New Uncertainty Relations in view of Weak Values

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Objective of this Talk

- To present a (hopefully) novel set of inequalities interpreted as the **uncertainty relations of approximation/estimation**.
- To see that the (1) position-momentum uncertainty relation and (2) time-energy uncertainty relation can be treated in **one framework**.
- To see that the best choice of proxy functions are given by Aharonov's **weak value**.

Program

5 min

1. Introduction: Various Uncertainty Relations (UR)

15 min

2. UR for Approximation/Estimation

   2.1. Non-commutativity in Depth

   2.2. Robertson-Kennard/Schrödinger ineq. Revisited

   2.3. UR between Generator and Parameter

5 min

3. Summary and Conclusion

5 min

1. Introduction

~ Various Uncertainty Relations ~

1.1. Review: URs in Quantum Mechanics

Uncertainty Relation between Error and Disturbance

- Heisenberg's Inequality (1927)

$$\epsilon(Q)\eta(P) \gtrsim \frac{\hbar}{2}$$

- Ozawa's Inequality (2003)

$$\epsilon_O(A)\eta_O(B) + \epsilon_O(A)\sigma(B) + \sigma(A)\eta_O(B) \geq \left| \left\langle \frac{[A,B]}{2i} \right\rangle_\psi \right|$$

- Watanabe-Sagawa-Ueda's Inequality (2010)

$$\epsilon_W(A)\eta_W(B) \geq \left| \left\langle \frac{[A,B]}{2i} \right\rangle_\psi \right|$$

Uncertainty Relations between **Observables**

- Robertson-Kennard's Inequality (1927-1929)

$$\sigma(A)^2\sigma(B)^2 \geq \left|\left\langle \frac{[A,B]}{2i} \right\rangle\right|^2$$

- Schrödinger's Inequality (1930)

$$\sigma(A)^2\sigma(B)^2 \geq \left|\left\langle \frac{\{A,B\}}{2} \right\rangle - \langle A\rangle\langle B\rangle\right|^2 + \left|\left\langle \frac{[A,B]}{2i} \right\rangle\right|^2$$

Uncertainty Relations between **Time** and **Energy**

- Mandelshtam-Tamm (1945)

$$\tau \cdot \Delta H \geq \frac{\hbar}{2}$$

15 min

2. Uncertaity Relations for Approximation/Estimation

- **Approximation** of Observables
- **Estimation** of Parameters

* New operator from old

Spectral Decomposition: $B = \int b \; |b\rangle\langle b| \; db$

(New operator)

Functional Calculus: $f(B) = \int f(b)|b\rangle\langle b| \, db,$

<u>e.g.</u>

$$B^2 = \int b^2 \; |b\rangle\langle b| \; db, \quad (f(b) = b^2)$$

$$e^{-isB} = \int e^{-isb} \; |b\rangle\langle b| \; db, \quad (f(b) = e^{-isb})$$

$$c \cdot \mathrm{Id} = \int c \; |b\rangle\langle b| \; db, \quad (f(b) = c \; (\text{const.}))$$

2.0. Starting Point: Versatile Inequality

- Robertson-Kennard's Inequality

$$\|A-\langle A\rangle\| \cdot \|B-\langle B\rangle\| \geq \frac{1}{2}\,|\,\langle[A,B]\rangle\,|\,,$$

Expectation Value: $\langle A\rangle = \langle\psi|A|\psi\rangle$
Operator Semi-norm: $\|X\| = \sqrt{\langle X^2\rangle}$

- Versatile Inequality

$$\|A-f(B)\| \cdot \|g(B)\| \geq \frac{1}{2}\,|\,\langle[A,g(B)]\rangle\,|\,,$$

Here, $f(b)$, $g(b)$ are real.

Operators created from B

- Proof of the Versatile Inequality

$$\|A - f(B)\| \cdot \|g(B)\| \geq \frac{1}{2} \, | \, \langle [A, g(B)] \rangle \, | \, ,$$

1. Given two self-adjoint operators X, Y, we have $\|X\|^2 \cdot \|Y\|^2 \geq |\langle XY \rangle|^2$ by the Cauchy-Schwarz (CS) inequality.

