Any time inhomogeneous quantum memoryless process ”converges”

NII
Keiji Matsumoto

Inhomogeneous memoryless process converges?

$$\rho^0 = \rho\ , \quad \rho^1 = \Gamma_1(\rho^0)\ , \cdots, \ \rho^t = \Gamma_t(\rho^{t-1}), \cdots$$

Γ_t : Completely Positive Trace Preserving (CPTP)

may depends on *t*

Typically, does not converge

1. May not be dissipative, $\Gamma_t(\rho) = U_t \rho U_t^*$
2. Dependency on time *t* :

   can move away once converged state

Positive and Completely Positive

Positive $T(\rho) \geqq 0$, $\rho \geqq 0$

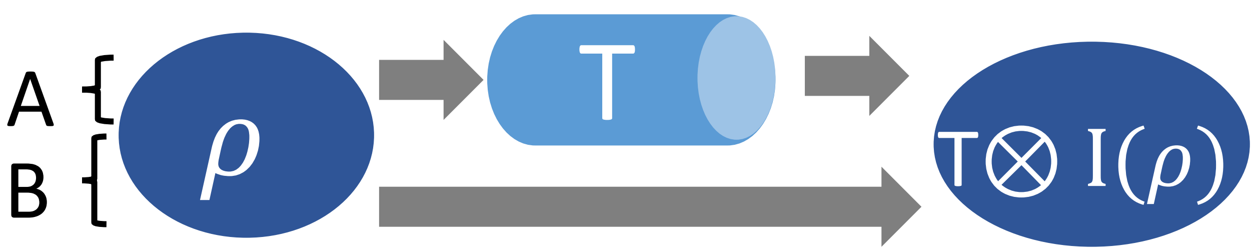
Trace Preserving tr $T(\rho)$ = trρ

Send density operators
To density operators

Completely positive $T \otimes 1(\rho) \geqq 0$,

Trace Preserving tr $T(\rho)$ = trρ

$T(\rho) = \mathrm{tr}_{H_0} U(\rho \otimes \rho_0) U^*$
U: unitary



Memoryless Process and CPTP maps

$$\rho^0 = \rho\,, \quad \rho^1 = \Gamma_1(\rho^0)\,, \cdots, \ \rho^t = \Gamma_t(\rho^{t-1}), \cdots$$

Γ_t : Completely Positive Trace Preserving (CPTP)

may depend on t

Γ_t is CPTP $\Leftrightarrow$ $\Gamma_t(\rho) = \mathrm{tr}_{H_0} U(\rho \otimes \sigma_t) U^*$ U: unitary

The process is memoryless, in the sense that σ_t has to be refreshed for each time

Quantum analogue of Markov chain

Inhomogeneous memoryless process converges?

$$\rho^0 = \rho\ , \quad \rho^1 = \Gamma_1(\rho^0)\ , \cdots,\ \rho^t = \Gamma_t(\rho^{t-1}), \cdots$$

Γ_t : Completely Positive Trace Preserving (CPTP)

