(\hat{\rho}_\mu) \le \mathcal{O}(Z^{-1/3})$$

Hierarchy of correlations suggests

$$C = \mathcal{O}(1/Z)\,, \quad \tau_1 = \mathcal{O}(1/Z)$$

P. Navez, F. Queisser, R.S., J. Phys. A **47**,225004 (2014).

Improved Mean-Field Ansatz

Idea: add a little bit of entanglement for 2 spins

$$|\Psi_{\mu\nu}\rangle = \mathcal{N}\left(1 + \hat{\boldsymbol{\sigma}}_\mu \cdot \boldsymbol{\xi} \cdot \hat{\boldsymbol{\sigma}}_\nu\right)|\uparrow\rangle_\mu |\uparrow\rangle_\nu$$

Generalisation to spin lattices

$$|\Psi\rangle_{\mathrm{imf}} = \mathcal{N}\left(\prod_{<\mu,\nu>} \exp\left\{\xi\hat{\sigma}_\mu^x\hat{\sigma}_\nu^x\right\}\right)\bigotimes_\mu |\uparrow\rangle_\mu \,,$$

Variational ansatz

$$\frac{\langle\hat{H}\rangle_{\mathrm{imf}}}{N} = -\frac{J}{2}\tanh(2\Re\xi) - B\left(\frac{\cos(2\Im\xi)}{\cosh(2\Re\xi)}\right)^Z$$

Energy minimum for

$$\xi_{\min} = \frac{J}{4BZ} + \mathcal{O}(1/Z^2) \rightsquigarrow C = \mathcal{O}(1/Z)$$

XY-Model

$$\hat{H} = -\frac{J}{Z}\sum_{<\mu,\nu>}\left(\frac{1+\gamma}{2}\,\hat{\sigma}^x_\mu\hat{\sigma}^x_\nu + \frac{1-\gamma}{2}\,\hat{\sigma}^y_\mu\hat{\sigma}^y_\nu\right) - B\sum_\mu \hat{\sigma}^z_\mu$$

Scaling with anisotropy parameter γ

$$\xi_{\mathrm{min}} = \gamma\,\frac{J}{4BZ} + \mathcal{O}(1/Z^2)$$

Scaling variable $\zeta = Z|\xi|$

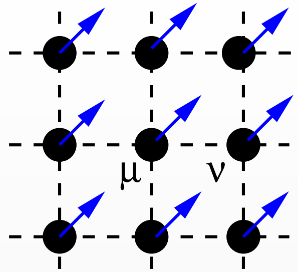
$$C = 2\,\frac{\zeta-\zeta^2}{Z}\,\Theta(1-\zeta) + \mathcal{O}(1/Z^2)$$

$\rightarrow$ ξ_{min} and C vanish in isotropic limit $\gamma = 0$

$$|\Psi_{\mathrm{imf}}\rangle = |\Psi_{\mathrm{mf}}\rangle = |\uparrow\uparrow\uparrow\ldots\rangle$$

$\leftrightarrow$ paramagnetic state is exact for $B > |J|$

Conclusions & Outlook



Concurrence C measures deviation from mean-field

$$\frac{\langle \hat{H} \rangle_{\rm mf}}{N} - \frac{\langle \hat{H} \rangle_{\rm exact}}{N} \leq (||\boldsymbol{J}|| + 2||\boldsymbol{B}||)\sqrt{C} + \mathcal{O}(C)$$

- $C = 0 \leftrightarrow$ mean-field yields exact ground state
- monogamy: $Z \gg 1 \rightarrow C \leq 1/\sqrt{Z} \ll 1$
- unique mean-field ground state: $C = \mathcal{O}(Z^{-2/3})$
- improved mean-field ansatz: $C = \mathcal{O}(1/Z)$ (note: not rigorously proven)

Outlook: bi-partite $\longrightarrow$ tri-partite entanglement. . .

A. Osterloh, R.S., `arXiv:1406.0311`