2. We also have $|\langle XY \rangle|^2 = |\langle [X, Y]/2 \rangle|^2 + |\langle \{X, Y\}/2 \rangle|^2 \geq |\langle [X, Y]/2 \rangle|^2$, where $\{X, Y\} = XY + YX$, hence

$$\|X\|^2 \cdot \|Y\|^2 \geq |\langle [X, Y]/2 \rangle|^2$$

by combining them.

3. Since X and Y are arbitrary, we may put $X = A - f(B)$ and $Y = g(B)$ and take the square-root to obtain

$$\|A - f(B)\| \cdot \|g(B)\| \geq \frac{1}{2} \, | \, \langle [A, g(B)] \rangle \, | \, ,$$

which was to be demonstrated.

2.1. Application 1: Non-commutativity in Depth

- Versatile Inequality

$$\|A - f(B)\| \cdot \|g(B)\| \geq \frac{1}{2} \left| \langle [A, g(B)] \rangle \right| ,$$

The semi-norm $\|A - f(B)\|$ gives a measure for the 'distance' or 'error' between the two observables. Specifically,

$$\min_f \|A - f(B)\| \geq \max_{\bar{g}} \frac{1}{2} |\langle [A, \bar{g}(B)] \rangle| ,$$

by normalising $\bar{g}(B) = g(B)/\|g(B)\|$. The minimal error in the approximation of A in terms of proxy functions $f(B)$, is dictated by the maximal degree of non-commutativity of A with respect to the family of all normalised self-adjoint operators generated by B.

~ Optima and the Weak Value ~

Q: Optimal choice of the proxy functions $f(b)$, $g(b)$?

$$\min_f \|A - f(B)\| \geq \max_{\bar{g}} \frac{1}{2} |\langle [A, \bar{g}(B)] \rangle| \, .$$

A: Real and Imaginary parts

$$f_{\mathrm{opt}}(b) = \mathrm{Re}\, A_w(b), \quad g_{\mathrm{opt}}(b) = \frac{\mathrm{Im}\, A_w(b)}{\|\mathrm{Im}\, A_w(B)\|}$$

of Aharonov's weak value

$$A_w(b) := \frac{\langle b|A|\psi\rangle}{\langle b|\psi\rangle}.$$

In such case, the inequality reduces to

$$\|A - \mathrm{Re}\, A_w(B)\| \geq \|\mathrm{Im}\, A_w(B)\|.$$

- Proof of the optimal Proxy Functions

$$\min_f \|A - f(B)\| \geq \max_{\bar{g}} \frac{1}{2} |\langle [A, \bar{g}(B)] \rangle| \,.$$

1. *Optimum of f.* An immediate consequence of the triangle inequality [1-2]

$$\|A - f(B)\|^2 = \|A - \mathrm{Re}A_w(B)\|^2 + \|\mathrm{Re}A_w(B) - f(B)\|^2$$

[1] M. J. W. Hall, Phys. Rev. A **64**, 052103 (2001).
[2] L. M. Johansen, Phys. Lett. A **322**, 298-300 (2004).

2. *Optimum of g.* First observe that

$$\langle g(B)A \rangle = \int_{\mathbb{R}} \langle \psi | g(B) | b \rangle \langle b | A | \psi \rangle \; db = \int_{\mathbb{R}} g(b) \cdot A_w(b) \; \rho(b) db,$$

and

$$\langle A g(B) \rangle = \int_{\mathbb{R}} g(b) \cdot A_w^*(b) \; \rho(b) db,$$

