Entanglement Behavior of 2D Quantum Models

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*VBS on symmetric graphs, J. Phys. A, **43**, 255303 (2010)*
*"VBS/CFT correspondence", Phys. Rev. B, **84**, 245128 (2011)*
*Quantum hard-square model, Phys. Rev. A, **86**, 032326 (2012)*
*Nested entanglement entropy, Interdisciplinary Information Sciences, **19**, 101 (2013)*

京都大学 KYOTO UNIVERSITY
YITP YUKAWA INSTITUTE FOR THEORETICAL PHYSICS

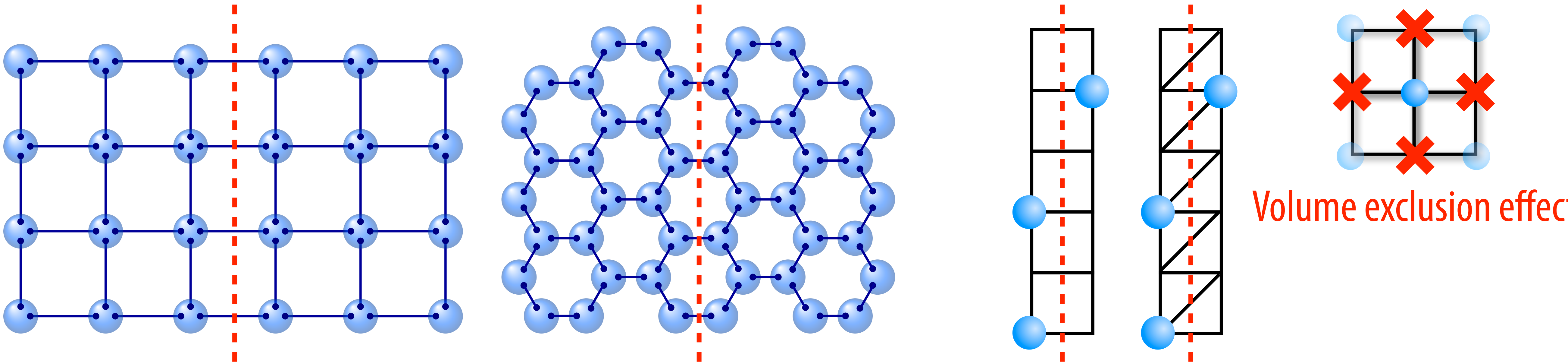
Digest

Entanglement properties of 2D quantum systems ⟷ **Physical properties of 1D quantum systems**

VBS on square lattice

VBS on hexagonal lattice

Quantum lattice gas on ladder



VBS state on 2D lattice

Total system	Entanglement Hamiltonian
Square lattice	**1D AF Heisenberg**
Hexagonal lattice	**1D F Heisenberg**

Quantum lattice gas on ladder

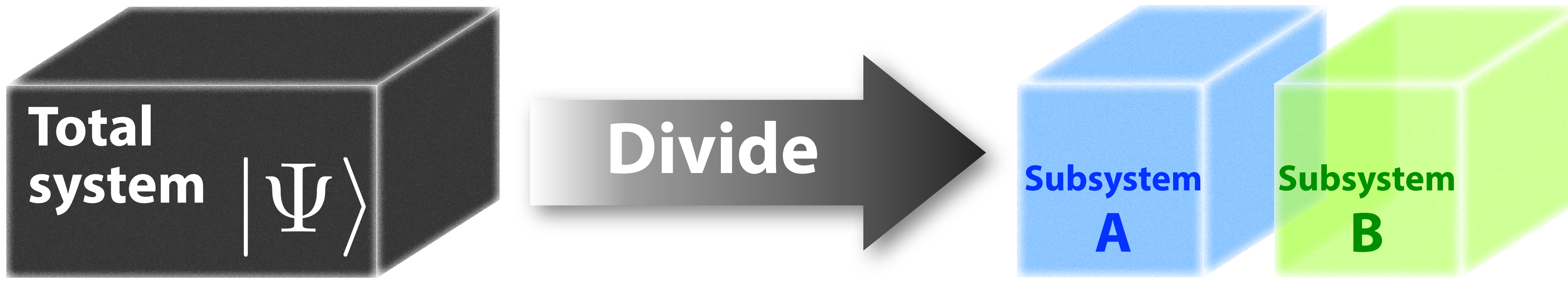
Total system	Entanglement Hamiltonian
Square ladder	**2D Ising**
Triangle ladder	**2D 3-state Potts**

Introduction

- ***Entanglement***
- ***Motivation***
- ***Preliminaries***

Introduction

EE is a measure to quantify entanglement.



Schmidt decomposition

$$|\Psi\rangle = \sum_{\alpha} \lambda_{\alpha} |\phi_{\alpha}^{[A]}\rangle \otimes |\phi_{\alpha}^{[B]}\rangle$$

Normalized GS

$\phi_{\alpha}^{[A]} \in \mathcal{H}_{A}, \phi_{\alpha}^{[B]} \in \mathcal{H}_{B}$

$\{|\phi_{\alpha}^{[A]}\rangle\}, \{|\phi_{\alpha}^{[B]}\rangle\}$: Orthonormal basis

Reduced density matrix

$$\rho_{A} = \mathrm{Tr}_{B} |\Psi\rangle\langle\Psi| = \sum_{\alpha} \lambda_{\alpha}^{2} |\phi_{\alpha}^{[A]}\rangle\langle\phi_{\alpha}^{[A]}|$$

von Neumann entanglement entropy

$$\mathcal{S} = \mathrm{Tr}\, \rho_{A} \ln \rho_{A} = -\sum_{\alpha} \lambda_{\alpha}^{2} \ln \lambda_{\alpha}^{2}$$

Introduction

Entanglement properties in **1D** quantum systems!!

1D **gapped** systems: EE **converges** to some value.

1D **critical** systems: EE **diverges** logarithmically with L.

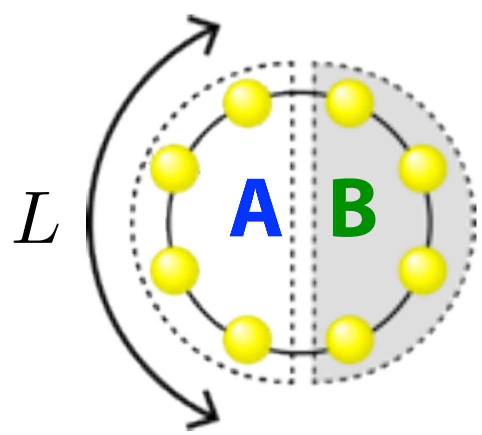
coefficient is related to the central charge.

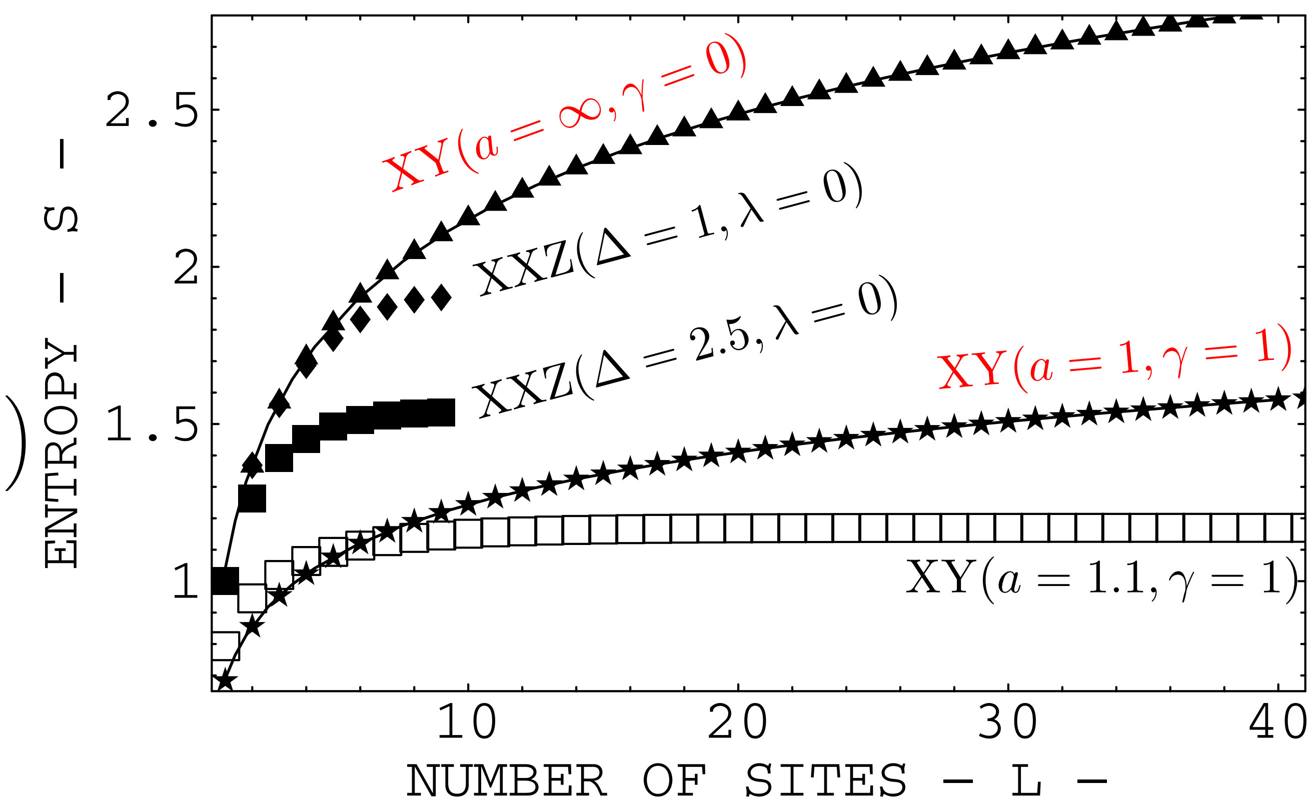
XXZ model under magnetic field

$$\mathcal{H}_{\mathrm{XXZ}} = \sum_i (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z - \lambda \sigma_i^z)$$

XY model under magnetic field

$$\mathcal{H}_{\mathrm{XY}} = -\sum_{i=0}^{N-1} \left(\frac{a}{2} \left[(1+\gamma)\sigma_i^x \sigma_{i+1}^x + (1-\gamma)\sigma_i^y \sigma_{i+1}^y \right] + \sigma_i^z \right)$$





G. Vidal et al. PRL ***90****, 227902 (2003)*

Entanglement properties in 2D quantum systems??

Preliminaries: reflection symmetric case

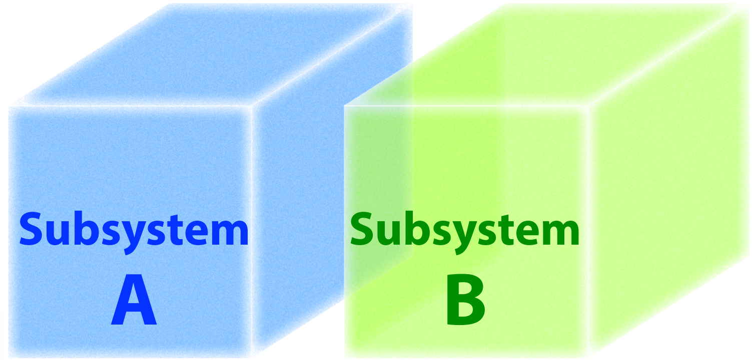
Pre-Schmidt decomposition

$$|\Psi\rangle = \sum_{\alpha} |\phi_{\alpha}^{[A]}\rangle \otimes |\phi_{\alpha}^{[B]}\rangle$$

$\{|\phi_{\alpha}^{[A]}\rangle\}, \{|\phi_{\alpha}^{[B]}\rangle\}$

Linearly independent
(but not orthonormal)

Reflection symmetry



Overlap matrix

$$(M^{[A]})_{\alpha\beta} := \langle \phi_{\alpha}^{[A]} | \phi_{\beta}^{[A]} \rangle, \ \ (M^{[B]})_{\alpha\beta} := \langle \phi_{\alpha}^{[B]} | \phi_{\beta}^{[B]} \rangle$$

Reflection symmetry $\Rightarrow$ $M^{[A]} = M^{[B]} = M$

Useful fact

If $M^{[A]} = M^{[B]} = M$ and M is real symmetric matrix,

$$S = -\sum_{\alpha} p_{\alpha} \ln p_{\alpha}, \qquad p_{\alpha} = \frac{d_{\alpha}^2}{\sum_{\alpha} d_{\alpha}^2}$$

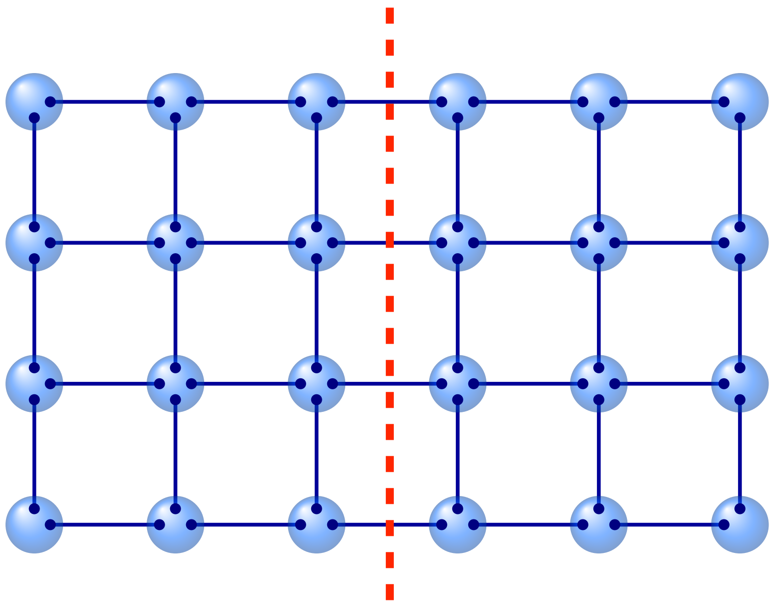
where d_{α} are the eigenvalues of M.

J. Phys. A, ***43***, *255303 (2010)*

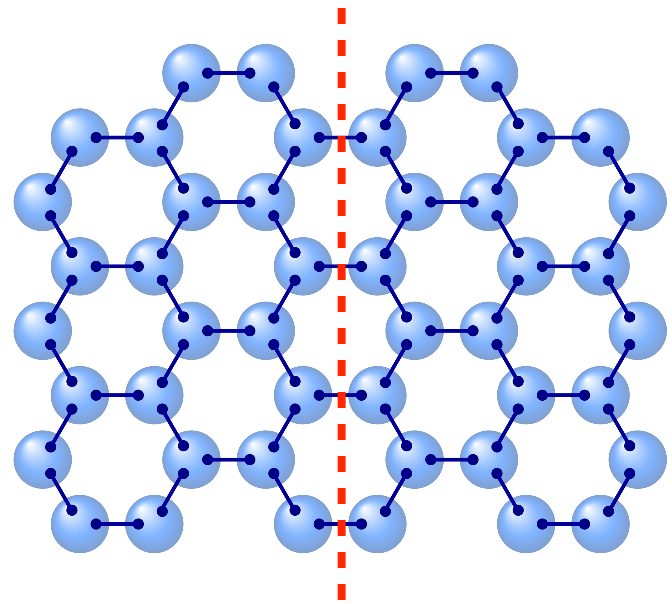
Digest

Entanglement properties of 2D quantum systems ⟷ **Physical properties of 1D quantum systems**

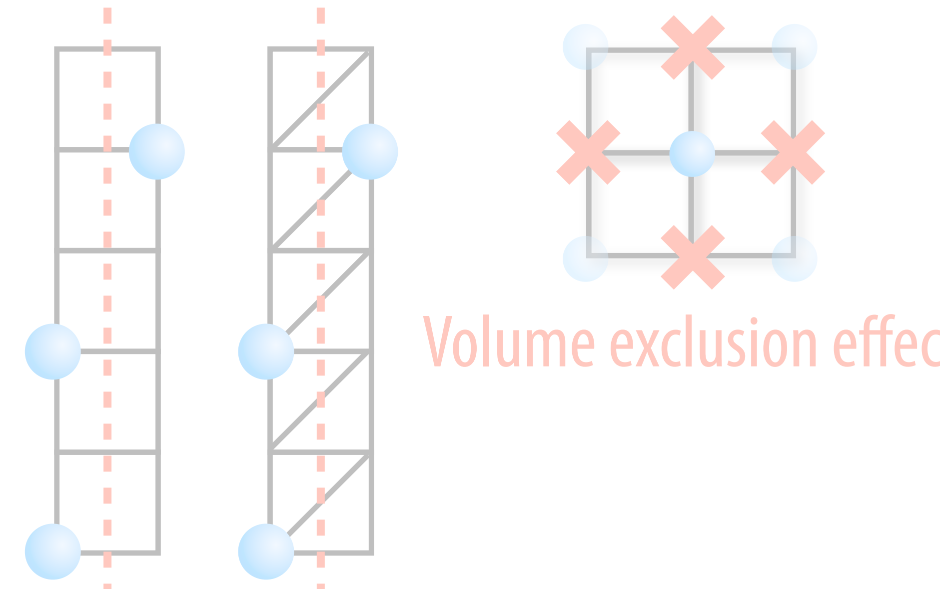
VBS on square lattice



VBS on hexagonal lattice



Quantum lattice gas on ladder



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Triangle ladder	2D 3-state Potts

VBS (Valence-Bond-Solid) state

Valence bond = Singlet pair $|\mathrm{s}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$

