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Generalized Pauli principle for Read-Rezayi non-Abelian Quantum Hall States

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A selection rule on "admissible configurations" in model FQHE wavefunctions

From polynomial wavefunctions to topological/conformal field theories

Numerical studies: effects on entanglement spectrum

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Collaborations with Emil Prodan, Andrei Bernevig, and Hui Li

After I had empirically found much of what I will describe in numerical studies, A. Bernevig found (by Google!) a remarkable earlier 2002 Kyoto mathematical paper (RIMS) that related a Bosonic variant of my results to Jack Polynomials (known earlier from the 1D integrable Sutherland-Calogero model)

International Mathematics Research Notices (2002) 2002:1223-1237 ,

A differential ideal of symmetric polynomials spanned by Jack polynomials at $\beta = -(k+1)/(r-1)$

B. Feigin, M. Jimbo, T. Miwa and E. Mukhin

For each pair of positive integers (k,r) such that k+1, r–1 are coprime, we introduce an ideal $I_n^{(k,r)}$ of the ring of symmetric polynomials $\mathbb{C}[x_1,\ldots,x_n]^{S_n}$. The ideal $I_n^{(k,r)}$ has a basis consisting of **Jack** polynomials with parameter rß=-(r–1)/(k+1), and admits an action of a family of differential operators of Dunkl type including the positive half of the Virasoro algebra. The space $I_n^{(k,2)}$ coincides with the space of all symmetric polynomials in n variables which vanish when k+1 variables are set equal. The space $I_n^{(2,r)}$ coincides with the space of correlation functions of an Abelian current of a vertex operator algebra related to Virasoro minimal series (3,r+2).

Laughlin FQHE state

$$\Psi = \Phi(z_1, z_2, \ldots, z_N) \prod_{i=1}^{N} e^{-\varphi(\boldsymbol{r}_i)}$$

lowest Landau level

N-variable (anti)symmetric polynomial

$$\nabla^2 \varphi(\boldsymbol{r}) = 2\pi B(\boldsymbol{r})/\Phi_0$$

- ν = 1/m Laughlin state

$$\Phi(z_1, z_2, \ldots, z_N) = \prod_{i<j} (z_i - z_j)^m$$

- "occupation number"-like representation in orbitals z^m, m = 0,1,..., N_Φ = m(N-1) orbitals

m=0 orbital → 1001001001001001001...1001 **(m=3)**

This is the "dominant" configuration of the Laughlin state

“Dominance”

“Dominance” is a partial ordering relations between partitions of a (non-negative) integer into N (non-negative) integer parts. (related to what was called “squeezing” in Sutherlands solution of the periodic Calogero model......)

- convert occupation pattern to a **partition** **λ**, “padded” with zeroes to length N:
- $1001001 \rightarrow \lambda = \{\lambda_1, \lambda_2, \lambda_3\} = \{6,3,0\}$
- λ dominates λ' if
  - $|\lambda| \equiv (\sum_i \lambda_i) = |\lambda'| = M$
  - $(\sum_{j \leq i} \lambda'_j) \leq (\sum_{j \leq i} \lambda_j)$ for all i $= 1,2,..N-1$

“dominance” and “squeezing”

- (pairwise) squeezing: move a particle from orbital m_1-1 to m_1 and another from m_2+1 to m_2 where $m_1 \leq m_2$.

$$1001001001001001001\ldots1001 \qquad \text{A}$$

$$1000101001001010001\ldots1001 \qquad \text{B}$$

A dominates B (A > B)

- dominance is a <u>partial ordering</u>: if A > B and B > C, then A > C.

- When expanded in occupation number states, the (polynomial) $1/m$ Laughlin state only contains configurations dominated by the most compressed (minimum M) "(1,m)-admissible configuration" where no group of m consecutive orbitals contains more than 1 particle.

- "admissibility" can be thought of as a generalized Pauli principle.

- For symmetric (Boson)states the 1/2 (circular droplet) Laughlin state is "dominated" by the configuration
  1010101010101...1010000000....

incompressible region

empty region outside droplet

center
of droplet

edge
of droplet

- polynomial part of 1/2 Laughlin wavefunction

negative jack parameter

$$J^{-2}_{\lambda_0}(z_1, z_2, \ldots, z_N)$$

partition corresponding to 1010101..

Laughlin state is a
Jack Polynomial!

Brief review of symmetric polynomials

- monomials $m_\lambda(\{z_i\})$ are analogs of free boson states ("Slater permanents").