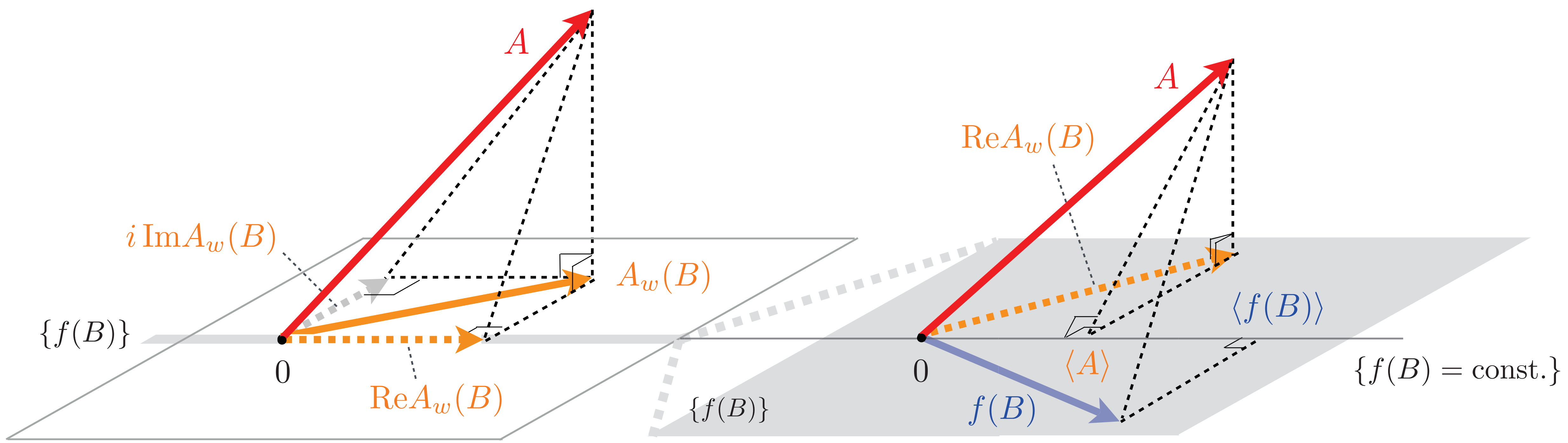
where we denote the probability by $\rho(b) := |\langle b | \psi \rangle|^2$.

Then, the CS inequality yields

$$\begin{aligned}
\frac{1}{2}\left|\langle[A,\bar{g}(B)]\rangle\right| &= \frac{1}{2}\left|\int_{\mathbb{R}} \overline{g}(b)\cdot A_w(b) - \overline{g}(b)\cdot A_w^*(b)\ \rho(b)db\right| \\
&= \left|\int_{\mathbb{R}} \overline{g}(b)\cdot \operatorname{Im} A_w(b)\ \rho(b)db\right| \\
&\leq \left(\int_{\mathbb{R}} |\overline{g}(b)|^2 \rho(b)db\right)^{\frac{1}{2}} \cdot \left(\int_{\mathbb{R}} |\operatorname{Im} A_w(b)|^2 \rho(b)db\right)^{\frac{1}{2}} \\
&= \|g_{\mathrm{opt}}(b)\| \cdot \|\operatorname{Im} A_w(B)\| \\
&= \|\operatorname{Im} A_w(B)\|.
\end{aligned}$$

Equality holds with the choice $g_{\mathrm{opt}}(b) = \operatorname{Im} A_w(b)/\|\operatorname{Im} A_w(B)\|$ (equality condition for the CS inequality).

~ Addendum: Geometric View ~



- Weak Value is the image of the Projection of A onto the subspace spanned by B.

- Quantum analogue of the L^2 structure on the space of classical random variables.

2.2. Application 2: RK Inequality Revisited

- Versatile Inequality

$$\|A - f(B)\| \cdot \|g(B)\| \geq \frac{1}{2} \left| \langle [A, g(B)] \rangle \right|,$$

- Robertson-Kennard's (RK) Inequality

$$\|A - \langle A \rangle\| \cdot \|B - \langle B \rangle\| \geq \frac{1}{2} \left| \langle [A, B] \rangle \right|,$$

for the choice $f(B) = \langle A \rangle$ and $g(B) = B - \langle B \rangle$.