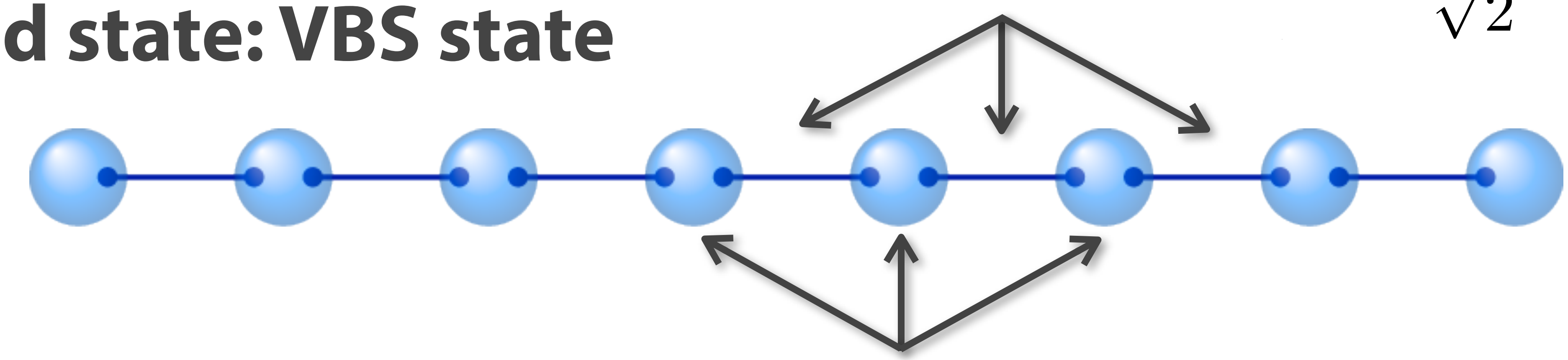
AKLT (Affleck-Kennedy-Lieb-Tasaki) model

*I. Affleck, T. Kennedy, E. Lieb, and H. Tasaki, PRL **59**, 799 (1987).*

$$\mathcal{H} = \sum_i \left[\vec{S}_i \cdot \vec{S}_{i+1} + \frac{1}{3}\left(\vec{S}_i \cdot \vec{S}_{i+1}\right)^2 \right] \qquad (S=1)$$

Ground state: VBS state

Valence bond $|\mathrm{s}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$



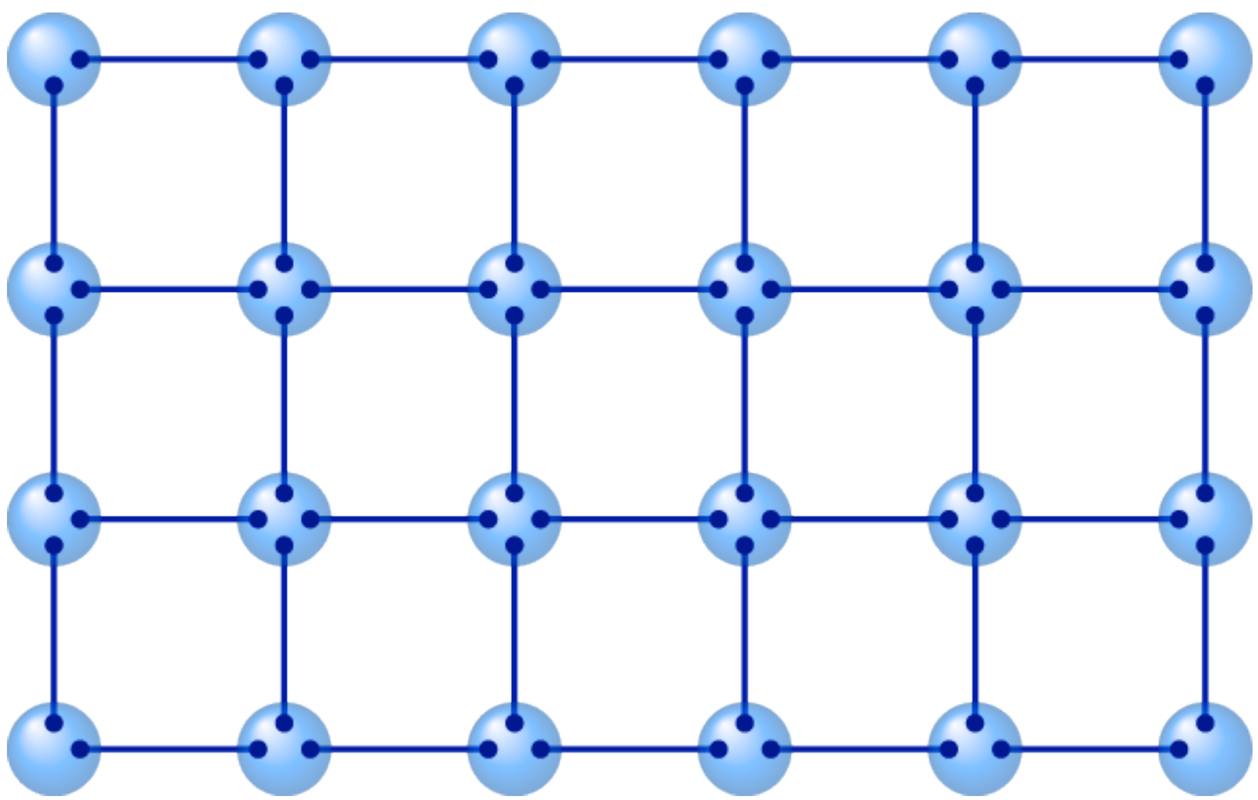
$S=1$ (projection)

- Exact unique ground state; S=1 VBS state
- Rigorous proof of the "Haldane gap"
- AFM correlation decays fast exponentially

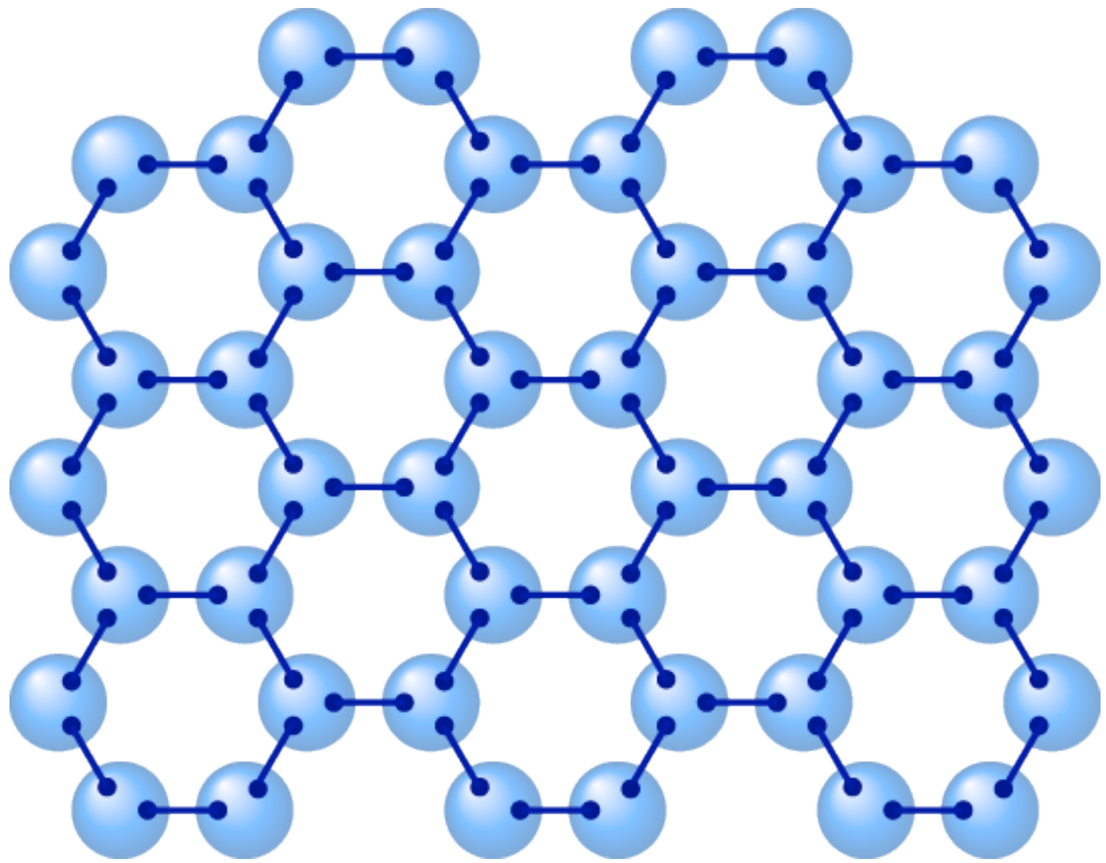
VBS (Valence-Bond-Solid) state

VBS state = Singlet-covering state

2D square lattice



2D hexagonal lattice



__MBQC using VBS state__

T-C. Wei, I. Affleck, and R. Raussendorf, Phys. Rev. Lett. ***106****, 070501 (2011).*

A. Miyake, Ann. Phys. ***326****, 1656 (2011).*

VBS (Valence-Bond-Solid) state

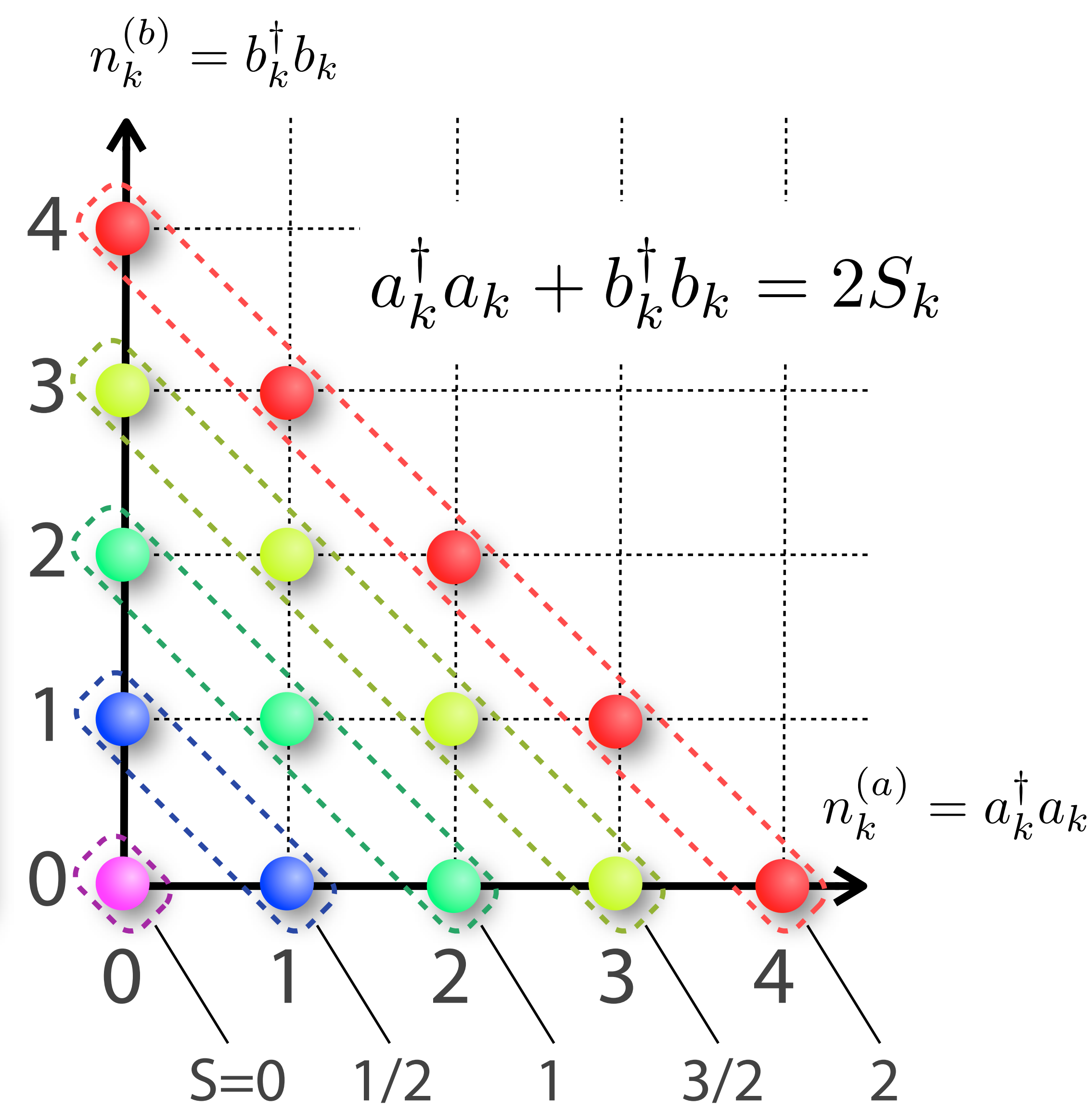
VBS state = Singlet-covering state

Schwinger boson representation

$$|\uparrow\rangle = a^\dagger|\mathrm{vac}\rangle, \quad |\downarrow\rangle = b^\dagger|\mathrm{vac}\rangle$$

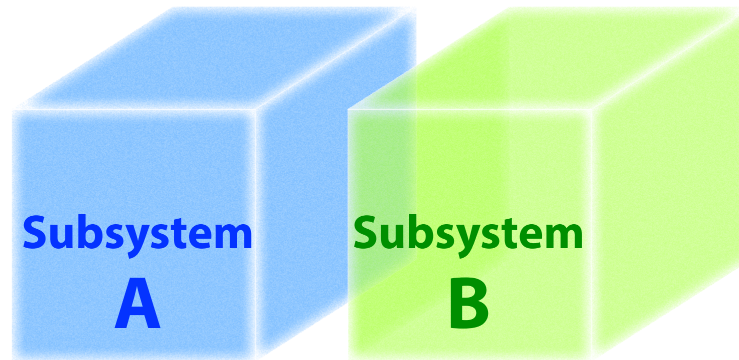
Valence bond solid (VBS) state

$$|\mathrm{VBS}\rangle = \prod_{\langle k,l\rangle}\left(a_k^\dagger b_l^\dagger - b_k^\dagger a_l^\dagger\right)|\mathrm{vac}\rangle$$