- They are eigenfunctions of differential operators

$$L^z = \sum_i z_i \frac{\partial}{\partial z_i} \qquad \Delta_0 = \frac{1}{N} \sum_{i<j} \left(z_i \frac{\partial}{\partial z_i} - z_j \frac{\partial}{\partial z_j} \right)^2$$

Total degree   variance

- Jacks $J^\alpha_\lambda(\{z_i\})$ are generalization of monomials which are eigenfunctions of L^z and $\Delta(\alpha)$:

$$\Delta(\alpha) = \alpha\Delta_0 + \sum_{i<j} \frac{z_i + z_j}{z_i - z_j} \left(z_i \frac{\partial}{\partial z_i} - z_j \frac{\partial}{\partial z_j} \right)$$

- Jacks are symmetric polynomials with the crucial property that their expansion in monomials has the form (with "monic normalization")

$$J_\lambda^\alpha = m_\lambda + \sum_{\mu<\lambda} A_{\lambda\mu}(\alpha) m_\mu$$

dominance

$$\lim_{\alpha\to\infty} J_\lambda^\alpha = m_\lambda$$

- Like monomials, they are a linearly-independent basis of symmetry polynomials (EXCEPT for negative rational λ , where for some α , Jacks are singular)

The Kyoto RIMS group seem to be the first to investigate Jacks when α is not real positive

- Multiply symmetric polynomials by Vandermonde determinant to get antisymmetric polynomials. This preserves the dominance property:

$$\prod_{i<j}(z_i - z_j)^q J_\lambda^\alpha = m_{\lambda'(q)} + \sum_{\mu<\lambda'(q)} A_{\lambda'\mu}(\alpha, q) m_\lambda$$

$$\lambda'(q) = \lambda + q\{N-1, N-2, \ldots, 1, 0\}$$

e.g., 1010101... becomes 1001001001.. when q=1

(same principles apply for fermions and bosons)

- RIMS group looked at Jacks with α = -(k +1)/(r-1) (relatively prime, and k > 0, which is all negative rationals except -1/(r-1))

- For r=2, they found that a SUBSET of the non-singular Jacks are a basis of polynomials that vanish if more than k coordinates coincide....

- The SUBSET is the "(k,r,N) admissible" partitions. For general r, these span space of polynomials that vanish as

$$\Psi(Z, \ldots, Z, z_{k+1}, \ldots z_N) \propto \prod_{j>k} (z_j - Z)^r$$

- More generally, multiply by q powers of the Vandermonde determinant to get "(k,q,r,N) admissible" polynomials.
- k=1 is the Laughlin states (can always choose r=2)
- r=2 is the Laughlin,Moore-Read,Read-Rezayi sequences. For k > 1, these are the non-Abelian FQHE states which are currently popular candidates for topological quantum computing!

- Polynomial equations can be solved using linear-independence, NOT orthogonality
- characterized by zeroes of polynomials:
- "quasihole" state defined by

$$\Psi_W(Z,\ldots,Z,z_k\ldots,z_N) \propto (W-Z)$$

↑ can't bring k particles together at quasihole position W!

Can find the representation of this in terms of (admissible) Jacks purely with polynomial algebra! (no quantum mechanics needed!)

quantum states corresponding to different monomials are orthogonal!

- where does quantum mechanics come in?
- through the mapping

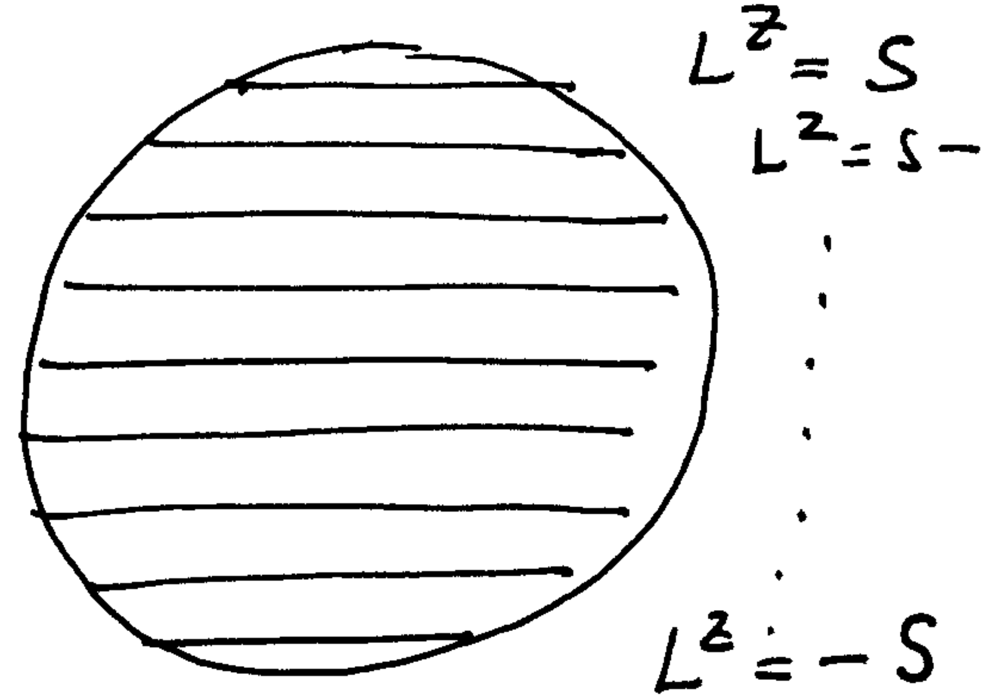
$$m_\lambda \to |m_\lambda\rangle$$

determined by geometry

$$\langle m_\lambda | m_{\lambda'} \rangle = \delta_{\lambda\lambda'} \prod_{m \geq 0} \frac{(\gamma_m)^{n_m(\lambda)}}{n_m(\lambda)!}$$

Compactification of the Lowest Landau level on the Riemann sphere.