- Tightened version of the RK Inequality

$$\|A - \mathrm{Re} A_w(B)\| \cdot \|B - \langle B \rangle\| \geq \frac{1}{2} \left| \langle [A, B] \rangle \right|$$

with the optimal choice $f(B) = f_{\mathrm{opt}}(B)$, $g(B) = B - \langle B \rangle$.

~ RK ineq. VS optimal ineq. ~

- Robertson-Kennard's (RK) Inequality

$$\|A - \langle A \rangle\| \cdot \|B - \langle B \rangle\| \geq \frac{1}{2}\, |\, \langle [A, B] \rangle\, |\,,$$

The optimal inequality reduces to the RK inequality if and only if

$$\mathrm{Re} A_w(B)|\psi\rangle = \langle A \rangle |\psi\rangle$$

($i.e.$, 'best approximation' is trivial), in which case the covariance,

$$\begin{aligned}\mathrm{Cov}[A, B] &= \frac{1}{2}\langle \{A, B\} \rangle - \langle A \rangle \langle B \rangle \\ &= \langle (\mathrm{Re} A_w(B) - \langle A \rangle)\, (B - \langle B \rangle) \rangle = 0,\end{aligned}$$

vanishes identically ($i.e.$, no 'correlation').

- Tightened version of the RK Inequality

$$\|A - \mathrm{Re} A_w(B)\| \cdot \|B - \langle B \rangle\| \geq \frac{1}{2}\, |\, \langle [A, B] \rangle\, |$$

~ 'Correlation' ~

Applying the CS inequality to $\mathrm{Cov}[A, B]$:

$$\|\mathrm{Re}A_w(B) - \langle A\rangle\| \cdot \|B - \langle B\rangle\| \geq \left| \frac{1}{2}\langle\{A, B\}\rangle - \langle A\rangle\langle B\rangle \right|.$$

(Classical Covariance inequality $\sigma(X)\sigma(Y) \geq \mathrm{Cov}[X, Y]$)

~ Schrödinger's Inequality Revisited ~

A tightened version of the Schrödinger's inequality

$$\|A_w(B) - \langle A\rangle\|^2 \cdot \|B - \langle B\rangle\|^2$$

$$\geq \left| \frac{1}{2}\langle[A, B]\rangle \right|^2 + \left| \frac{1}{2}\langle\{A, B\}\rangle - \langle A\rangle\langle B\rangle \right|^2,$$

by combining the two.

2.3. Application 3: Parameter Estimation

Consider a parametrised family of states

$$|\psi(t)\rangle = e^{-itA/\hbar}|\psi\rangle, \quad t \in \mathbb{R}$$

generated by A for a fixed $|\psi\rangle$.

Objective: How well can one obtain the information of both the generator A and the parameter t through the measurement of B?

Strategy. We try to minimise the distances

$$\|A - f(B)\|, \quad \|t - g(B)\|$$

by freely choosing the proxy functions f, g.

~ Cramére-Rao Inequality ~

Recall the CS inequality

$$\|\mathrm{Im}\, A_w(B)\| \cdot \|B - \langle B\rangle\| \geq \left| \frac{1}{2} \langle [A, B]\rangle \right|$$

1. The imaginary part of the weak value corresponds to the *Fisher Information*

$$\begin{aligned} I(t) &= \int \left[\frac{d}{dt} \ln p(b,t)\right]^2 p(b,t)\, db \\ &= \left(\frac{1}{\hbar}\right)^2 \int \left[\frac{\langle\psi(t)|b\rangle\langle b|A\psi(t)\rangle}{|\langle b|\psi(t)\rangle|^2} - \frac{\langle A\psi(t)|b\rangle\langle b|\psi(t)\rangle}{|\langle b|\psi(t)\rangle|^2}\right]^2 p(b,t)\, db \\ &= \left(\frac{2}{\hbar}\right)^2 \int [\mathrm{Im} A_w(b)]^2\, p(b,t)\, db = \left(\frac{2}{\hbar}\right)^2 \|\mathrm{Im} A_w(B)\|^2, \end{aligned}$$

for the estimation of t, where $p(b,t) = |\langle b|\psi(t)\rangle|^2$ is the probability distribution.