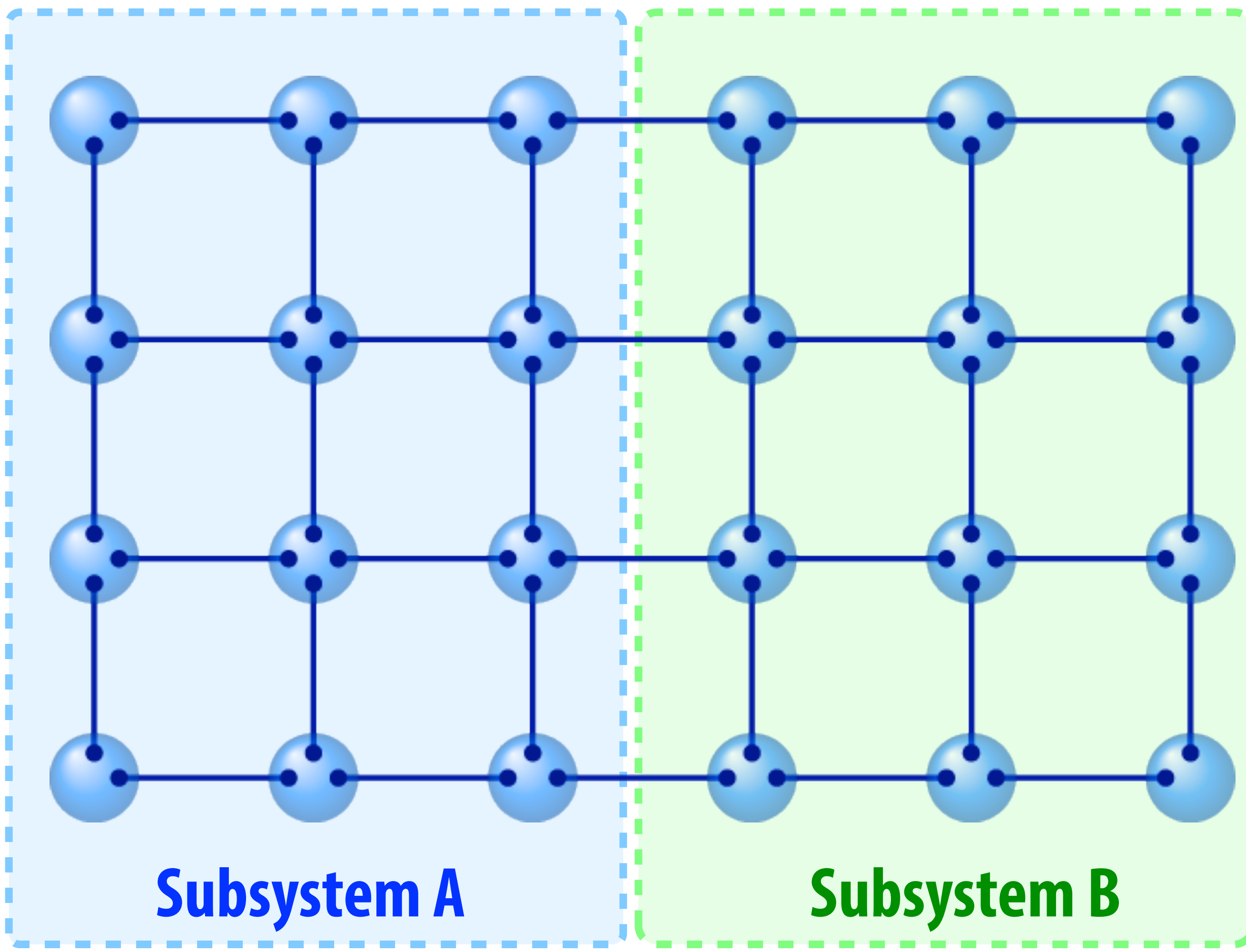


VBS (Valence-Bond-Solid) state

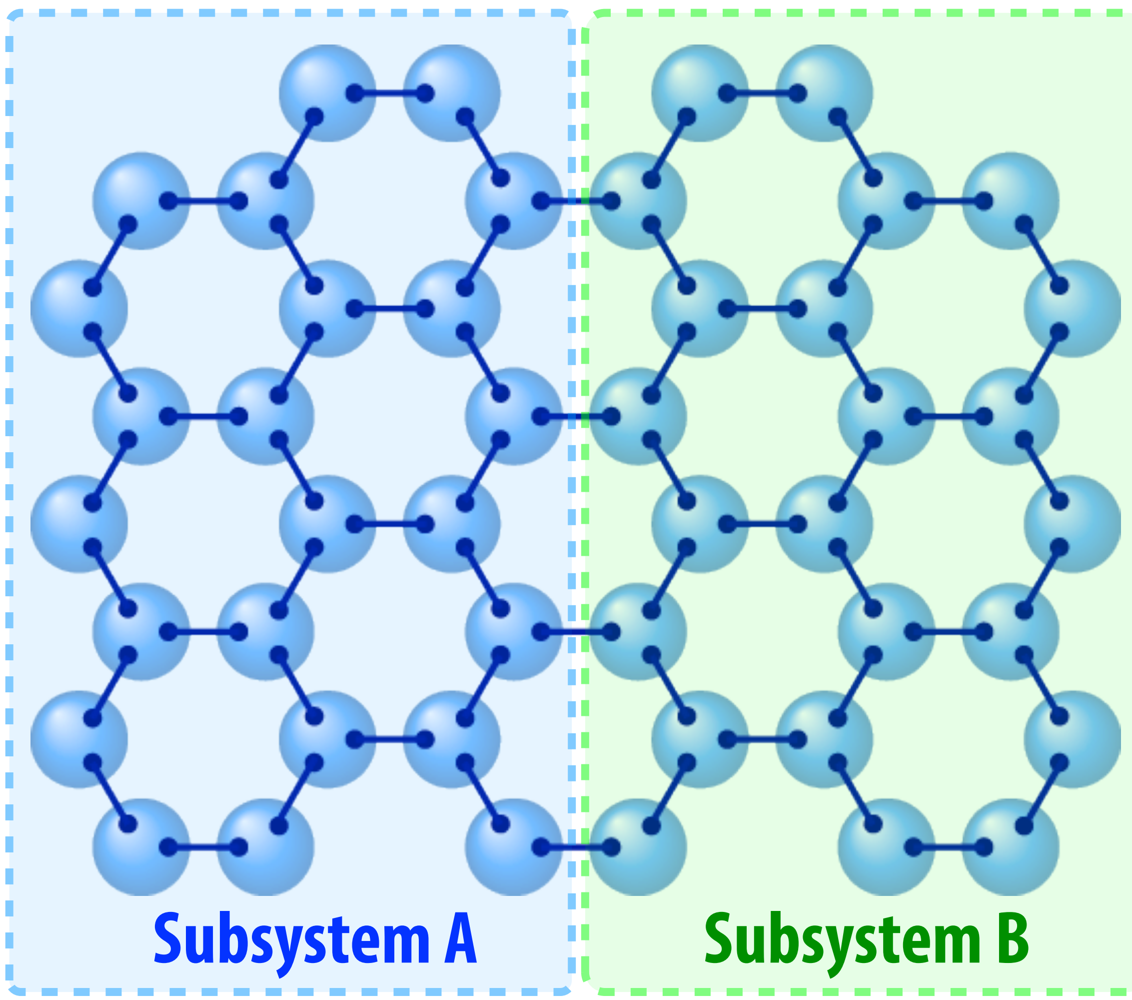
Reflection symmetry



2D square lattice



2D hexagonal lattice



VBS (Valence-Bond-Solid) state

$$|\mathrm{VBS}\rangle = \prod_{\langle k,l\rangle} \left(a_k^\dagger b_l^\dagger - b_k^\dagger a_l^\dagger\right)|\mathrm{vac}\rangle$$

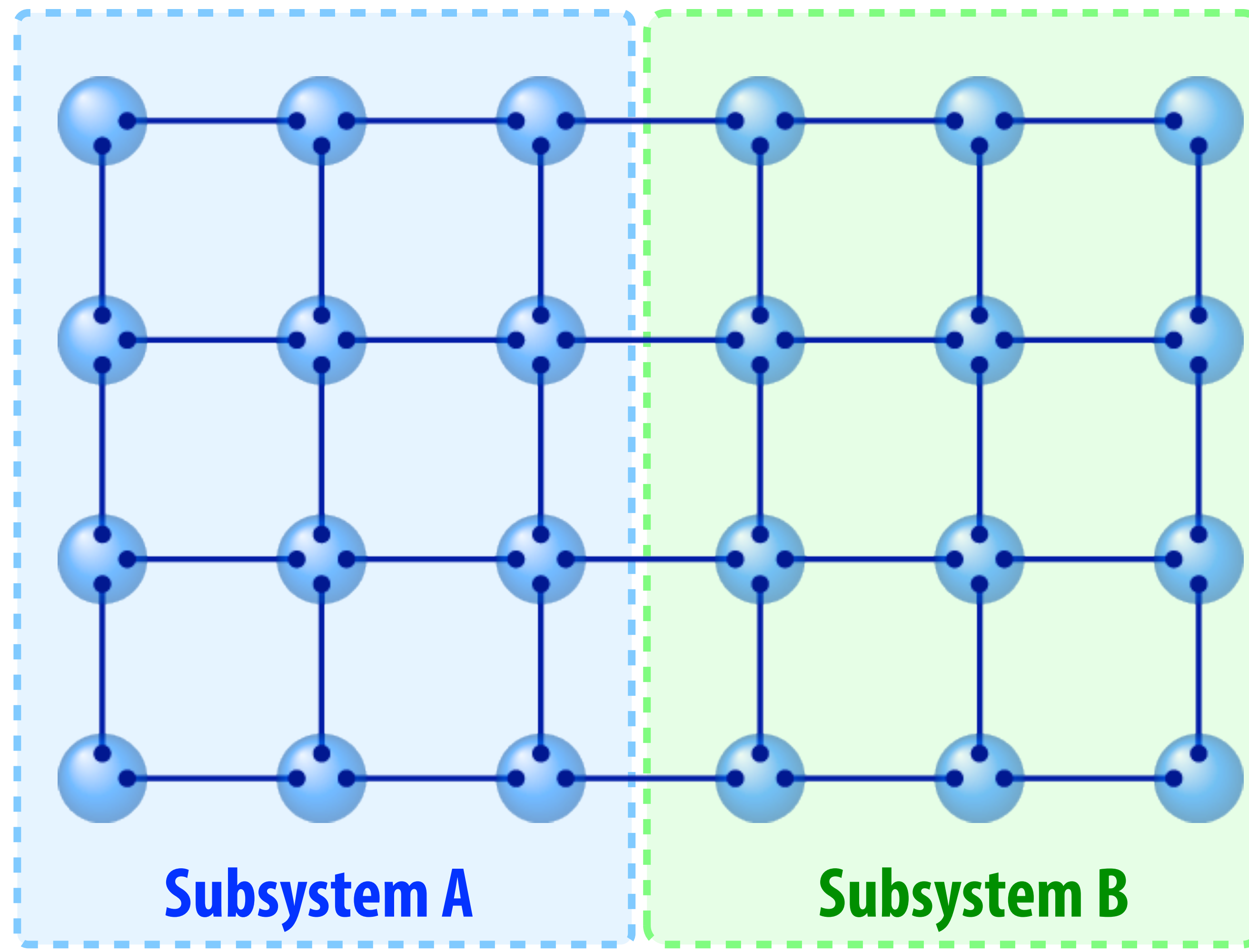
$$= \sum_{\{\alpha\}} |\phi_\alpha^{[\mathrm{A}]}\rangle \otimes |\phi_\alpha^{[\mathrm{B}]}\rangle$$

- Local gauge transformation
- Reflection symmetry

$\{\alpha\} = \{\alpha_1, \cdots, \alpha_{|\Lambda_\mathrm{A}|}\}$

Auxiliary spin: $\alpha_i = \pm 1/2$

#bonds on edge: $|\Lambda_\mathrm{A}|$



Overlap matrix

$M_{\{\alpha\},\{\beta\}}$: $2^{|\Lambda_\mathrm{A}|} \times 2^{|\Lambda_\mathrm{A}|}$ matrix

Each element can be obtained by Monte Carlo calculation!!

SU(N) case can be also calculated.

Phys. Rev. B, ***84***, *245128 (2011)*

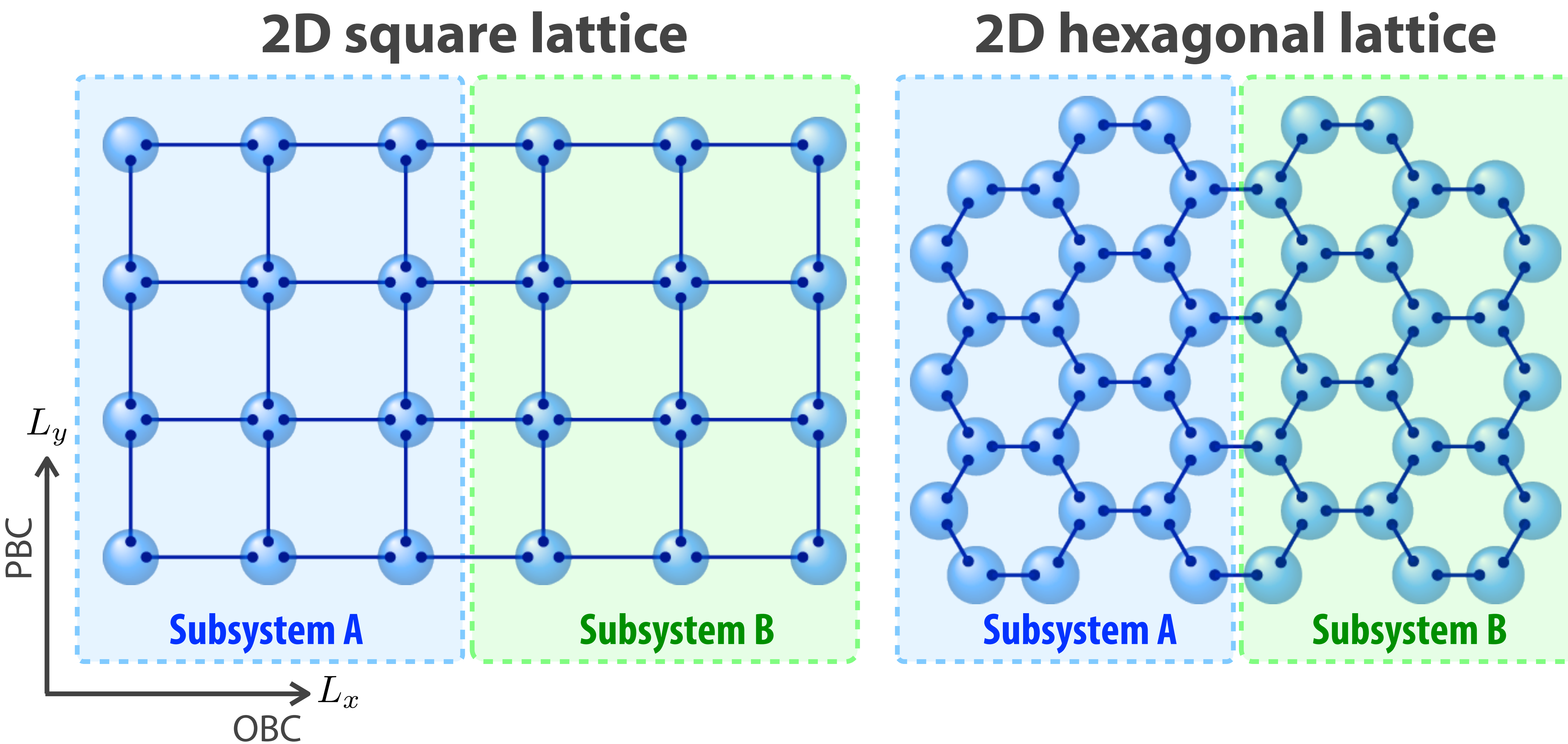
cf. H. Katsura, arXiv:1407.4262

Entanglement properties

- ***Entanglement entropy***
- ***Entanglement spectrum***
- ***Nested entanglement entropy***

Entanglement properties of 2D VBS states

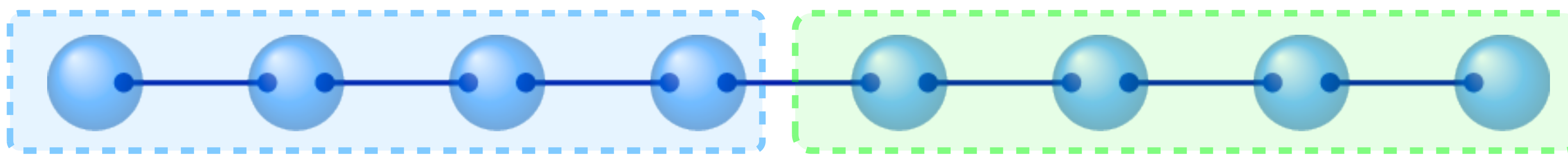
VBS state = Singlet-covering state

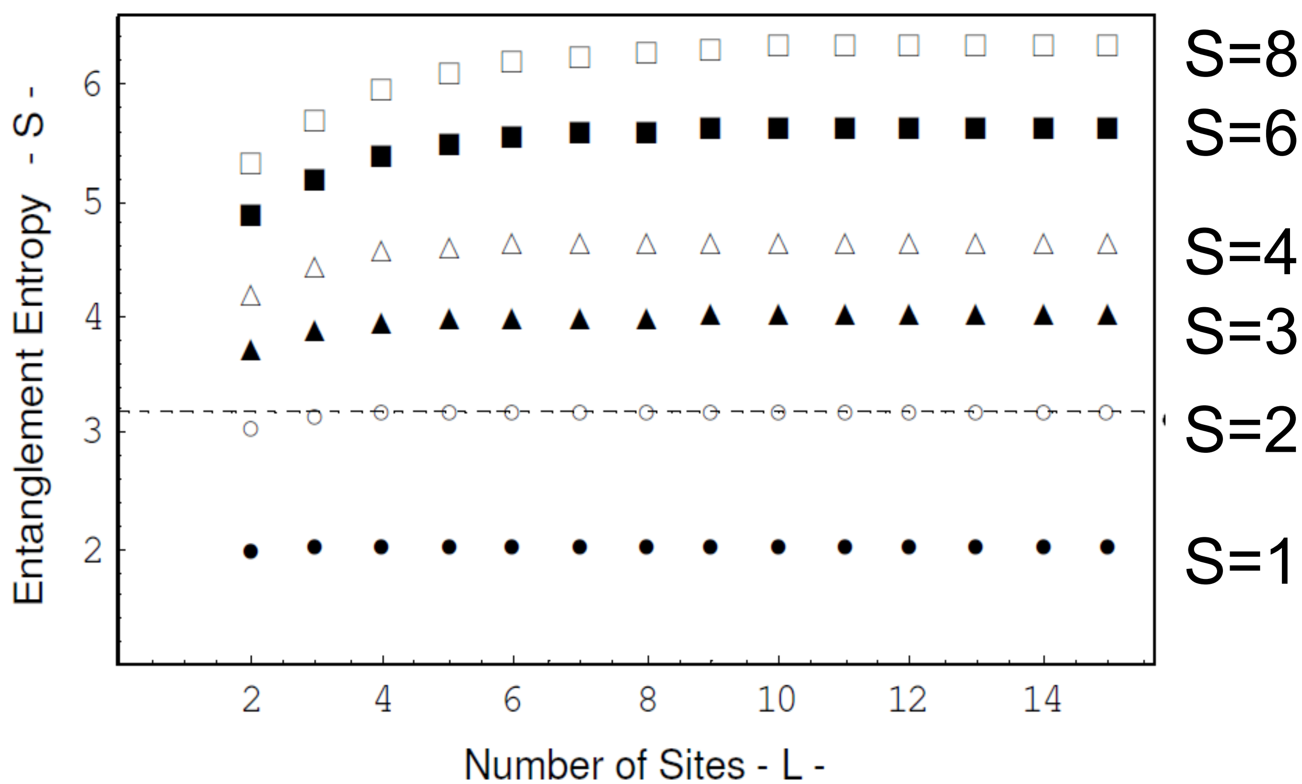


Entanglement entropy of 2D VBS states

cf. Entanglement entropy of **1D** VBS states

$$|\mathrm{VBS}\rangle = \prod_{i=0}^{N} \left(a_i^\dagger b_{i+1}^\dagger - b_i^\dagger a_{i+1}^\dagger \right)^S |\mathrm{vac}\rangle$$





*H. Katsura, T. Hirano, and Y. Hatsugai, PRB **76**, 012401 (2007).*

S = ln (# Edge states)