Identify orbitals m = 0,1,...,N_Φ with orbitals Lz = S,S-1,...,-S on a sphere enclosing magnetic monopole charge N_Φ= 2S



Uniform QHE states are rotationally-invariant, L_{tot} = 0.

Beyond "standard" occupation number formalism

- $\boldsymbol{k}$-particle $1/\boldsymbol{m}$ Laughlin droplet creation operator (circular droplet centered at $\boldsymbol{R}$):

Gaussian centered at $\boldsymbol{R}$

$$\eta_{km}(\boldsymbol{R})^{\dagger}|vac\rangle \propto \prod_{i>j}(z_i - z_j)^m \prod_{i=1}^{k} \psi_{\boldsymbol{R}}(\boldsymbol{r}_i)$$

- For $k = 1$, $(m \ has \ no \ meaning \ in \ this \ case)$, this is just the standard lowest Landau-level single-particle creation operator

$$c(\boldsymbol{R})^{\dagger}|vac\rangle \propto \psi_{\boldsymbol{R}}(\boldsymbol{r}_1)$$

- Read-Rezayi (includes Laughlin, Moore-Read) FQHE states are defined by

$$\nu = \frac{k}{km+2}$$

(k+1)-particle destruction → $\eta_{k+1,m}(\boldsymbol{R})|\Psi\rangle = 0$

$\eta_{2,m'}(\boldsymbol{R})|\Psi\rangle = 0, m' < m$

for all $\boldsymbol{R}$

k-particle destruction → $\eta_{k,m}(\boldsymbol{R}_i)|\Psi\rangle = 0$

at locations $\boldsymbol{R}_i$ of pinned elementary quasiholes

"Admissible" configurations:

Not more than k particles in km+2 consecutive orbitals
For m >0, not more than one particle in m consecutive orbitals

- On the sphere, the number of charge -e/(km+2) elementary quasiholes for a given N, N_Φ is

$$N_{qh} = k(N_\Phi - \tfrac{1}{2}mk(k-1)) - (km+2)(N-k)$$

- The size of the basis set of quantum states (with unpinned quasi holes) is equal to the number of admissible configurations.
- The states can be completely constructed out of configurations dominated by the dominant admissible configuration ("top" configuration).
- These are a very small subset of lowest Landau level states!

Fermionic 2/4=1/2 Moore-Read state

uniform vacuum state on sphere:

1100110011001100110011001100110011

even fermion number -e/2 double quasihole (h/e vortex) at North Pole:

∘∘01100110011001100110011001100110011

odd fermion number -e/2 double quasihole (h/e vortex) at North Pole:

∘∘100110011001100110011001100110011

fractionalization: one -e/4 quasihole (h/2e vortex) at North Pole, one near equator.

∘10101010101010∘1001100110011001100110011

These translate into explicit wavefunctions that can be calculated in finite-size systems

3/5 (fibonacci) Read-Rezayi state primary configurations

111001110011100111001110011100 . . .

elementary -e/5 vortex at North pole

11010110101101011010110101101 . . .

110011100111001110011100111001 . . .

10101101011010110101101011010 . . .

01110011100111001110011100111 . . .

0101101011010110101101011010 . . .

1001110011100111001110011100111 . . .

01101011010110101101011010110 . . .

vortex moves by hopping 5 orbitals at a time

For charge -ne/5, n > 1 there are always 2 orthogonal primary states.

example: $SU(2)_{k=2}$ (= Moore Read state)

- Counting of states determines all "trace formulas" of cft, fusion rules, etc.
-

$\ldots 202020202020200000000000$	S=0 primary
$\ldots 111111111110000000000$ $\ldots 111111111100000000000$	S=1/2 primary
$\ldots 02020202020202020000000000$ $\ldots 02020202020202010000000000$ $\ldots 02020202020202000000000000$	S=1 primary

- For general (k,r) sequence (q=0)

$$\nu = \frac{k}{r} \qquad c^{eff} = \frac{k(r+1)}{k+r}$$

electrical QHE thermal QHE

Non-unitary "W-minimal" cft for r > 2!