may depends on *t*

Typically, does not converge

1. May not be dissipative, $\Gamma_t(\rho) = U_t \rho U_t^*$
2. Dependency on time *t* :

   can move away once converged state

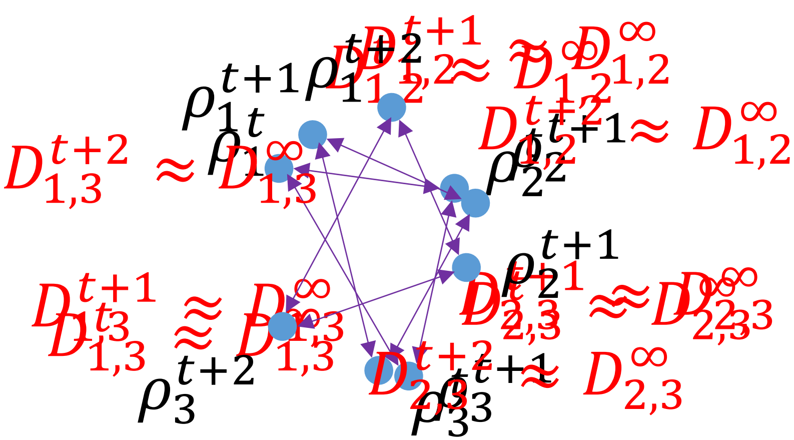
However …

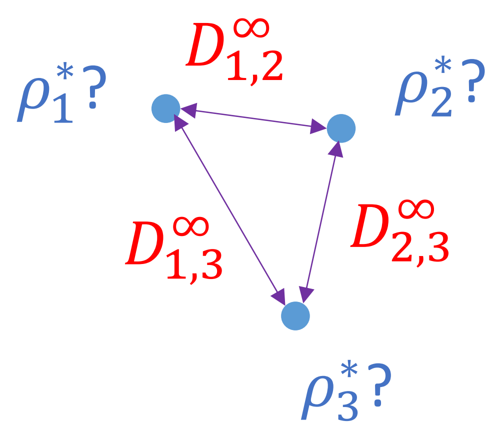
Any "good" distance btw any two states converges

$$\lim_{t\to\infty} D(\rho_1^t,\rho_2^t) = \lim_{t\to\infty} D_{1,2}^t = D_{1,2}^\infty$$

"good"⇔ $D(\rho_1,\rho_2) \geq D(\Gamma(\rho_1),\Gamma(\rho_2))$, $\forall\Gamma$: CPTP

e.g. $\mathrm{tr}\,\rho_1(\ln\rho_1 - \ln\rho_2), 1 - \mathrm{tr}\rho_1^\alpha\rho_2^{1-\alpha}$, $\|\rho_1 - \rho_2\|_1$,





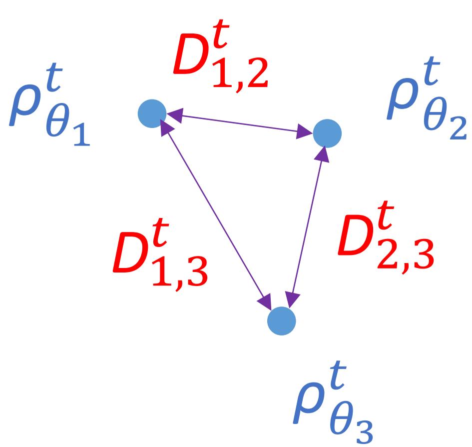
$\rho_1^t \to \rho_1^*$ mod unitary ??

Formulation (I)

$E := \{\rho_\theta; \theta \in \Theta\}$: set of possible initial states
(may be all sates over Hilbert space H)

1. Consider convergence of the collection of states
rather than convergence of each state

2. Want to view $\{\rho_\theta; \theta \in \Theta\} \sim \{U\rho_\theta U^*; \theta \in \Theta\}$



Formulation (II) : equivalence relation

$$E := \{\rho_\theta ; \theta \in \Theta\} \quad F := \{\sigma_\theta ; \theta \in \Theta\}$$

$$E \sim F \Leftrightarrow E \geq F , E \leq F$$

Def (Ver.1)

$$E \geq F \Leftrightarrow D(\rho_{\theta_1}, \rho_{\theta_2}, \cdots, \rho_{\theta_k}) \geq D(\sigma_{\theta_1}, \sigma_{\theta_2}, \cdots, \sigma_{\theta_k})$$

D : Monotone non-increasing by CPTP maps

e.g. Holevo's capacity

$$\inf_{\rho_*} \sup_{\theta} D(\rho_\theta, \rho_*)$$

$[E]$ embodies all the information quantities defined on E

Formulation (III) : equivalence relation

$$E := \{\rho_\theta ; \theta \in \Theta\} \qquad F := \{\sigma_\theta ; \theta \in \Theta\}$$

$$E \sim F \;\Leftrightarrow\; E \geq F\,,\, E \leq F$$

Def (Ver.2)

$$E \geq F \;\Leftrightarrow\; \sigma_\theta = \Gamma(\rho_\theta),\ \exists\Gamma\text{: CPTP}$$

$E \sim F$ iff E can be mapped to F reversibly

e.g. $\sigma_\theta = U\rho_\theta U^*$

(If E= all the states, reversible ⇔ unitary)