Entanglement entropy of 2D VBS states

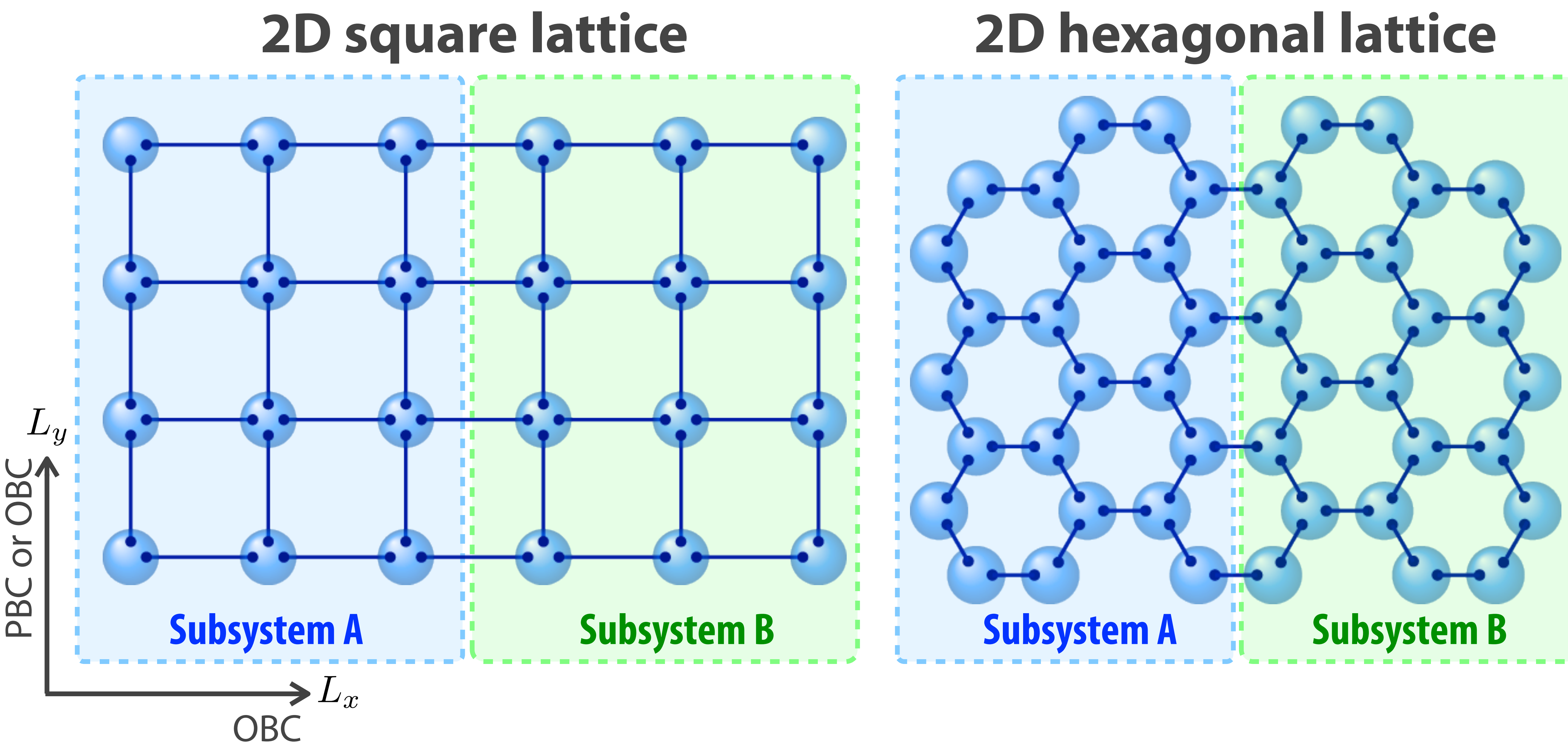
$$\frac{\mathcal{S}}{|\Lambda_{\mathrm{A}}|} = \ln 2 - \sigma$$

$\sigma \geq 0$

$\sigma_{1\mathrm{D}} = 0$

$\sigma_{\mathrm{square}} > \sigma_{\mathrm{hexagonal}}$ ⟸ $\xi_{\mathrm{square}} > \xi_{\mathrm{hexagonal}}$

#bonds on edge: $|\Lambda_{\mathrm{A}}|$



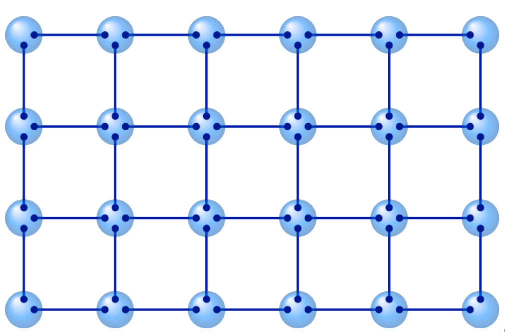
Entanglement spectra of 2D VBS states

*H. Li and F. D. M. Haldane, Phys. Rev. Lett. **101**, 010504 (2008).*

Reduced density matrix $\rho_{\rm A} = \sum_{\alpha} {\rm e}^{-\lambda_\alpha} |\phi_\alpha^{[{\rm A}]}\rangle\langle\phi_\alpha^{[{\rm A}]}|$

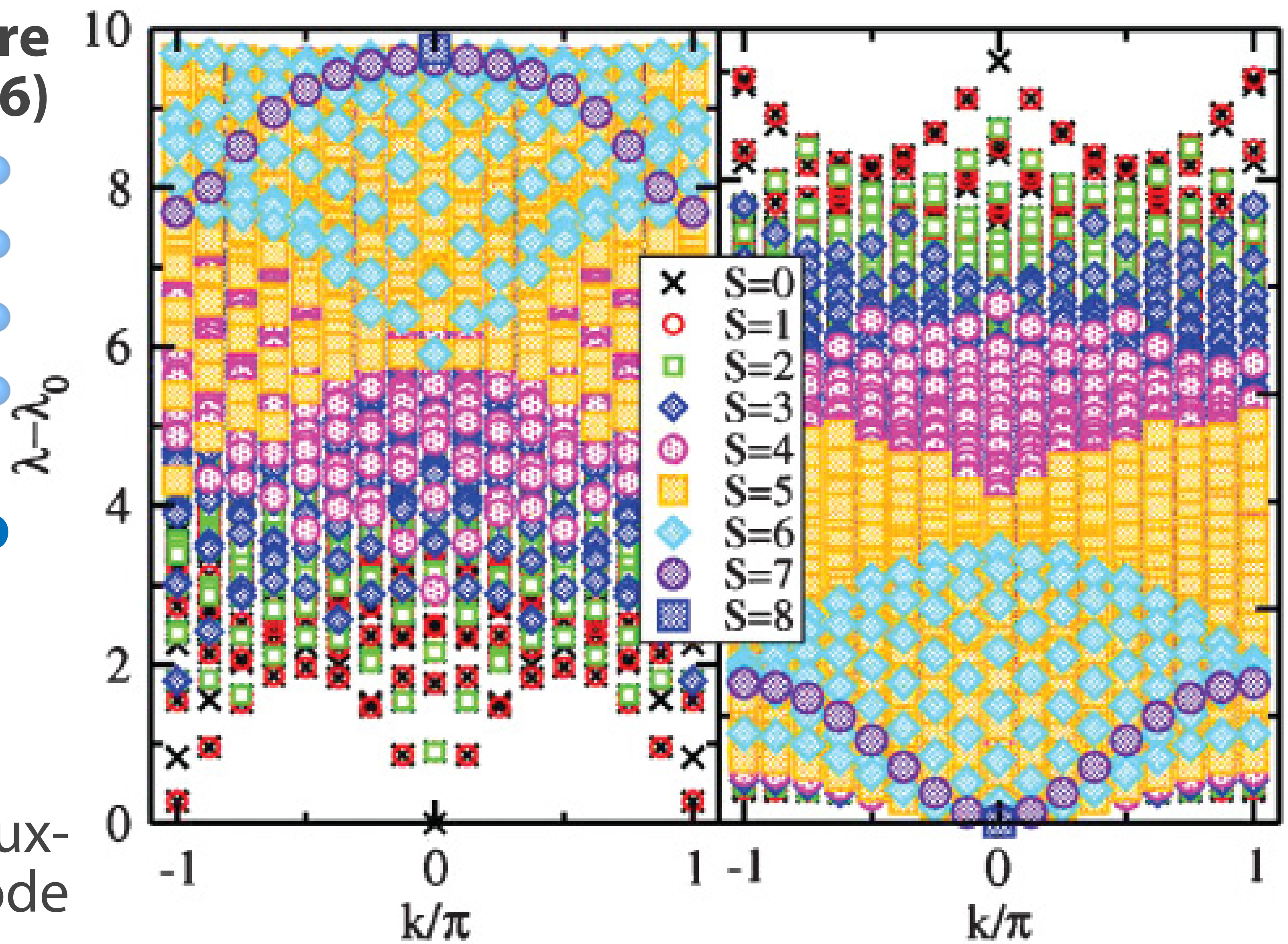
Entanglement Hamiltonian $\rho_{\rm A} = {\rm e}^{-\mathcal{H}_{\rm E}} \quad (\mathcal{H}_{\rm E} = -\ln \rho_{\rm A})$

Square
(L_x=5, L_y=16)

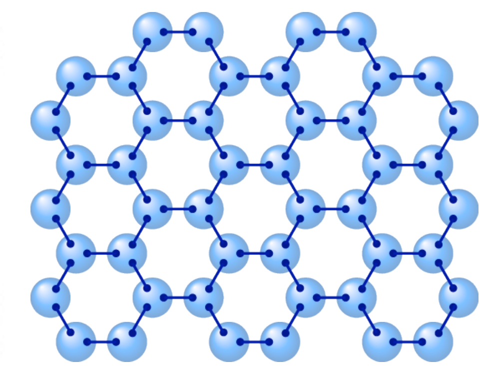


1D antiferro Heisenberg

des Cloizeaux-Pearson mode



Hexagonal
(L_x=5, L_y=32)



1D ferro Heisenberg

Spin wave

*cf. J. I. Cirac, D. Poilbranc, N. Schuch, and F. Verstraete, Phys. Rev. B **83**, 245134 (2011).*

Nested entanglement entropy

"Entanglement" ground state := g.s. of $\mathcal{H}_{\rm E}$: $|\Psi_0\rangle$

$$\mathcal{H}_{\rm E} = -\ln \rho_{\rm A}$$

$$\mathcal{H}_{\rm E}|\Psi_0\rangle = E_{\rm gs}|\Psi_0\rangle \Longleftrightarrow \rho_{\rm A}|\Psi_0\rangle = \underline{\underline{\rho_0}}|\Psi_0\rangle$$

Maximum eigenvalue

Nested reduced density matrix

$$\rho(\ell) := \mathrm{Tr}_{\ell+1,\cdots,L}\left[|\Psi_0\rangle\langle\Psi_0|\right]$$

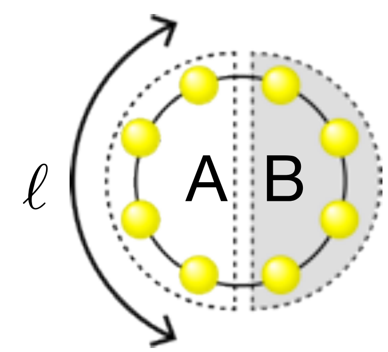
Nested entanglement entropy

$$\mathcal{S}(\ell, L) = -\mathrm{Tr}_{1,\cdots,\ell}\left[\rho(\ell)\ln\rho(\ell)\right]$$

1D quantum critical system (periodic boundary condition)

$$\mathcal{S}^{\rm PBC}(\ell, L_y) = \frac{c}{3}\ln[f(\ell)] + s_1$$

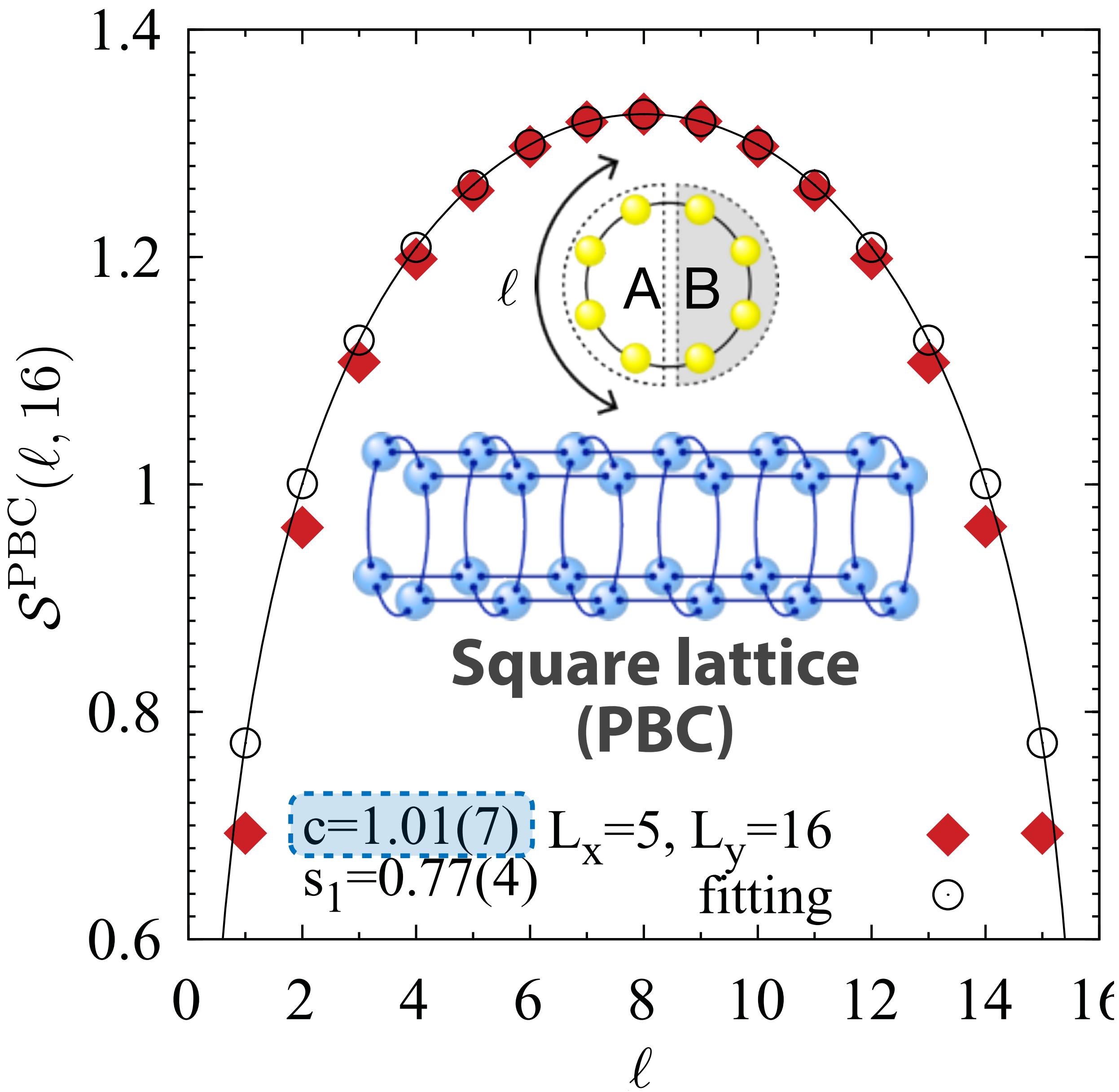
$$f(\ell) = \frac{L_y}{\pi}\sin\left(\frac{\pi\ell}{L_y}\right)$$



P. Calabrese and J. Cardy, J. Stat. Mech. (2004) P06002.

Nested entanglement entropy

$$\mathcal{S}(\ell, L) = -\mathrm{Tr}_{1,\cdots,\ell}[\rho(\ell)\ln\rho(\ell)]$$



Central charge: $c = 1$ **1D antiferromagnetic Heisenberg**

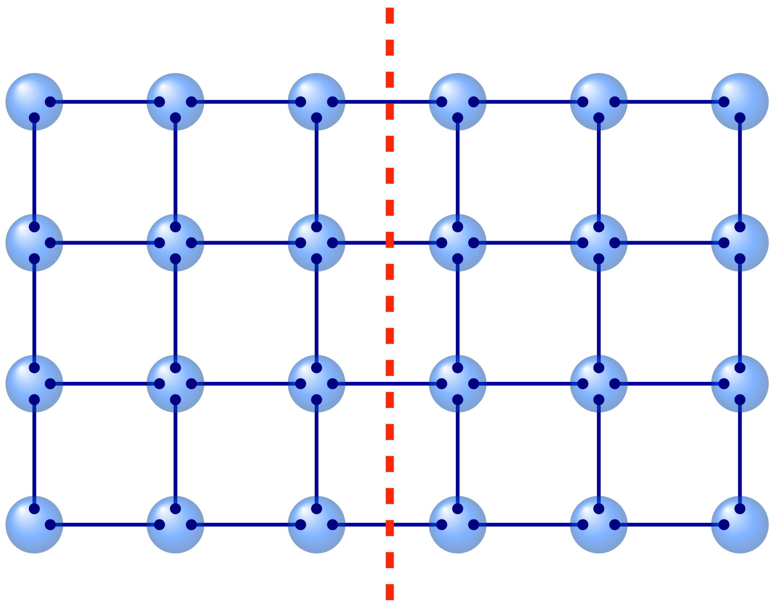
des Cloizeaux-Pearson mode in ES supports this result.