$$W_n^{p,p'} = W_k^{k+1,k+r}$$

explicit numerical calculations

- Strategy: obtain full set of highest-weight states by solving

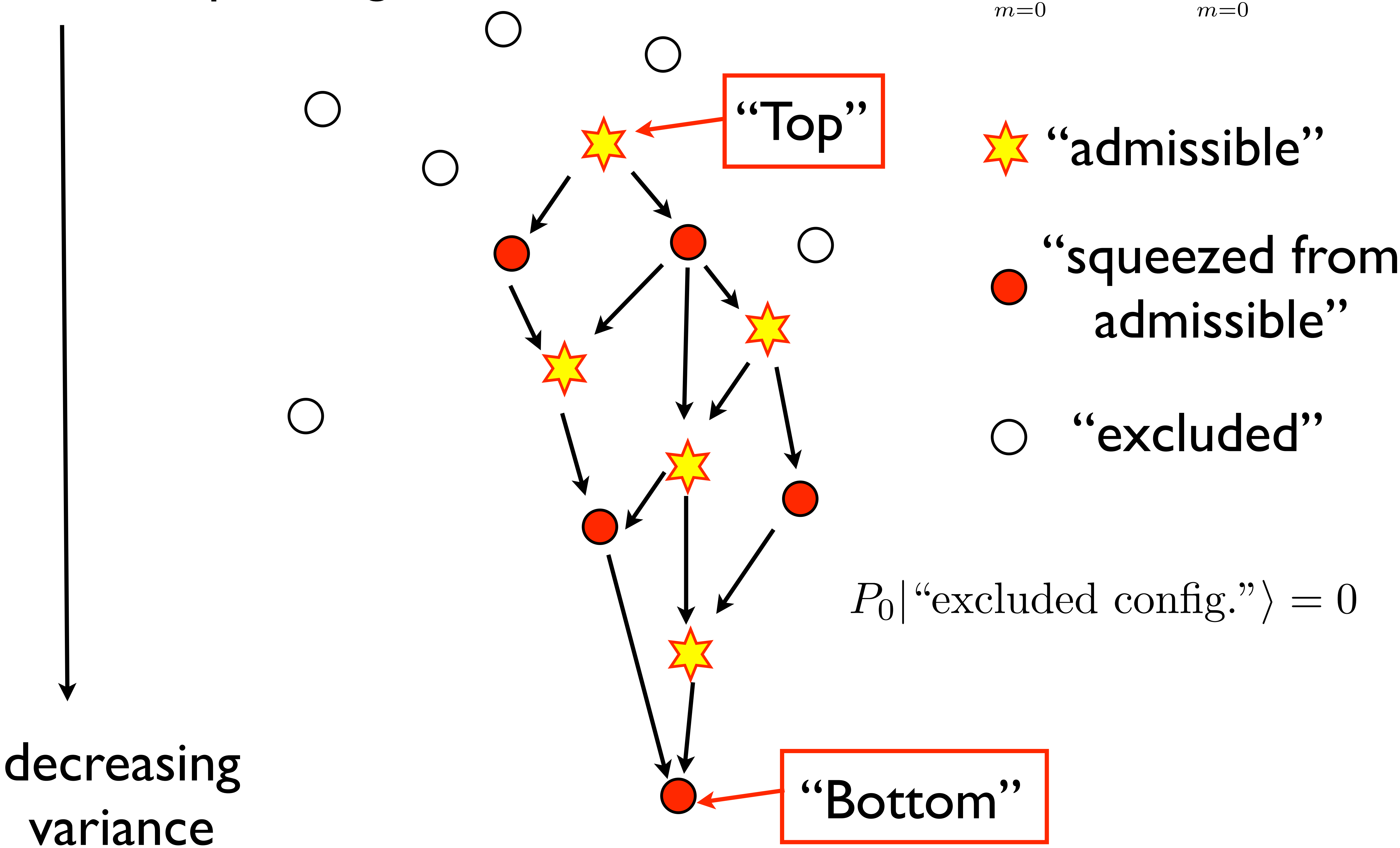
$$L_{tot}^{+}|\Psi\rangle = 0$$

- The number of admissible configs at each Lz tells us how many we need. We exclude from the basis set configs not dominated by the dominant admissible config. This gives a highly overdetermined system of equations!

- Within the full basis set thus obtained, impose the condition that pins the quasiholes at the desired locations.

Partial ordering of occupation number configurations with fixed L_z

- squeezing decreases the variance $\sum_{m=0}^{m_{max}} m^2 n_m - (\sum_{m=0}^{m_{max}} m n_m)^2$



$$P_0|\text{"excluded config."}\rangle = 0$$

key point:

Null space is invariant under the Euclidean group

- Disk: $[P_0, a] = 0$
- Sphere: $[P_0, L^+] = 0$

- Use Wigner-Eckert: need to (simultaneously) solve

$$L^+|\Psi\rangle = 0 \text{ and } P_0|\Psi\rangle = 0$$

highest weight null modes

- In the full basis this is an undetermined problem (more columns than rows)

- After "excluded" states are removed, it is overdetermined (more rows than columns)!

- (can efficiently solve with a variant Lanczos-type technique to full floating-point accuracy.)

example:

16 electrons on sphere, maximum ν=1/2 Moore-Read density, plus 2h/e extra flux (single qubit when vortices are fixed)

```
 16 spinless fermions on the sphere with   32 orbitals:
full basis: 601080390 projected basis:       825 (summmed over LZ)
=========================================================================================
Ltot=     0.0     1.0     2.0     3.0     4.0     5.0     6.0     7.0     8.0     9.0    10.0    11.0
all:     6235   17625   30017   41207   53324   64172   75813   86131   97177  106789  117059  125864
nul:        3       0       6       2       7       4       7       4       7       3       5       2
=========================================================================================
Ltot=    12.0    13.0    14.0    15.0    16.0    17.0    18.0    19.0    20.0    21.0    22.0    23.0
all:   135218  143078  151432  158231  165486  171188  177258  181794  186683  190016  193686  195853
nul:        3       1       2       0       1       0       0       0       0       0       0       0
=========================================================================================
```