2. The commutator

$$\left|\frac{d}{dt}\langle g(B)\rangle_t\right| = \frac{1}{\hbar}\left|\langle Ag(B)\rangle_t - \langle g(B)A\rangle_t\right|$$
$$= \frac{1}{\hbar}|\langle [A, g(B)]\rangle_t|$$

corresponds to the derivative of the average of g.

- CS Inequality

$$\|\mathrm{Im}\, A_w(B)\|^2 \cdot \|g(B)\|^2 \geq \frac{1}{4}|\langle [A, g(B)]\rangle|^2$$

$$g(B) \to g(B) - \langle g(B)\rangle$$

- Cramére-Rao Inequality

$$I(t)\cdot \mathrm{Var}[g(B)] \geq \left(\frac{d}{dt}\langle g(B)\rangle\right)^2,$$

Definition (Locally unbiased estimator)**.** *A function g is called a locally unbiased estimator of a function $\varphi(t)$ at the point $t_0 \in \mathbb{R}$, if its statistical average*

$$\langle g(B)\rangle_t := \langle \psi(t)|g(B)|\psi(t)\rangle.$$

coincides with φ

$$\langle g(B)\rangle_t = \varphi(t_0) + \varphi'(t_0)\cdot(t-t_0) + o(t-t_0)$$

at least for the first order expansion at the point t_0.

If we plug the conditions for the locally unbiased estimator

$$I(t)\cdot \mathrm{Var}[g(B)] \geq \left(\frac{d}{dt}\langle g(B)\rangle\right)^2,$$

$$\Downarrow \quad \langle g(B)\rangle_{t_0} = t_0, \quad \frac{d}{dt}\langle g(B)\rangle_{t_0} = 1,$$

$$\|t_0 - g(B)\| \geq \frac{1}{I(t_0)} = \left(\frac{\hbar}{2}\right)\frac{1}{\|\mathrm{Im}\, A_w(B)\|}$$

~ UR between Generator and Parameter ~

Combining the previous result

$$\|t_0 - g(B)\| \geq \left(\frac{\hbar}{2}\right) \frac{1}{\|\mathrm{Im}\, A_w(B)\|}$$

and the inequality from application 1

$$\|A - f(B)\| \geq \|A - \mathrm{Re}\, A_w(B)\| \geq \|\mathrm{Im}\, A_w(B)\|$$

we arrive at

$$\boxed{\|A - f(B)\| \cdot \|t_0 - g(B)\| \geq \frac{\hbar}{2} \cdot \frac{\|A - \mathrm{Re}\, A_w(B)\|}{\|\mathrm{Im}\, A_w(B)\|} \geq \frac{\hbar}{2}}$$

Note added: A similar inequality [3] is known from a different context and argument. Our inequality is tighter.

[3] H. F. Hofmann, Phys. Rev. A **83**, 022106 (2011).

Theorem (Uncertainty Relation between Generator and Parameter). *Let A, B be self-adjoint, and consider $|\psi(t)\rangle = e^{-itA/\hbar}|\psi\rangle$.*

1. *Locally unbiased estimators of t at t_0 exists if and only if*

$$I(t_0) = \|\mathrm{Im}\, A_w(B)\|_{t_0} \neq 0.$$

2. *Provided that $\|\mathrm{Im}\, A_w(B)\|_{t_0} \neq 0$, the inequality*

$$\|A - f(B)\|_{t_0} \cdot \|t_0 - g(B)\|_{t_0} \geq \frac{\hbar}{2} \cdot \frac{\|A - \mathrm{Re}\, A_w(B)\|_{t_0}}{\|\mathrm{Im}\, A_w(B)\|_{t_0}} \geq \frac{\hbar}{2}$$

holds.