VBS/CFT correspondence

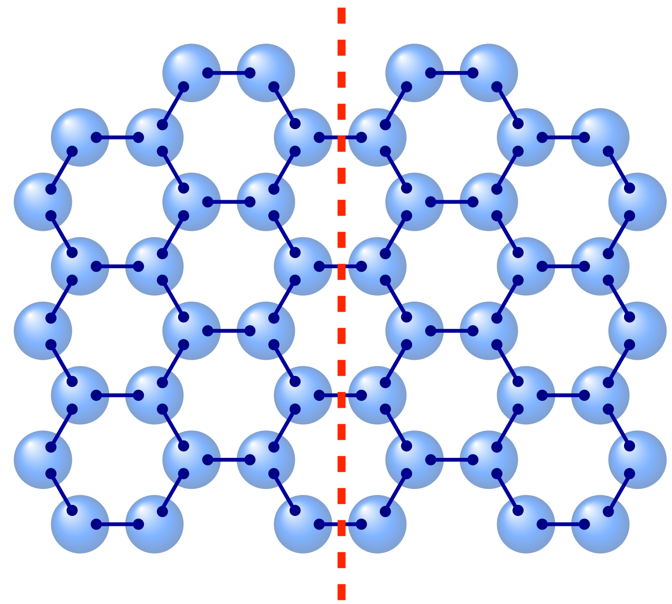
Digest

Entanglement properties of 2D quantum systems ⟷ **Physical properties of 1D quantum systems**

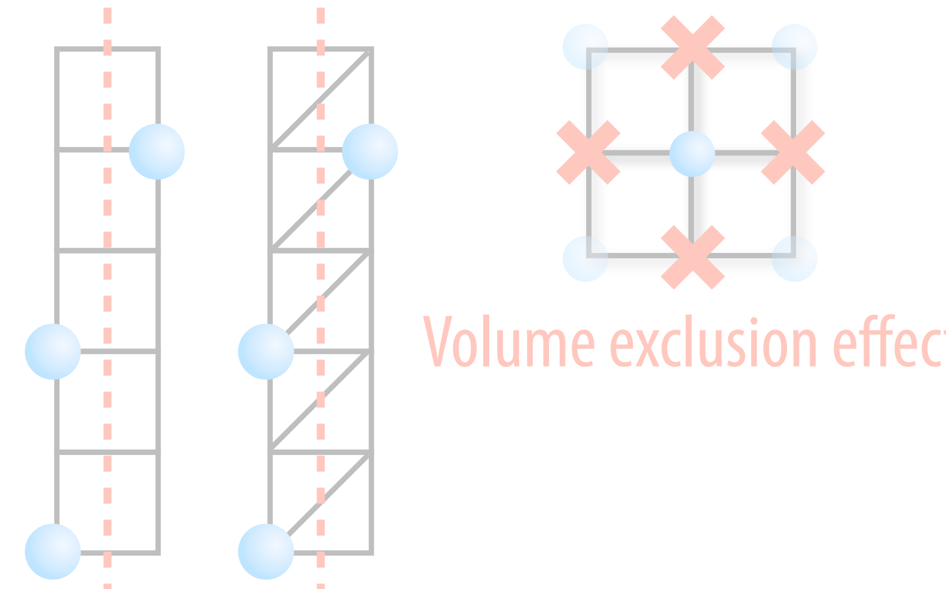
VBS on square lattice



VBS on hexagonal lattice



Quantum lattice gas on ladder



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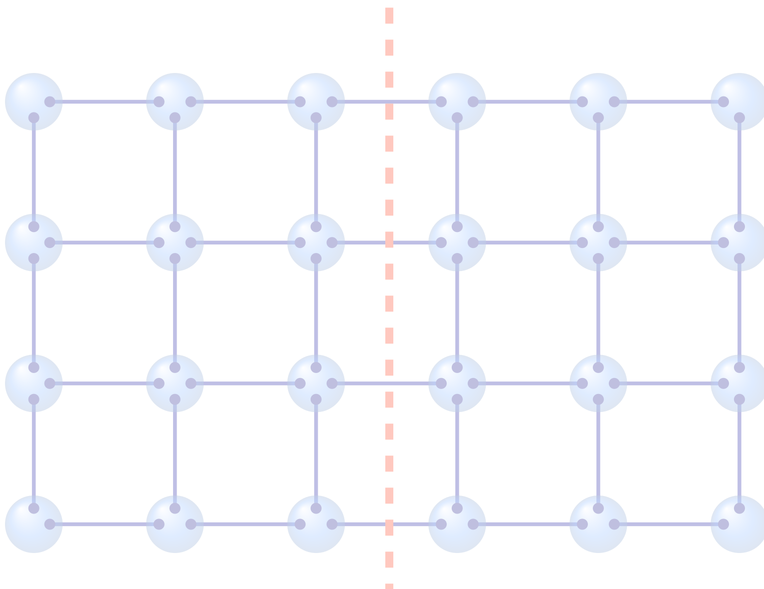
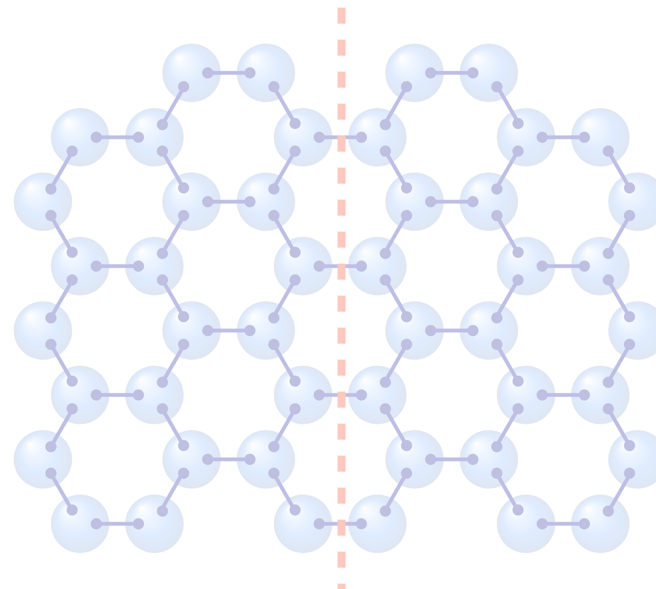


VBS on hexagonal lattice



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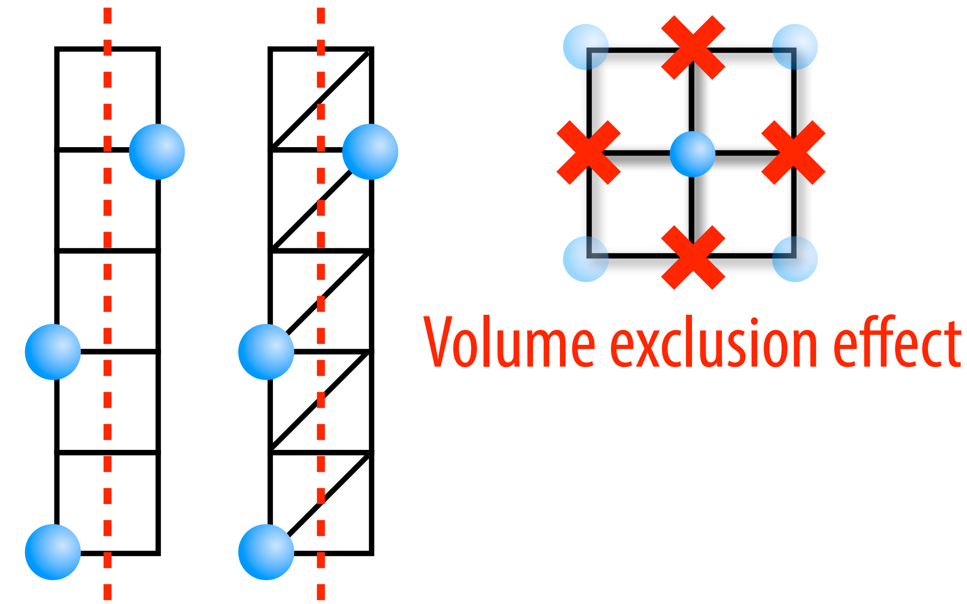

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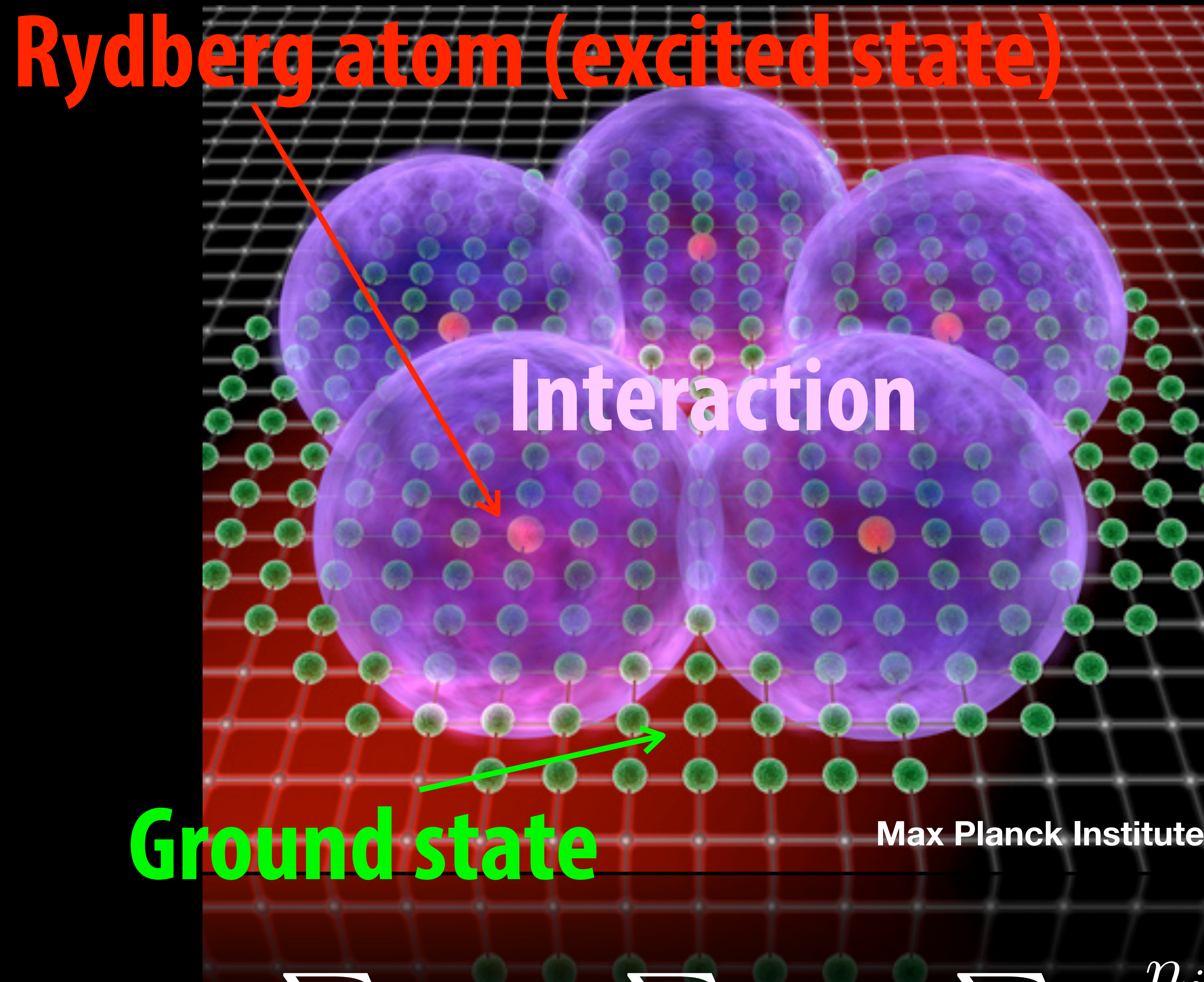
Rydberg Atom



$$\mathcal{H} = \Omega \sum_{i\in\Lambda} \sigma_i^x + \Delta \sum_{i\in\Lambda} n_i + V \sum_{i,j} \frac{n_i n_j}{|\mathbf{r}_i - \mathbf{r}_j|^\gamma}$$

Quantum hard-core lattice gas model

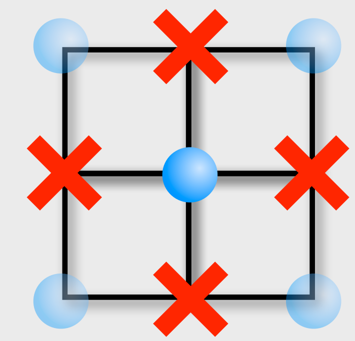
Construct a solvable model

$$\mathcal{H}_{\rm sol} = \sum_{i\in\Lambda} h_i^\dagger(z) h_i(z), \quad h_i(z) := [\sigma_i^- - \sqrt{z}(1-n_i)]\mathcal{P}_{\langle i\rangle}$$

$$\mathcal{H}_{\rm sol} = \underbrace{-\sqrt{z}\sum_{i\in\Lambda}(\sigma_i^+ + \sigma_i^-)\mathcal{P}_{\langle i\rangle}}_{\text{Creation/annihilation}} + \underbrace{\sum_{i\in\Lambda}[(1-z)n_i + z]\mathcal{P}_{\langle i\rangle}}_{\text{Interaction btw particles \& chemical potential}}$$

$$n_i = \frac{\sigma_i^z + 1}{2}$$

$$\mathcal{P}_{\langle i\rangle} := \prod_{j\in G_i}(1-n_j)$$



1-dim chain

$$\mathcal{H} = \sum_{i=1}^{L} \mathcal{P}\left[-\sqrt{z}\sigma_i^x + (1-3z)n_i + z n_{i-1}n_{i+1} + z\right]\mathcal{P}$$

Transverse Ising model with constraint

Hamiltonian is positive semi-definite. → Eigenenergies are non-negative.

Zero-energy state (ground state)

$$|z\rangle = \frac{1}{\sqrt{\Xi(z)}}\prod_{i\in\Lambda}\exp(\sqrt{z}\sigma_i^+\mathcal{P}_{\langle i\rangle})\,|{\downarrow\downarrow}\cdots\downarrow\rangle$$

$|{\downarrow\downarrow}\cdots\downarrow\rangle$: Vacuum state

Unique (Perron-Frobenius theorem)

GS of the quantum hard-core lattice gas model

unnormalized ground state: $|\Psi(z)\rangle := \sqrt{\Xi(z)}|z\rangle = \sum_{\mathcal{C}\in\mathcal{S}} z^{n_\mathcal{C}/2}|\mathcal{C}\rangle$

$\mathcal{C}$: classical configuration of particle on Λ

$\langle\mathcal{C}|\mathcal{C}'\rangle = \delta_{\mathcal{C},\mathcal{C}'}$ ($|\mathcal{C}\rangle$ is orthonormal basis)

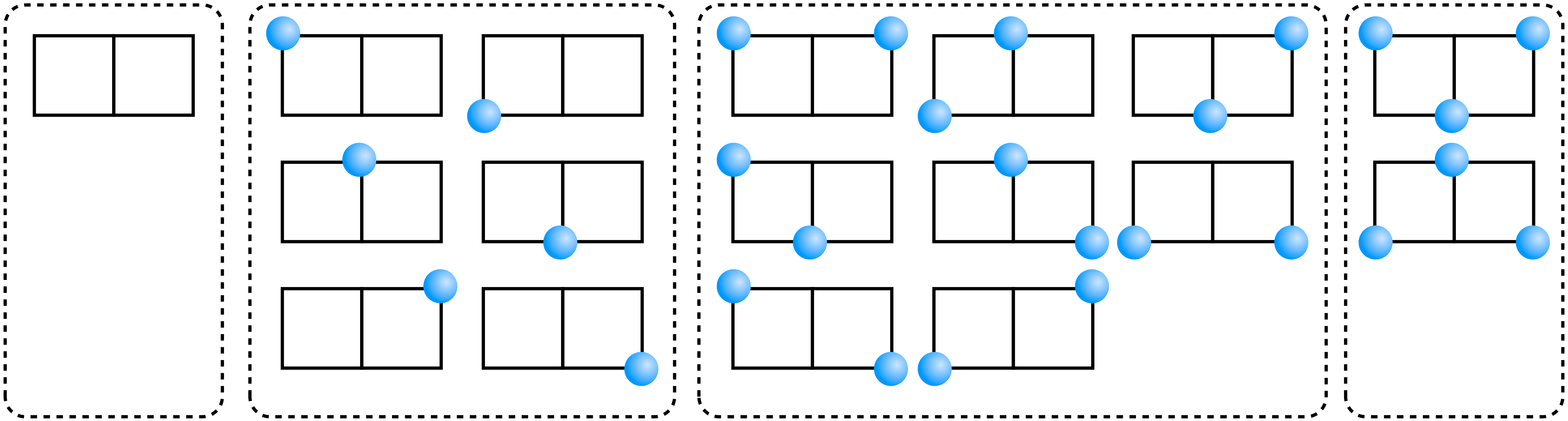
$\mathcal{S}$: set of classical configurations with "constraint"

$n_\mathcal{C}$: number of particles in the state $\mathcal{C}$

Normalization factor
= Partition function of classical hard-core lattice gas model

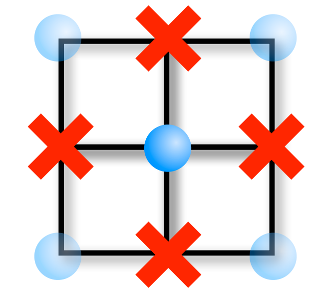
$$\Xi(z) = \langle\Psi(z)|\Psi(z)\rangle = \sum_{\mathcal{C}\in\mathcal{S}} z^{n_\mathcal{C}}$$

z : chemical potential

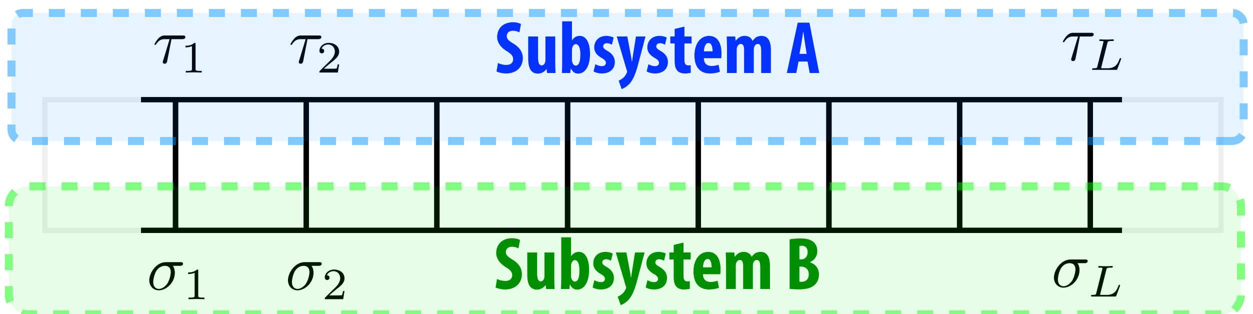


GS of the quantum hard-core lattice gas model

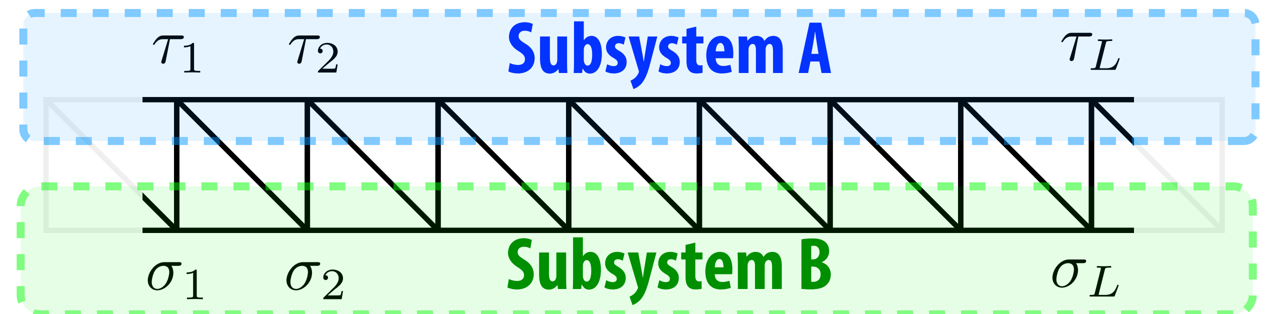
Periodic boundary condition is imposed in the leg direction.