$[E\,] := \{E' ; E' \sim E\}$ correction of E' equivalent to E

Formulation (IV)

Th (quantum randomization criterion, [M 2010])

Def ver. 1 $\Leftrightarrow$ Def ver. 2

$D(\rho_1, \rho_2, \ldots, \rho_d)$ can be thought as average gain of certain operational task

$E \geq F$ means E is more informative than F

$E_1 \geq E_2 \geq \cdots \geq E_t \geq \cdots$
monotone decreasing sequence

$$E_t := \{\rho_\theta^t ; \theta \in \Theta\}$$

Formulation (V)

To define limit, define distance btw $[E]$ and $[F]$

$$\delta([E],[F]) := \inf_{\Gamma:\mathrm{CPTP}} \sup_{\theta\in\Theta} \left\| \Gamma(\rho_\theta) - \sigma_\theta \right\|_1$$

But $\delta([E],[F]) \neq \delta([F],[E])$,

$$\Delta([E],[F]) := \max\{\delta([E],[F]), \delta([F],[E])\}$$

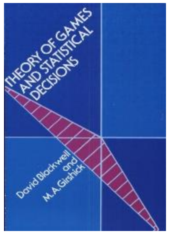
Δ is distance :

$$\Delta([E],[F]) = 0 \Leftrightarrow [E] = [F]$$

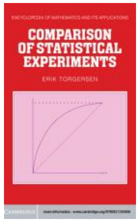
$$\Delta([E],[E']) + \Delta([E'],[E'']) \geq \Delta([E],[E''])$$

Background: comparison of statistical experiments

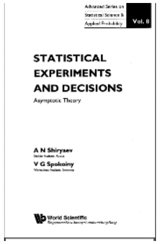
Blackwell, D. and Girshick, M. A. *Theory of Games and Statistical Decisions* (1954)



Torgersen, E. *Comparison of Statistical Experiments,* Cambridge University Press(1991)



Shiryaev A N , Spokoiny, V G,
STATISTICAL EXPERIMENTS AND DECISIONS: Asymptotic Theory, World Scientific (2000)



Goel P K
"When is one experiment 'always better than' another?"
The Statistician 52 , Part 4 , pp. 515–537 (2003)

Background: comparison of statistical experiments

Statistical experiments $E = \{p_\theta\}$
= parametrized family of probability distributions

Statistical Inference (test, estimation, etc)
= data processing to obtain information about ϑ

Compare "information" contained in $E = \{p_\theta\}$ and $F = \{q_\theta\}$

Applications :

1. Experimental design
2. Approximation of probability distribution family

   ex. Approximation of $E^n = \{p_\theta^n\}$ by Gaussian shift family

Assertion

Th

If $\dim H < \infty$, $\exists E_\infty = \{\rho_\theta^\infty ; \theta \in \Theta\}$

$$\lim_{t \to \infty} \Delta([E_t], [E_\infty]) = 0$$

Th

If $\dim H < \infty$, $\Gamma_t = \Gamma$,

$$\Gamma(E_\infty) \sim E_\infty$$

$\Gamma(E_\infty) = \left\{\Gamma\left(\rho_\theta^\infty\right); \theta \in \Theta\right\}$ is a fixed point

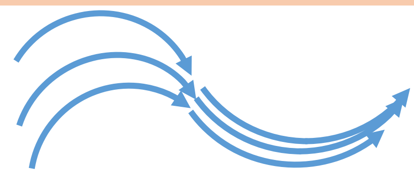
Strongly and Weakly Ergodic

Def The process $\{\Gamma_t\}$ is strongly ergodic iff

$$\forall \rho \quad \lim_{t\to\infty} \left\| \Gamma_t \circ \cdots \circ \Gamma_2 \circ \Gamma_1(\rho) - \rho_* \right\|_1 = 0$$

weakly ergodic iff

$$\lim_{t\to\infty} \sup_{\rho,\rho'} \| \Gamma_t \circ \cdots \circ \Gamma_2 \circ \Gamma_1(\rho) - \Gamma_t \circ \cdots \circ \Gamma_2 \circ \Gamma_1(\rho') \|_1 = 0$$