3. *The optimal proxy functions are respectively given by*

$$f_{\mathrm{opt}}(B) = \mathrm{Re}\, A_w(B), \quad g_{\mathrm{opt}}(B) = -\frac{2}{\hbar I(t_0)} \mathrm{Im} A_w(B) + t_0.$$

Equality holds for the optimal choice.

3 min

3. Summary and Conclusion

Summary

- By considering a problem of approximating/estimating an observable or a parameter from the measurement of another observable, we have derived several inequalities.

- For the convenience of presentation, we first presented a versatile inequality

$$\|A - f(B)\| \cdot \|g(B)\| \geq \frac{1}{2} \, | \, \langle [A, g(B)] \rangle \, | \, ,$$

and derived three types of inequalities as its special cases.

1. **Non-commutativity in Depth**

$$\min_{f} \|A - f(B)\| \geq \max_{\bar{g}} \frac{1}{2} \, |\langle [A, \bar{g}(B)] \rangle| \, .$$

Maximal degree of non-commutativity provides the lower bound to the distance of approximation.

2. **Robertson-Kennard/Schrödinger Inequalities revisited**

In view of approximation/estimation, inequalities tighter than the RK inequality are presented:

$$\|\mathrm{Re}\, A_w(B) - \langle A\rangle\| \cdot \|B - \langle B\rangle\| \geq \left| \frac{1}{2}\langle\{A, B\}\rangle - \langle A\rangle\langle B\rangle \right|$$

$$\begin{aligned}\|A - \mathrm{Re}\, A_w(B)\| \cdot \|B - \langle B\rangle\| &\geq \|\mathrm{Im}\, A_w(B)\| \cdot \|B - \langle B\rangle\| \\ &\geq \left| \frac{1}{2}\langle[A, B]\rangle \right|\end{aligned}$$

Adding both hand sides of the two inequalities

$$\begin{aligned}\|A - \langle A\rangle\|^2 \cdot \|B - \langle B\rangle\|^2 &\geq \|A_w(B) - \langle A\rangle\|^2 \cdot \|B - \langle B\rangle\|^2 \\ &\geq \left| \frac{1}{2}\langle[A, B]\rangle \right|^2 + \left| \frac{1}{2}\langle\{A, B\}\rangle - \langle A\rangle\langle B\rangle \right|^2\end{aligned}$$

we obtained a tighter version of the Schrödinger Inequality.

3. Uncertainty Relation between Parameter and Generator (incl. Time-Energy Uncertainty Relation)

We considered the problem of approximating/estimating both the generator and the parameter of a unitary transformation

$$|\psi(t)\rangle = e^{-itA/\hbar}|\psi\rangle, \quad t \in \mathbb{R}$$

from the measurement of B, and found the inequality

$$\|A - f(B)\|_{t_0} \cdot \|t_0 - g(B)\|_{t_0} \geq \frac{\hbar}{2} \cdot \frac{\|A - \mathrm{Re}\, A_w(B)\|_{t_0}}{\|\mathrm{Im}\, A_w(B)\|_{t_0}} \geq \frac{\hbar}{2}$$

valid for any locally unbiased estimator g of the parameter t.

[4] J. Lee and I. Tsutsui, *Uncertainty Relations for Approximation and Estimation*, arXiv:1511.08052 (2015).

Conclusion & Discussion

- Position-momentum and time-energy UR are treated in one framework.
- Aharonov's weak value appears as the optimal choices of the proxy functions in all the inequalities presented.
- One may obtain a better understanding of the whole argument in view of **quasi-probabilities**.

1. It provides a unified framework for the discussion, makes comparison with the classical theory easier, and better accounts for the significance of non-commutativity.

2. It offers a geometric/statistical understanding to account for the reason why weak values appear.

[5] J. Lee and I. Tsutsui, *Quasi-probabilities of Quantum Observables and a Geometric/Statistical Interpretation of the Weak Value*, PTEP, (appearing).

Thank you for your attention