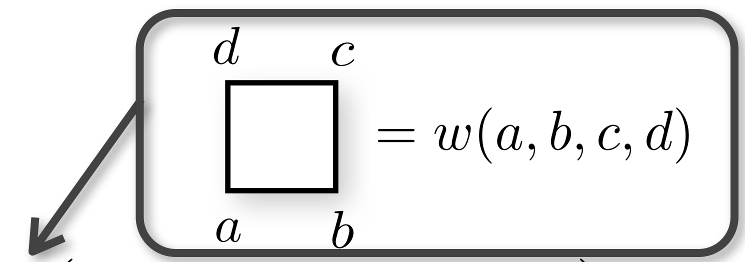
Square ladder



Triangle ladder

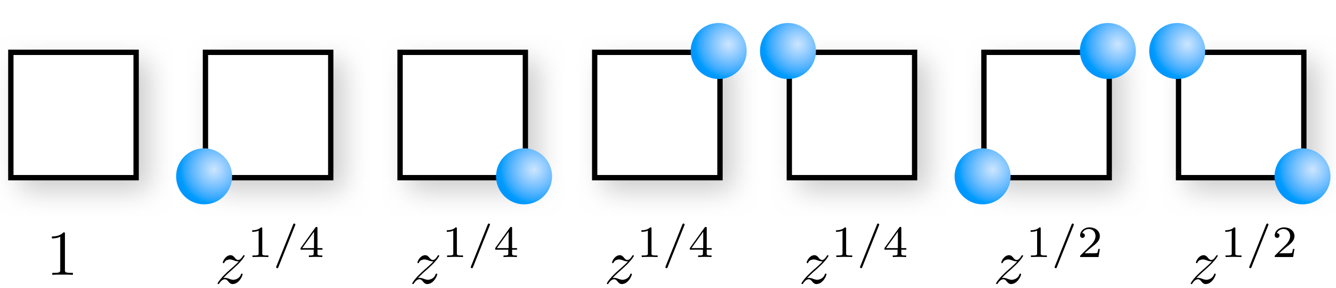


unnormalized ground state:



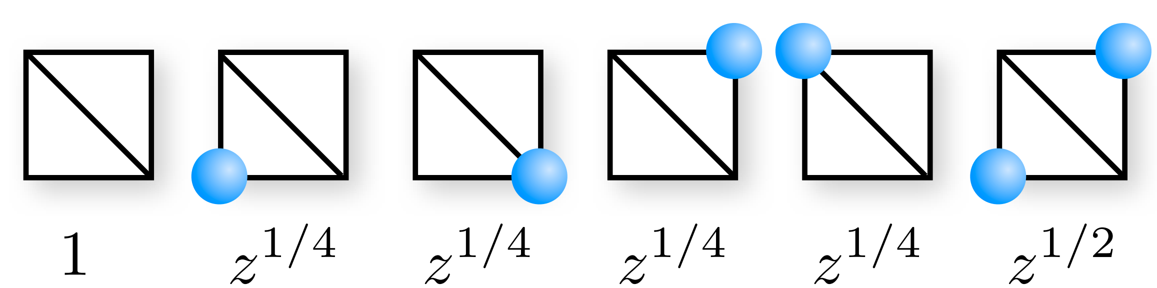
$$|\Psi(z)\rangle = \sum_{\sigma}\sum_{\tau}[T(z)]_{\tau,\sigma}|\tau\rangle\otimes|\sigma\rangle, \qquad [T(z)]_{\tau,\sigma} := \prod_{i=1}^{L} w(\sigma_i,\sigma_{i+1},\tau_{i+1},\tau_i)$$

Square ladder



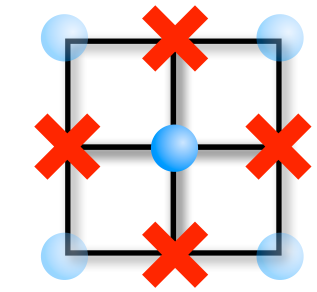
$$[T(z)]_{\tau,\sigma} = \prod_{i=1}^{L} z^{(\sigma_i+\tau_i)/2}(1-\sigma_i\tau_i)(1-\sigma_i\sigma_{i+1})(1-\tau_i\tau_{i+1})$$

Triangle ladder



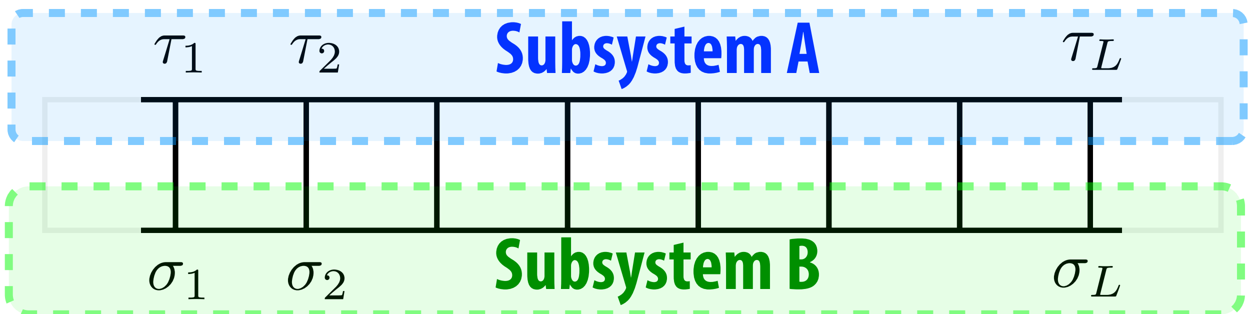
$$[T(z)]_{\tau,\sigma} = \prod_{i=1}^{L} z^{(\sigma_i+\tau_i)/2}(1-\sigma_i\tau_i)(1-\sigma_i\sigma_{i+1})(1-\tau_i\tau_{i+1})(1-\tau_i\sigma_{i+1})$$

GS of the quantum hard-core lattice gas model

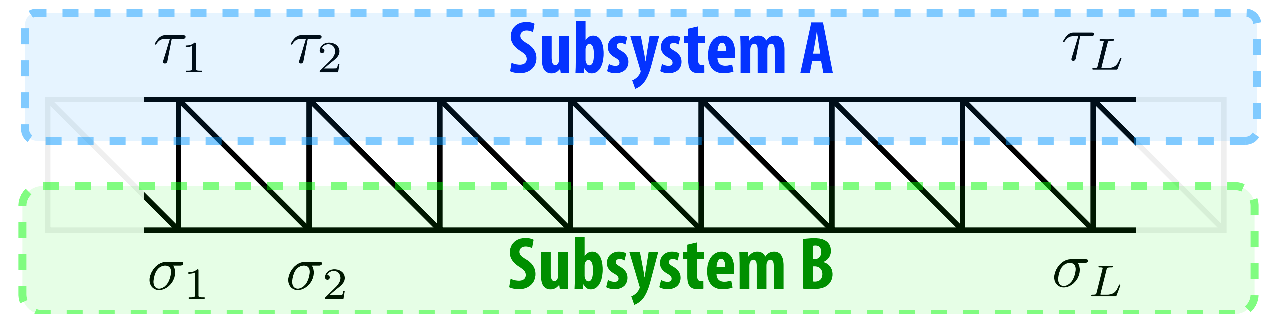


Periodic boundary condition is imposed in the leg direction.

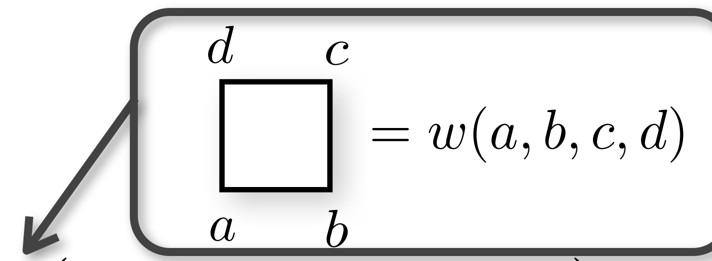
Square ladder



Triangle ladder



unnormalized ground state:



$$|\Psi(z)\rangle = \sum_{\sigma}\sum_{\tau}[T(z)]_{\tau,\sigma}|\tau\rangle\otimes|\sigma\rangle, \qquad [T(z)]_{\tau,\sigma} := \prod_{i=1}^{L} w(\sigma_i,\sigma_{i+1},\tau_{i+1},\tau_i)$$

$$|z\rangle = \frac{1}{\sqrt{\Xi(z)}}\sum_{\sigma}\sum_{\tau}[T(z)]_{\tau,\sigma}|\tau\rangle\otimes|\sigma\rangle$$

Overlap matrix

$$M = \frac{1}{\Xi(z)}[T(z)]^{\mathrm{T}}T(z)$$

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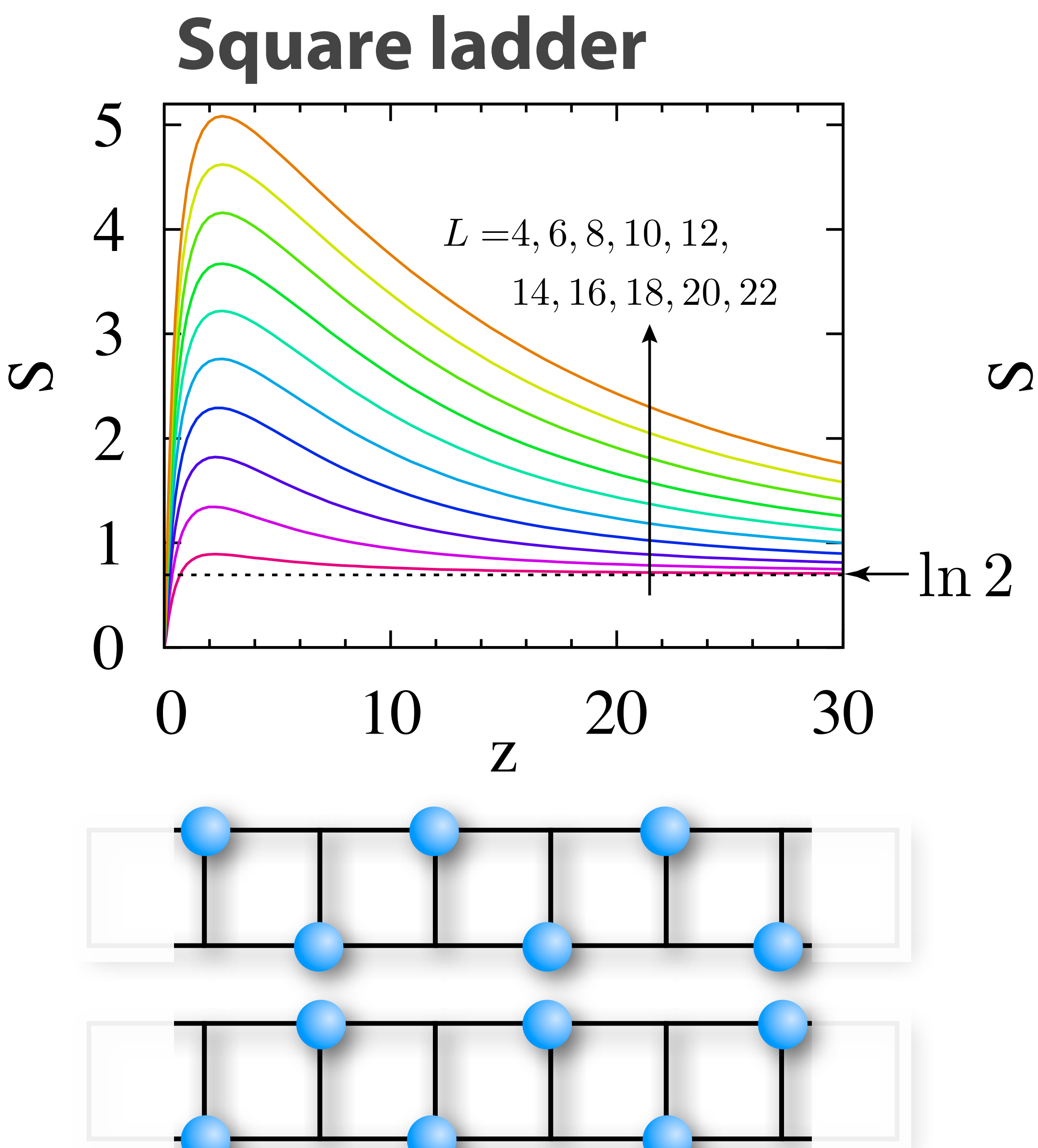
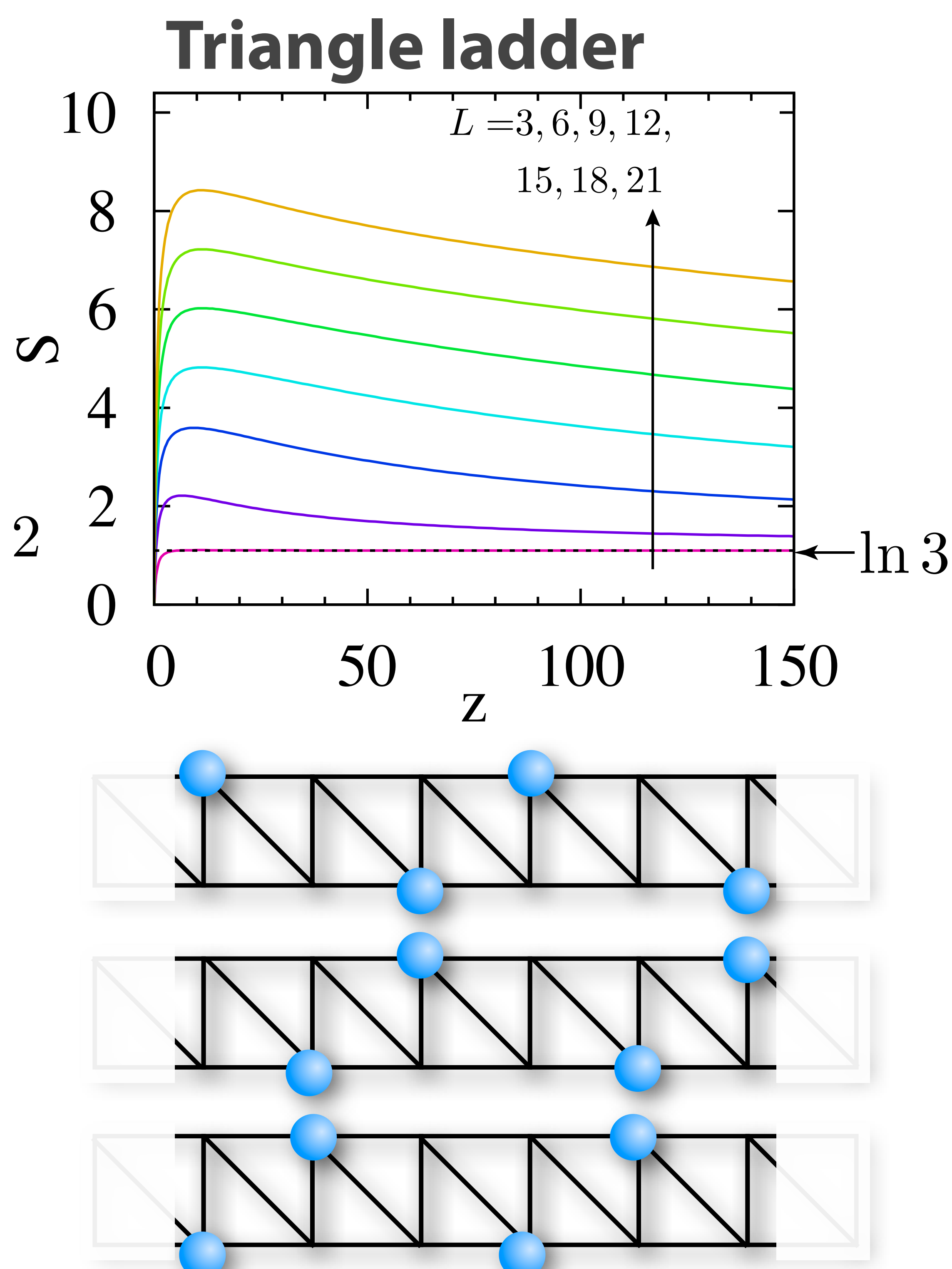
Entanglement entropy

$$\mathcal{S} = -\mathrm{Tr}\left[M \ln M\right] = -\sum_{\alpha} p_\alpha \ln p_\alpha \qquad p_\alpha \left(\alpha = 1, 2, \cdots, \underline{\underline{N_L}}\right)$$