L_z = 2: find 6 zero modes of a sparse 5,800,384 x 6,170,810 overdetermined matrix (52×10^6 non-zero matrix elements)

```
PURGE: nstate =    8884686 LZ=     2.0 root =    11001100110011001001001100110011
nstate before purge=    8884686 after purge =    5800384
 BINARY: registered binary code total size =      2:
     1 components with sizes:       2
PURGE: nstate =    8854669 LZ=     3.0 root =    11001100110011001010001100110011
L+ has    52060614 +   21167057 non-zero elements
   6170810 constraints,    52060614 nonzero matrix elements, and    370432 linear dependencies
 second representation of L+:    16 distinct values   48727308 elements
```

601,080,390 lowest LL states

825 MR null-mode states, of which

57 are highest weight

```
zero mode #     1: maximum error       3.0D-18
zero mode #     2: maximum error       5.2D-18
zero mode #     3: maximum error       4.8D-18
zero mode #     4: maximum error       6.2D-18
zero mode #     5: maximum error       3.8D-18
zero mode #     6: maximum error       5.5D-18
final overlap matrix eigenvalues:
    1.00000000      1.00000000      1.00000000      1.00000000      1.00000000
    1.00000000
two-body interaction energies:
     -7.4892956319233095      -7.4689790893071946      -7.4496977681804388
     -7.4039194994824538      -7.3871308896580405      -7.3668521272231633
   6 highest weight zero modes found
```

two Moore-Read vortices (fused)

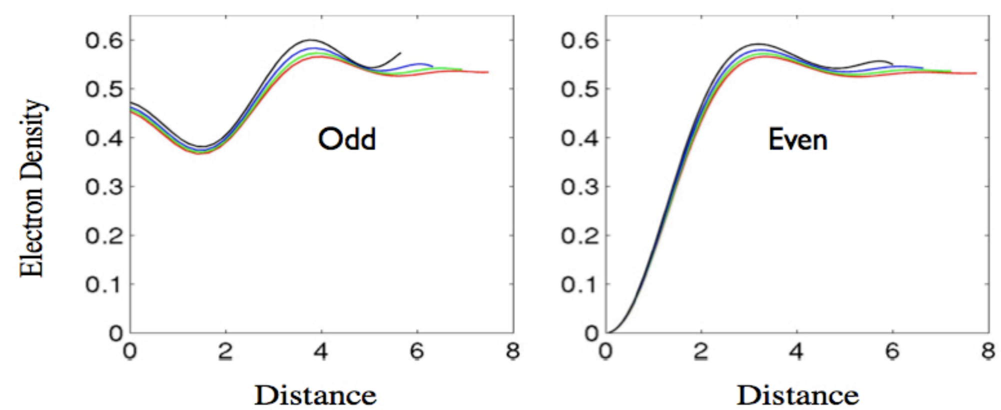


FIG. 4: The particle density for fused probes as function of distance from the fusing point. Left/right panel refers to odd/even number of electrons. On the left, the different curves correspond to N/N_ϕ=9/16, 11/20, 13/24, 15/28 and, on the right, the different curves correspond to N/N_ϕ=10/18, 12/22, 14/26, 16/30.

$1001100110011001 1 0 0 1 1 \ldots$

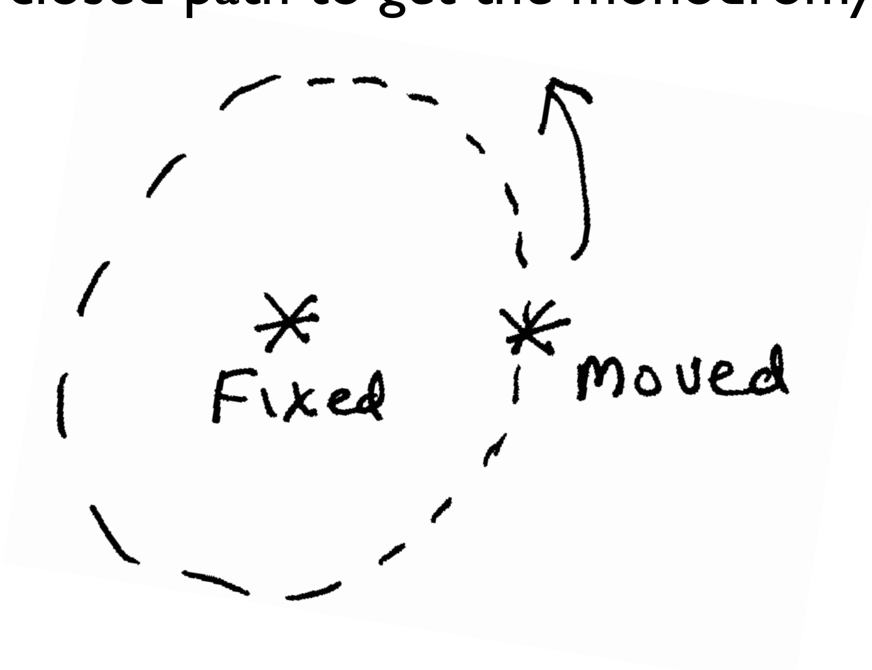
$0110011001100110 0 1 1 0 0 \ldots$

unpaired electron at North pole

Monodromy

- Hold one vortex at the north pole, and move the other in infinitesimal loops to map out the Berry curvature, in the two cases of even and odd fermion number.