Fact The process $\{\Gamma_t\}$ is weakly ergodic iff

$$E_\infty = \{\rho_\theta^\infty = \rho_*;\ \theta \in \Theta\} \quad (\ \rho_\theta^\infty \text{ is independent of } \theta\ )$$

Sketch of the Proof

⚠ eigenvalue analysis does NOT work, for Γ_t depends on t

Suffices to show the assertion for $|\Theta|=(\dim H)^2$ ($<\infty$)

choose $\rho_{\theta_1}, \rho_{\theta_2}, \cdots, \rho_{\theta_{(\dim H)^2}}$ spanning the state space

$\{[E]; E$ is a family over $\Theta\}$ is compact

$\therefore$ $\{[E_t] ; t \in \mathbb{N}\}$ has accumulation point $[E_\infty]$

Show $\lim_{t\to\infty} \Delta([E_t],[E_\infty]) = 0$ using

properties of Δ, $E_1 \geq E_2 \geq \cdots \geq E_t \geq \cdots$

Representation of $[E]$: classical case

Motivation: equivalence class is too abstract

For simplicity, let Θ={0,1} $E=\{p_0, p_1\}$ $F=\{q_0, q_1\}$

Relative Information Spectrum

⇔Distribution of $\ln\left(\frac{p_0}{p_1}\right)$ when $x \sim p_0(x)$

1. expectation is relative entropy , contains more info.
2. Essential in information theory of non-stationary process
3. Characteristic function (Hellinger transform)

$$\phi_E(\alpha) = E_{p_0}\left[\exp\left\{\alpha \ln\left(\frac{p_0}{p_1}\right)\right\}\right] = \int p_0^{1-\alpha}(x) p_1^{\alpha}(x)\, dx$$

Fact

$$E \sim F \Leftrightarrow \phi_E(\alpha) = \phi_F(\alpha), \forall \alpha, 0 \leq \alpha \leq 1$$

On mathematics …

- Proof heavily relies on dim H<∞
- In classical case, can remove this conditon , though the statement becomes a bit weaker.

  (convergence w.r.t. “weak topology”)

  Uses representation of [*E*]
- In quantum case, can we ?

  No good representation of [*E*] so far

Discussion

Assertion

$$\dim H < \infty, \qquad E_t := \{\rho_\theta^t; \theta \in \Theta\}$$

Th $\exists E_\infty \quad \lim_{t\to\infty} \Delta([E_\infty],[E_t]) = 0$

Def The process $\{\Gamma_t\}$ is weakly ergodic iff

$$\lim_{t\to\infty} \sup_{\rho,\rho'} \|\Gamma_t \circ \cdots \circ \Gamma_2 \circ \Gamma_1(\rho) - \Gamma_t \circ \cdots \circ \Gamma_2 \circ \Gamma_1(\rho')\| = 0$$