# of states





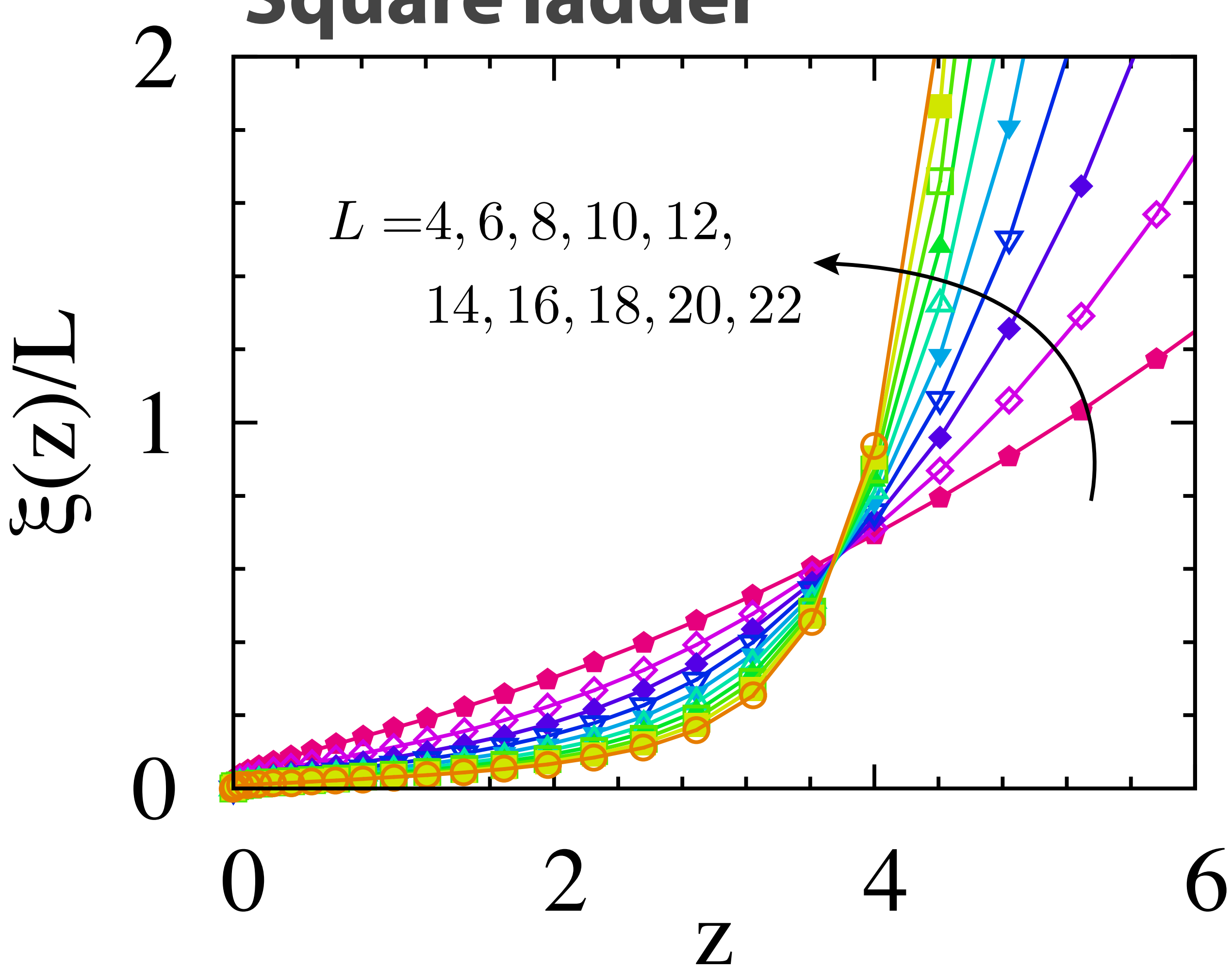
Estimation of zc

$$\xi(z) := \frac{1}{\ln[p^{(1)}(z)/p^{(2)}(z)]}$$

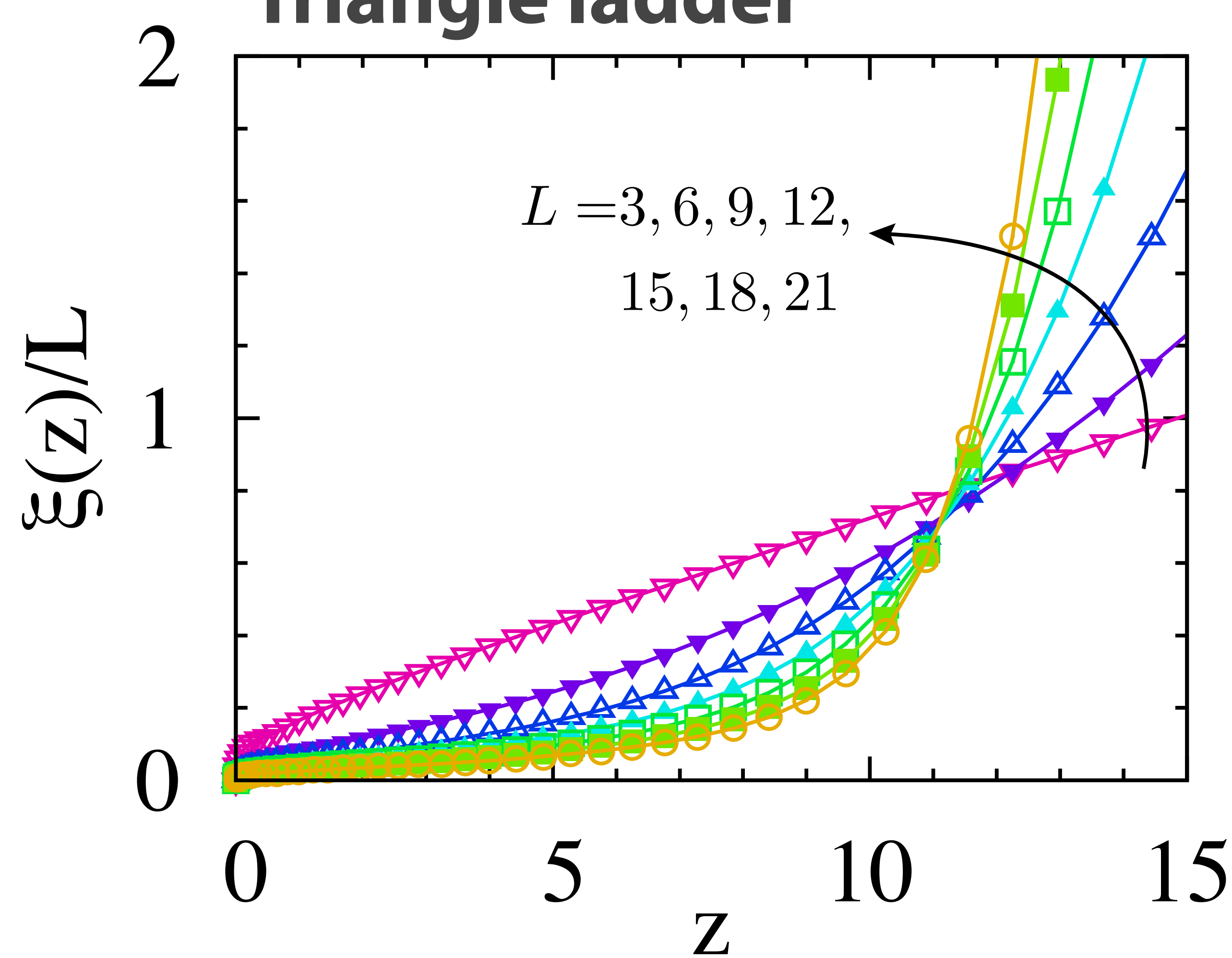
$p^{(1)}(z)$: the largest eigenvalue of M

$p^{(2)}(z)$: the second-largest eigenvalue of M

Square ladder



Triangle ladder

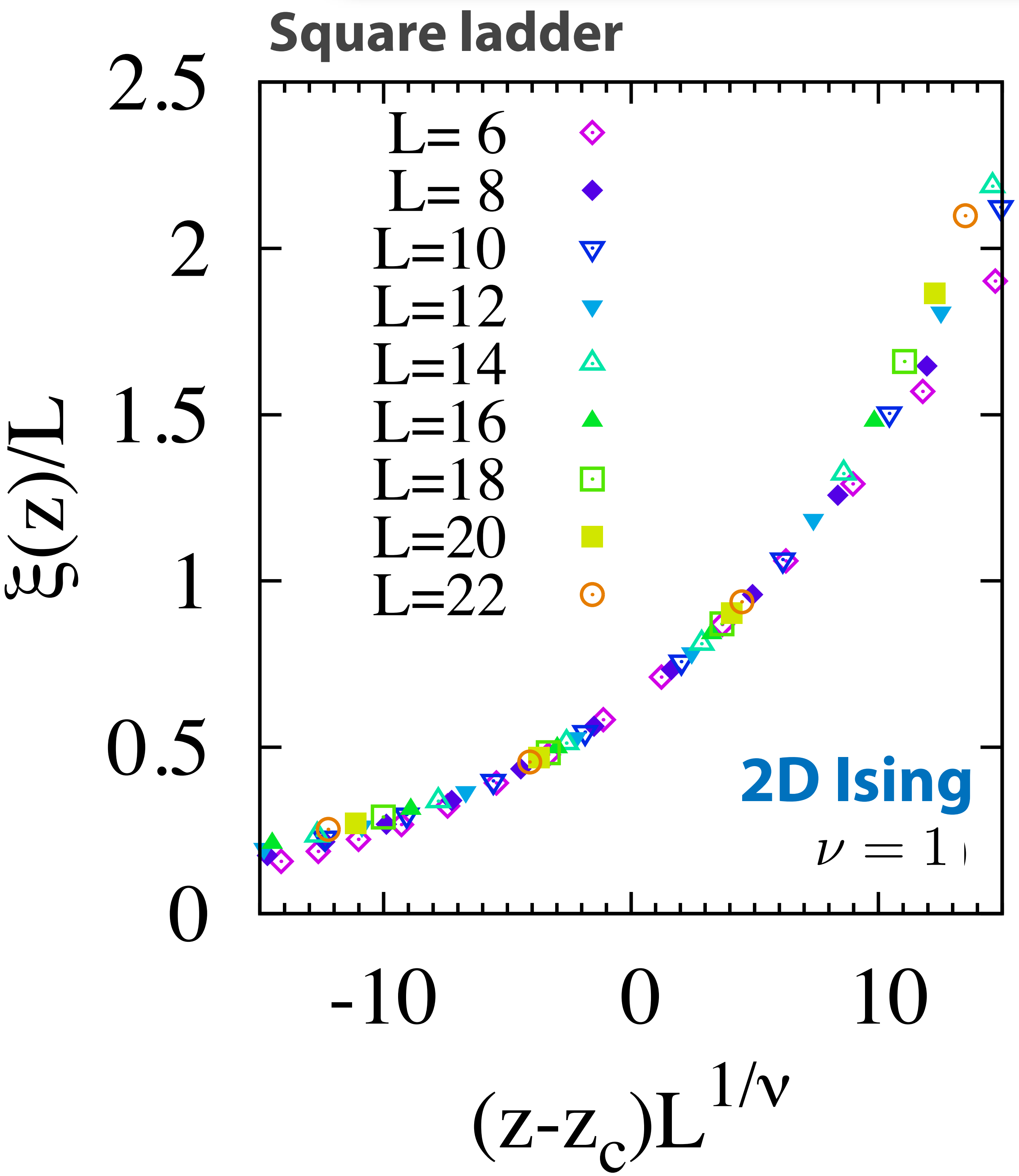


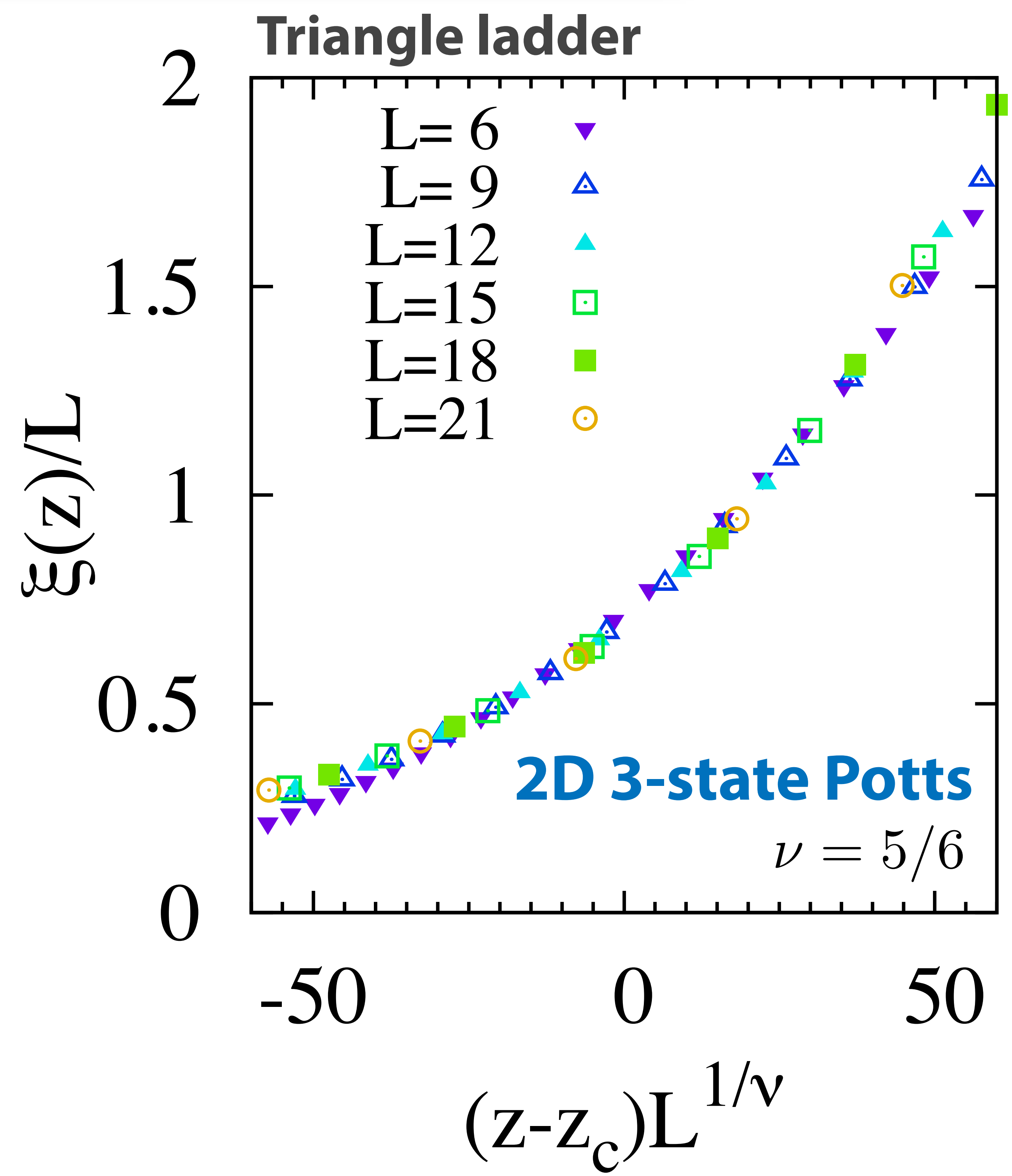
correlation length crosses at z_c

Finite-size scaling for correlation length

Finite-size scaling

Finite-size scaling relation: $\xi(z)/L = f[(z-z_c)L^{1/\nu}]$



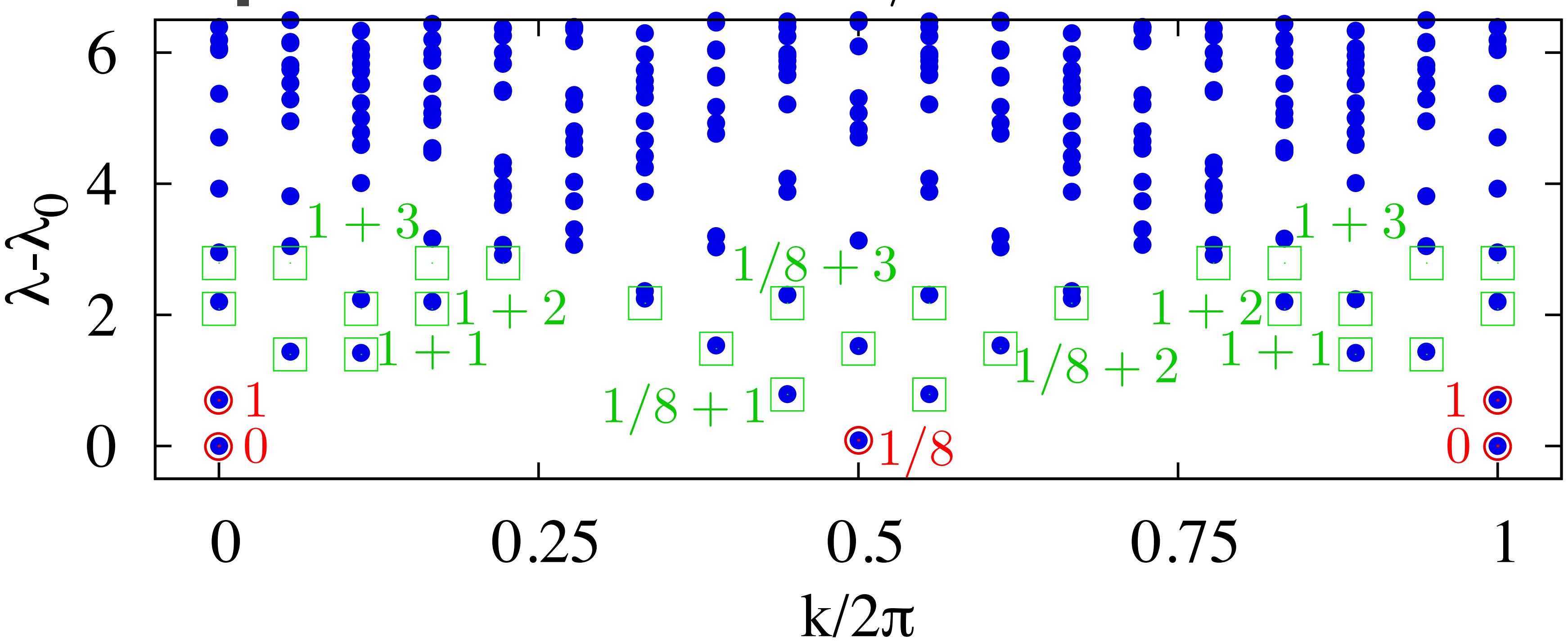


Entanglement spectra at z=zc

Eigenvalues of entanglement Hamiltonian at $z = z_c$

Square ladder $c = 1/2$ (**2-dim Ising**)

$$\lambda_\alpha - \lambda_0 = \frac{2\pi v}{L}(h_{L,\alpha} + h_{R,\alpha})$$



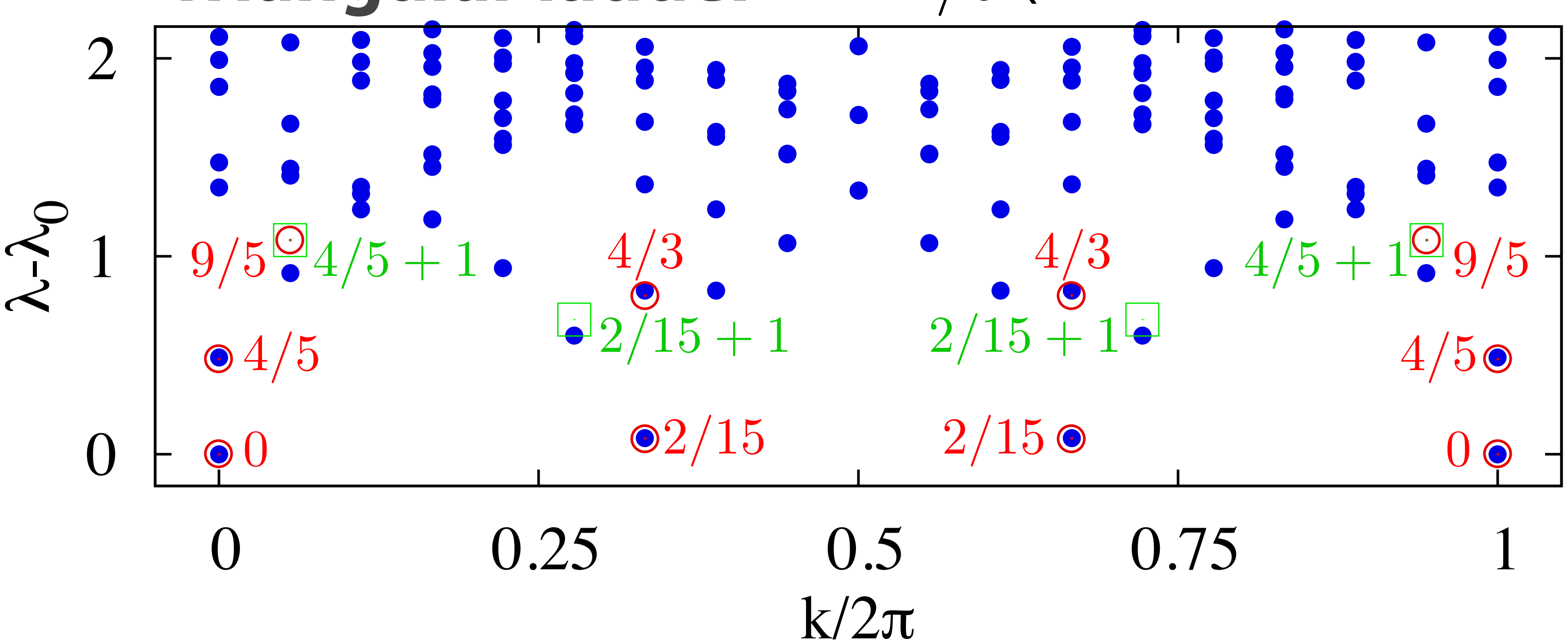
v : Velocity

$h_{L,\alpha}$: Holomorphic conformal weight

$h_{R,\alpha}$: Antiholomorphic conformal weight

$h_{L,\alpha} + h_{R,\alpha}$ Scaling dimension

Triangular ladder $c = 4/5$ (**2-dim 3-state Potts**)