- Integrate the berry curvature inside a closed path to get the monodromy.



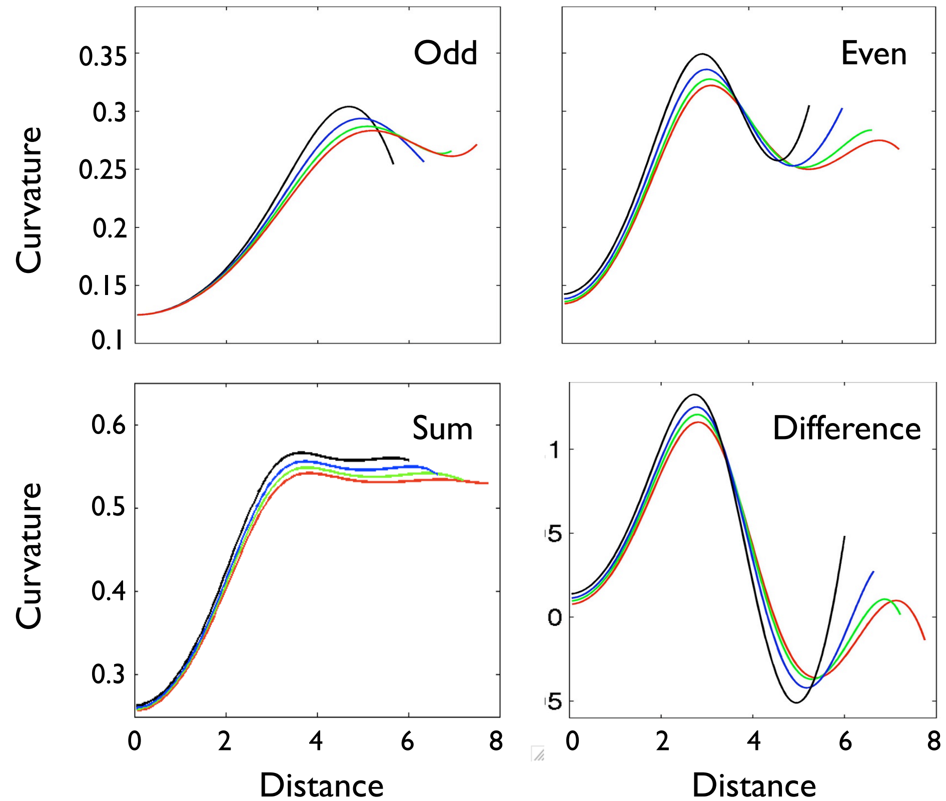


FIG. 10: (Color online.) The Berry curvature obtained by moving one anyon while keeping the other fixed. Left-upper panel shows the results for odd number of electrons: N/N_ϕ=9/16, 11/20, 13/24, 15/28 and the right-upper panel shows the results for even number of electrons: N/N_ϕ=10/18, 12/22, 14/26, 16/30. The lower-left and lower-right panels show the sum and the difference between the odd and even results, respectively. For example, we added and subtracted the result for N/N_ϕ=10/18 and N/N_ϕ=9/16, and then the results for N/N_ϕ=12/22 and N/N_ϕ=11/20, etc..

for a path with a large radius, the relative Berry phase factor between the even and odd fermion number cases approaches -1 (as predicted!)