Fact The process $\{\Gamma_t\}$ is weakly ergodic iff

$$E_\infty = \{\rho_\theta^\infty = \rho_*;\ \theta \in \Theta\}$$

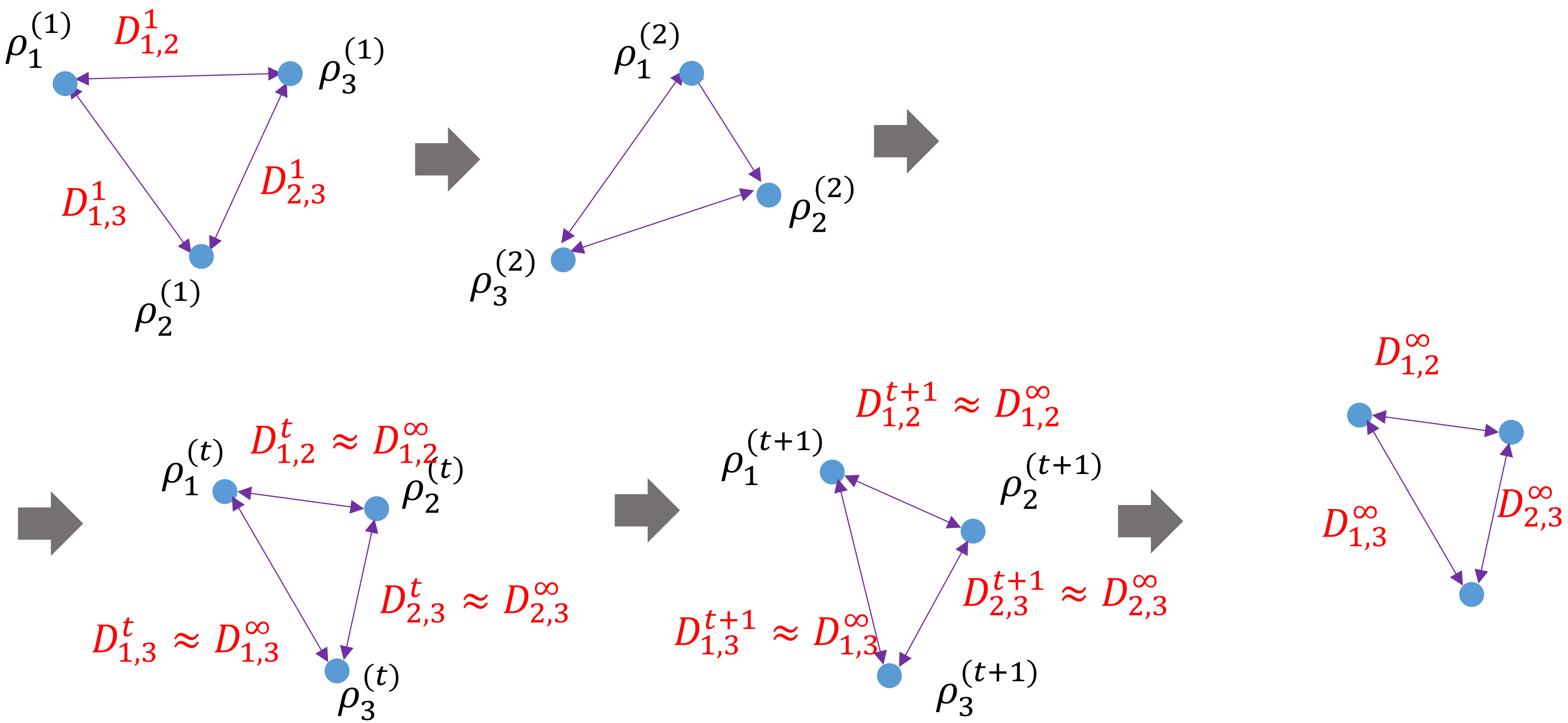
(ρ_θ^∞ is independent of θ)

But⋯

Any "good" distance btw any two states converges

$$\lim_{t\to\infty} D\left(\rho_1^{(t)}, \rho_2^{(t)}\right) = \lim_{t\to\infty} D_{1,2}^{t} = D_{1,2}^{\infty}$$

$$D(\rho_1, \rho_2) \geq D(\Gamma(\rho_1), \Gamma(\rho_2)) \ , \ \forall\Gamma : \mathrm{CPTP}$$