Primary field

Descendant field

M. Henkel "Conformal invariance and critical phenomena" (Springer)

Nested entanglement entropy at z=zc

$|\psi_0\rangle$: Ground state of entanglement Hamiltonian ($z = z_c$)

nested reduced density matrix: $\rho(\ell) := \mathrm{Tr}_{\ell+1,\cdots,L}[|\psi_0\rangle\langle\psi_0|]$

nested entanglement entropy: $s(\ell, L) := -\mathrm{Tr}_{1,\cdots,\ell}[\rho(\ell)\ln\rho(\ell)]$

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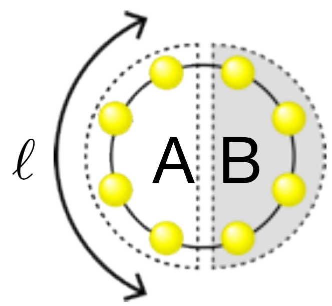
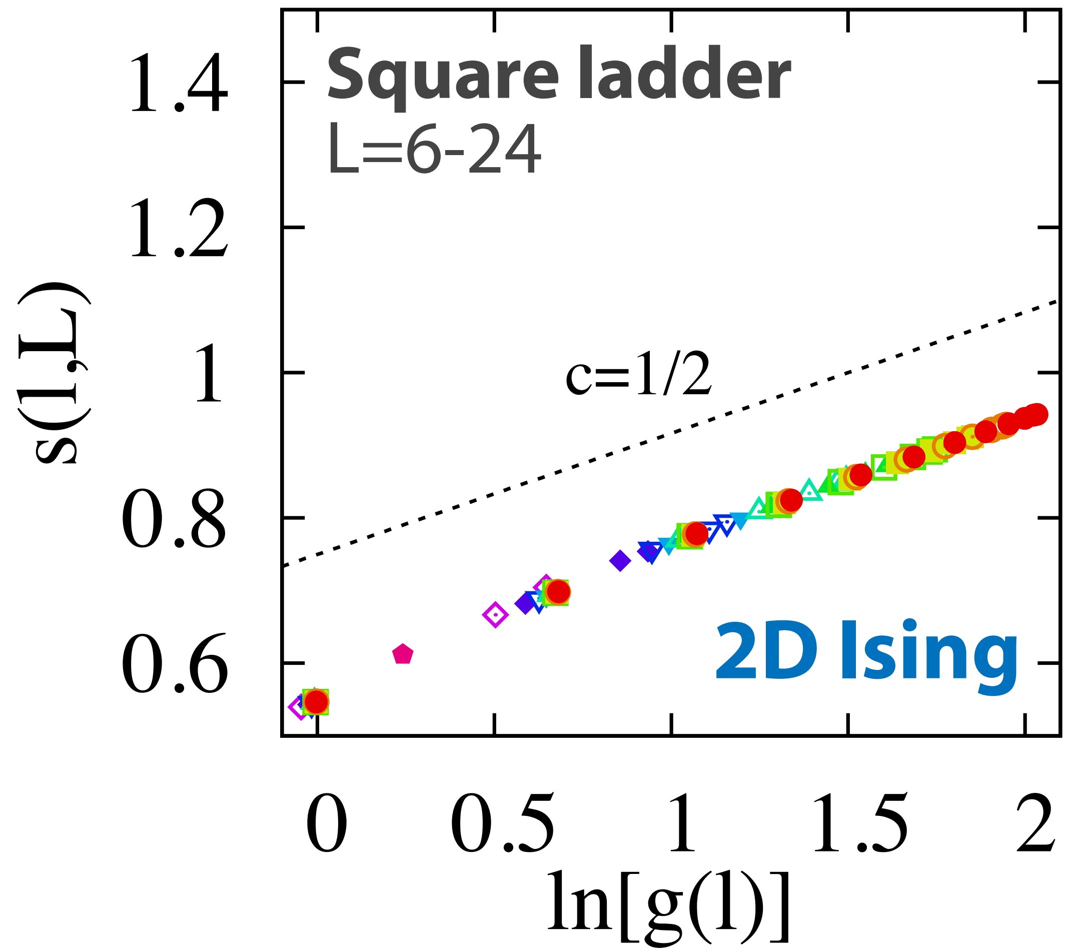
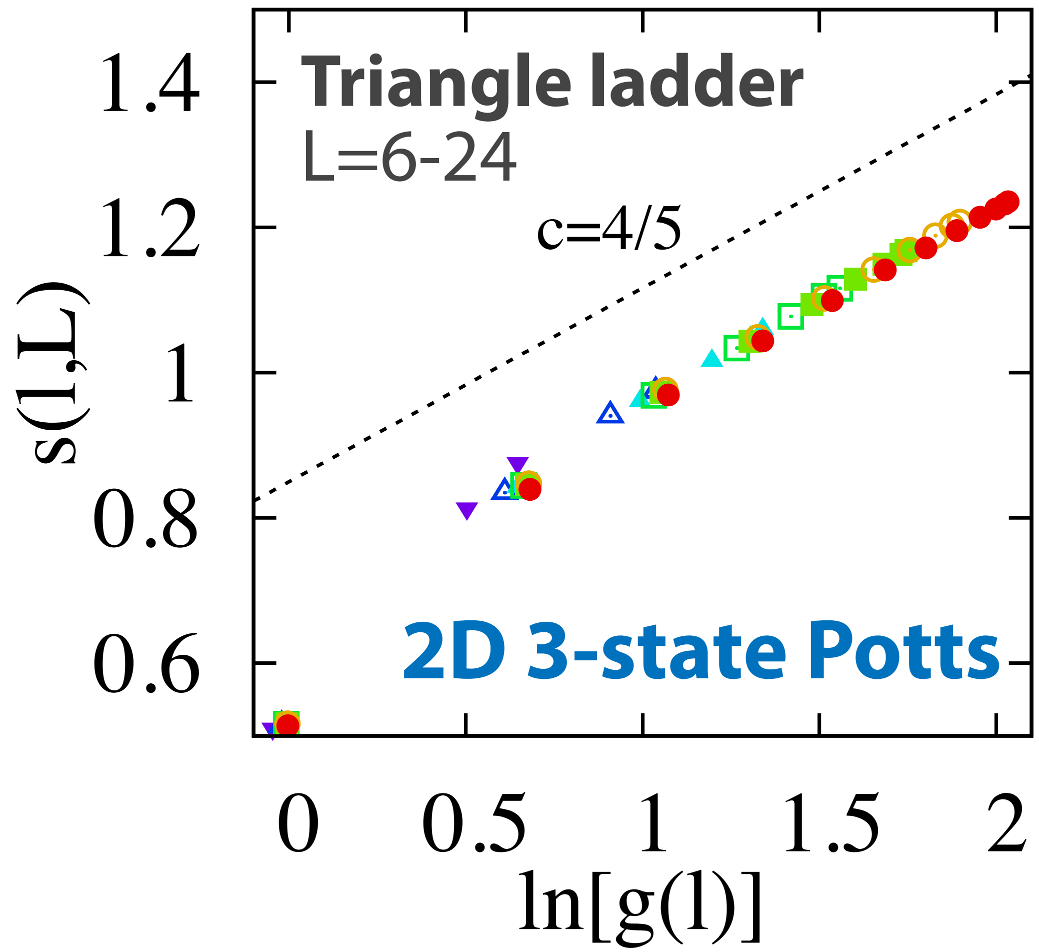
$$s(\ell, L) = \frac{c}{3}\ln[g(\ell)] + s_1, \quad g(\ell) = \frac{L}{\pi}\sin\left(\frac{\pi\ell}{L}\right)$$

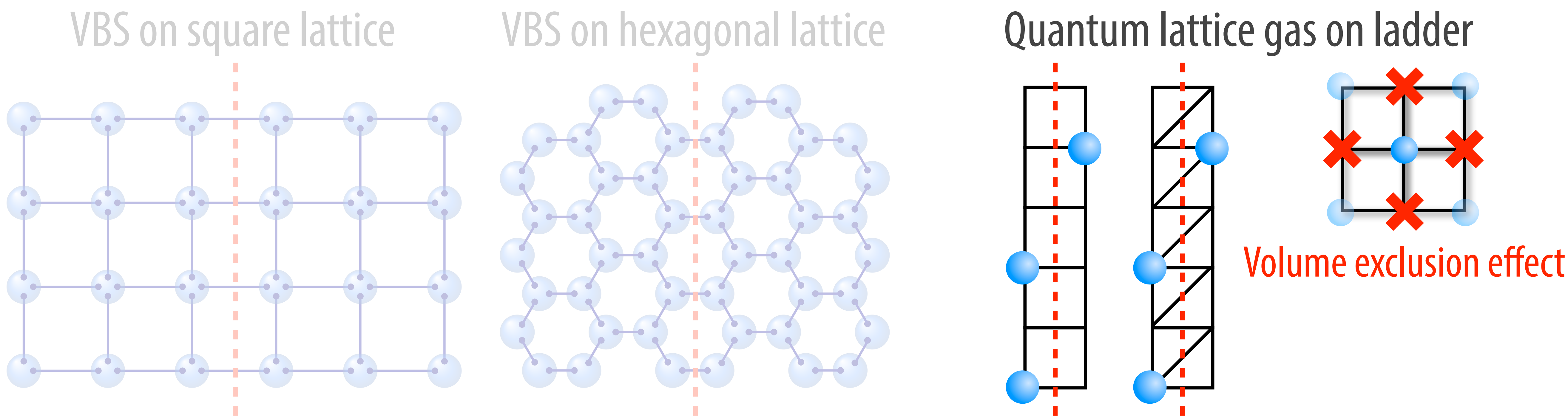






Digest

Entanglement properties of 2D quantum systems ⟷ **Physical properties of 1D quantum systems**



VBS state on 2D lattice

Total system	Entanglement Hamiltonian
Square lattice	1D AF Heisenberg
Hexagonal lattice	1D F Heisenberg

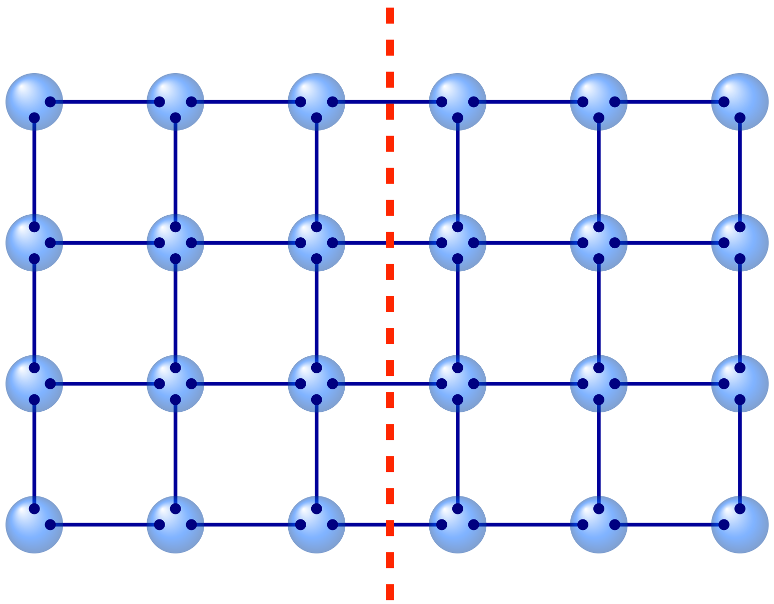
Quantum lattice gas on ladder

Total system	Entanglement Hamiltonian
Square ladder	2D Ising
Triangle ladder	2D 3-state Potts

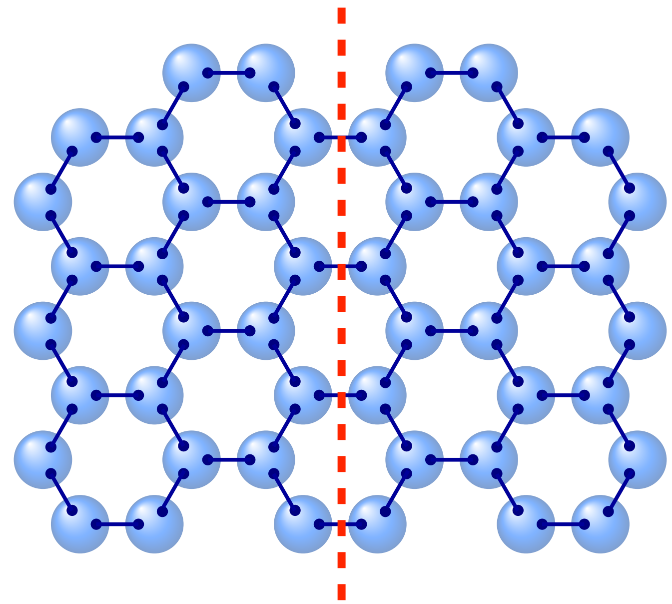
Conclusion

Entanglement properties of 2D quantum systems ⟷ **Physical properties of 1D quantum systems**

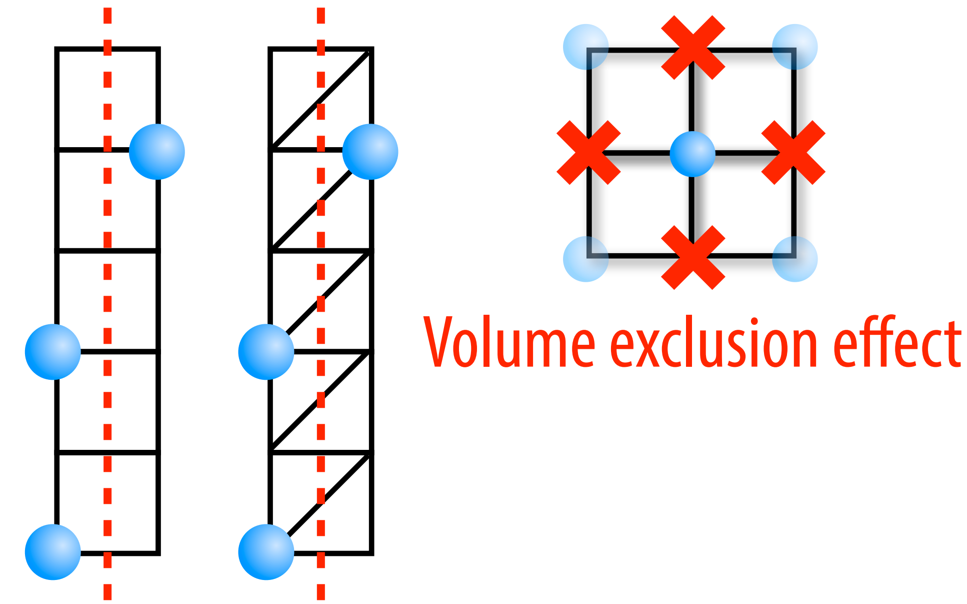
VBS on square lattice



VBS on hexagonal lattice



Quantum lattice gas on ladder



VBS state on 2D lattice

Total system	Entanglement Hamiltonian
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Quantum lattice gas on ladder

Total system	Entanglement Hamiltonian
Square ladder	**2D Ising**
Triangle ladder	**2D 3-state Potts**

Thank you for your attention!!

*VBS on symmetric graphs, J. Phys. A, **43**, 255303 (2010)*
*“VBS/CFT correspondence”, Phys. Rev. B, **84**, 245128 (2011)*
*Quantum hard-square model, Phys. Rev. A, **86**, 032326 (2012)*
*Nested entanglement entropy, Interdisciplinary Information Sciences, **19**, 101 (2013)*