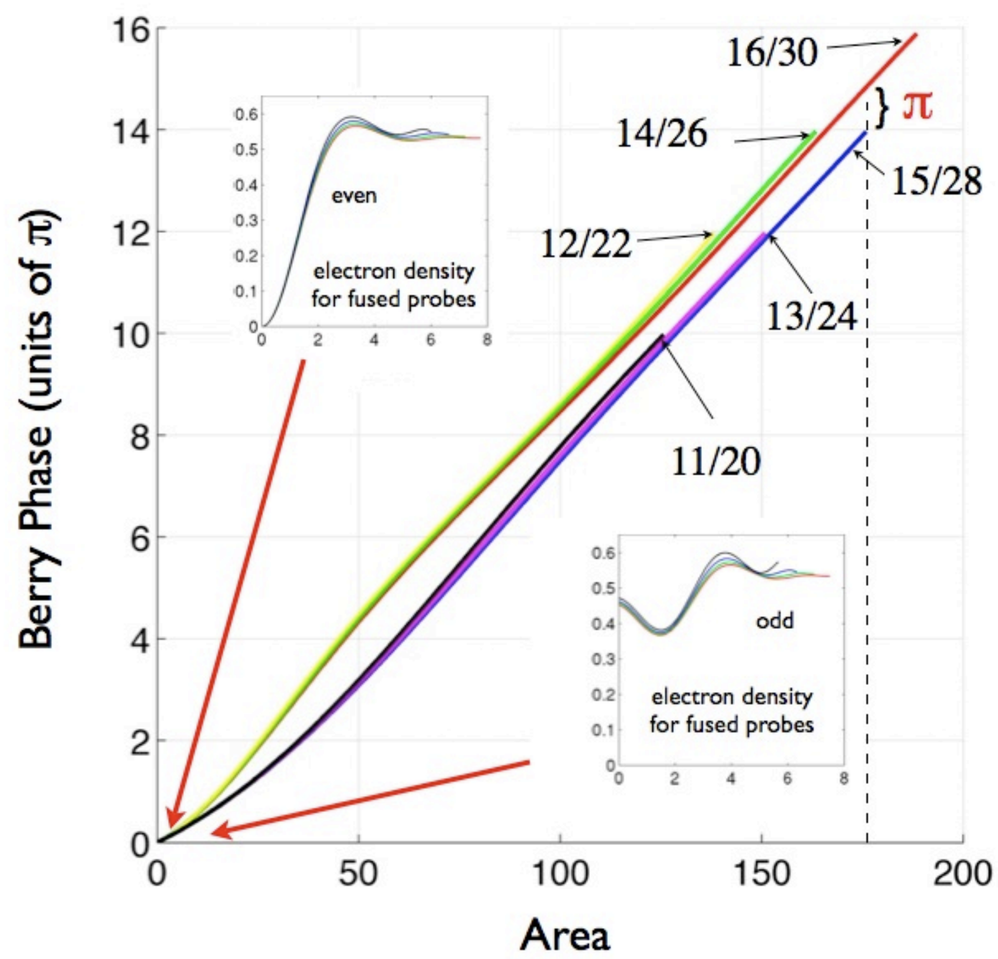


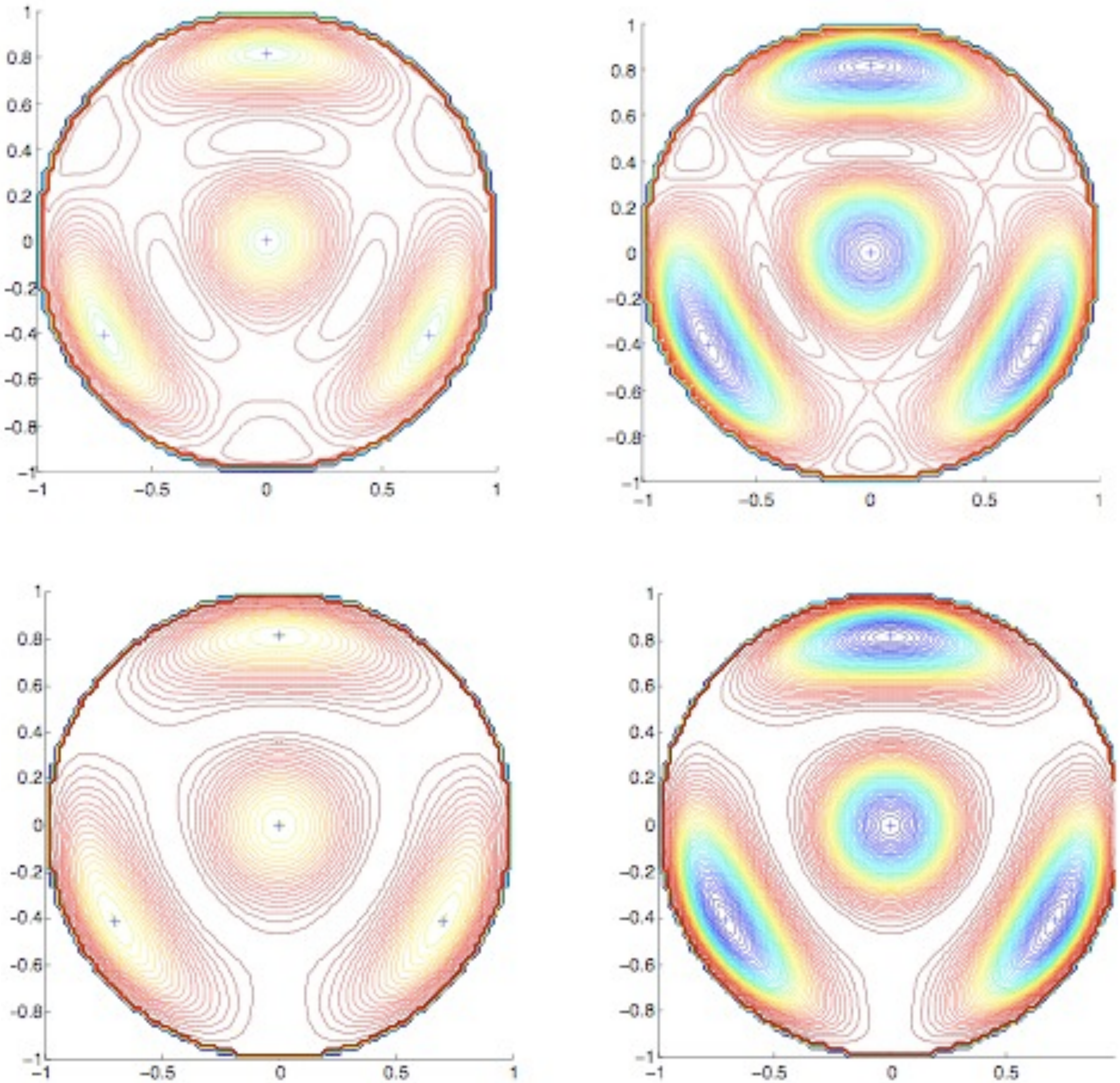
FIG. 11: (Color online.) The Berry phase accumulated by an anyon when moved along a path θ=const, with the other anyon fixed at the North pole. The Berry phase is plotted against the area enclosed by the paths. Each curve is marked with the corresponding N/N_ϕ numbers. The insets show the electron density for the fused anyons, computed in Fig. 4, which one can use, experimentally, to distinguish between even/odd cases.