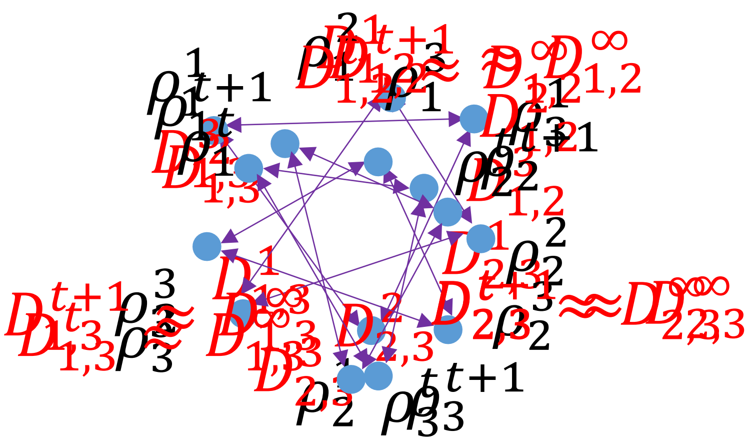
But…

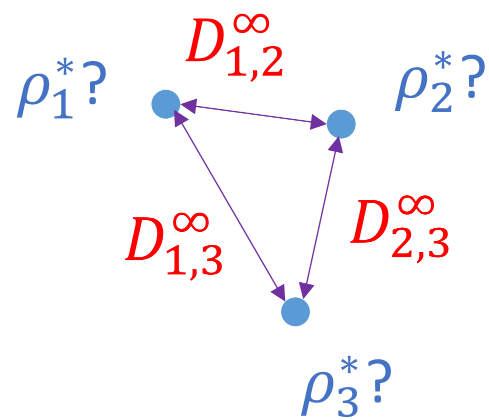
Any "good" distance btw any two states converges

$$\lim_{t\to\infty} D(\rho_1^t, \rho_2^t) = \lim_{t\to\infty} D_{1,2}^t = D_{1,2}^\infty$$

$$D(\rho_1, \rho_2) \geq D(\Gamma(\rho_1), \Gamma(\rho_2)) \ , \ \forall\Gamma : \text{CPTP}$$

e.g. $\operatorname{tr} \rho_1(\ln\rho_1 - \ln\rho_2), \|\rho_1 - \rho_2\|_1, 1 - \left\|\sqrt{\rho_1}\sqrt{\rho_2}\right\|_1$





$\rho_1^{(t)} \to \rho_1^*$??

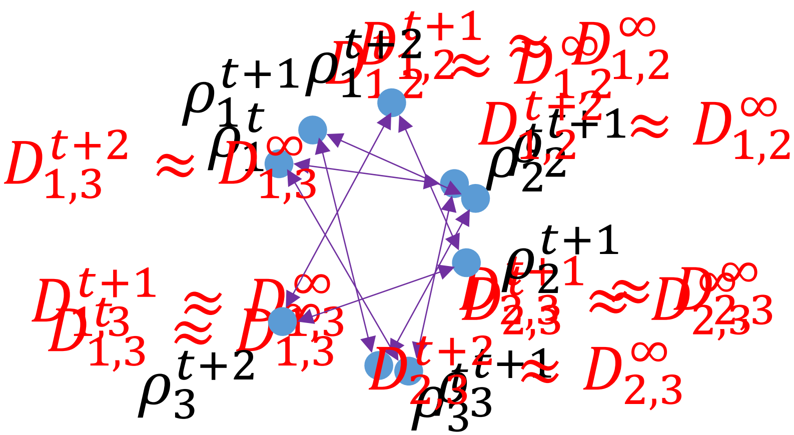
But…

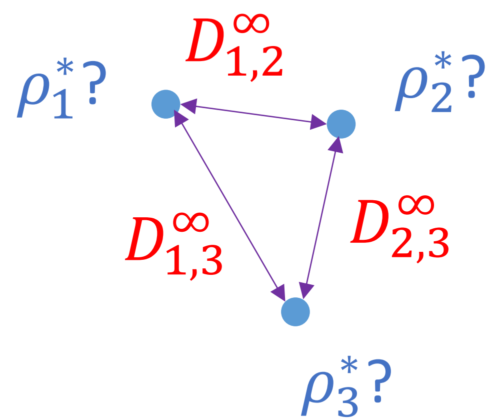
Any "good" distance btw any two states converges

$$\lim_{t\to\infty} D(\rho_1^t, \rho_2^t) = \lim_{t\to\infty} D_{1,2}^t = D_{1,2}^\infty$$

$$D(\rho_1, \rho_2) \geq D(\Gamma(\rho_1), \Gamma(\rho_2)) \ , \ \forall \Gamma : \text{CPTP}$$

e.g. $\mathrm{tr}\, \rho_1(\ln\rho_1 - \ln\rho_2), \|\rho_1 - \rho_2\|_1 \, , \, 1 - \left\|\sqrt{\rho_1}\sqrt{\rho_2}\right\|_1$





$\rho_1^t \to \rho_1^*$ mod unitary ??