4 well-separated vortices (a qubit)

Note that the two state have slightly different "interference ripple" patterns in the electron density that will be exponentially small as the distance between the vortices increases, but which is a residual local physical difference between the states.

single-particle density

$m=1$ two-particle density



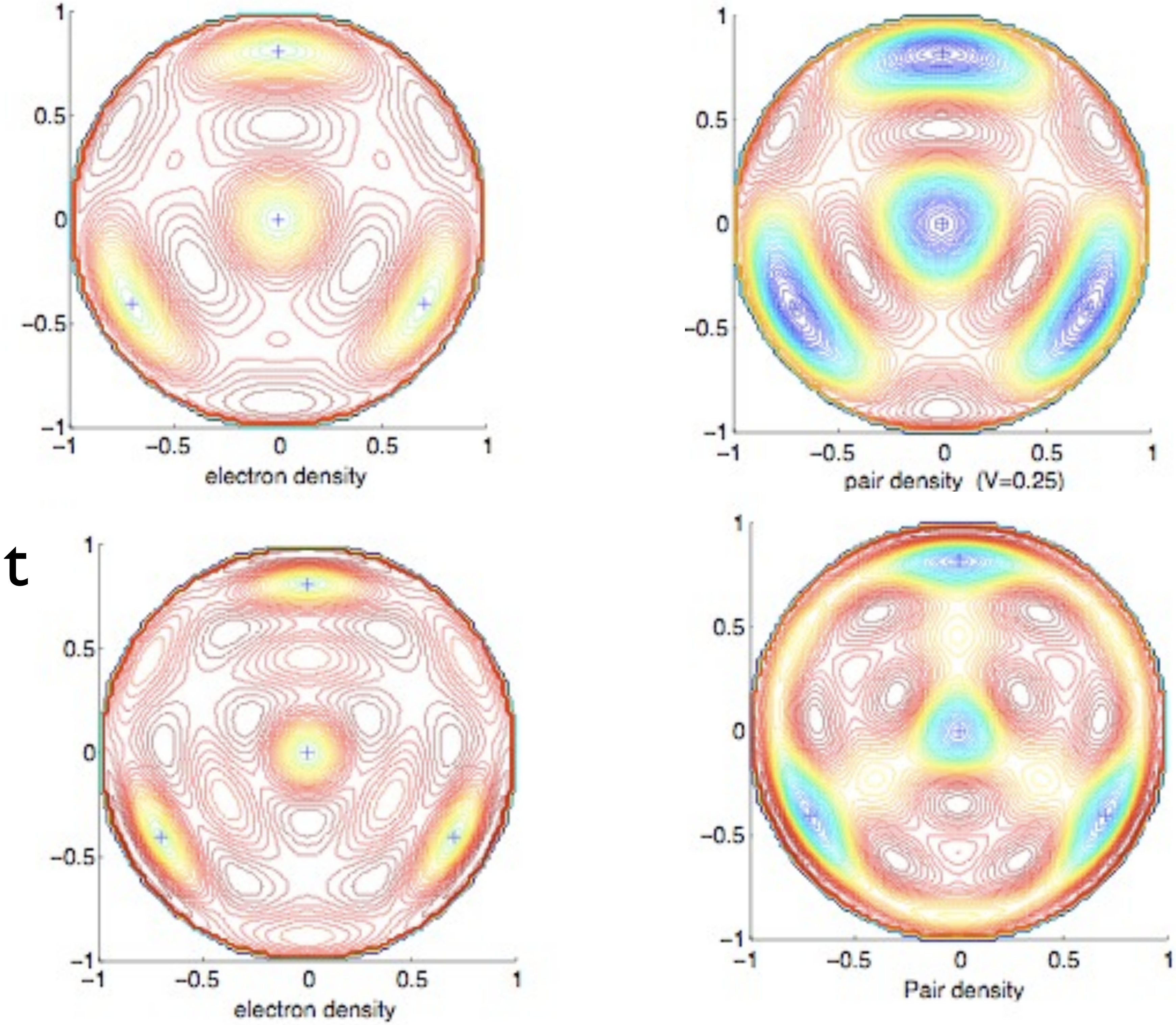
Tetrahedral arrangement of 4 MR h/2e vortices, (14 electrons, 28 orbitals)

One qubit is left after positions of vortices are fixed.

Sphere is mapped to unit disk.

the qubit doublet is split by the Coulomb interaction, both states are shown. THE SPLITTING AND LOCAL DIFFERENCE BETWEEN THE TWO STATES IS EXPECTED TO DISAPPEAR AS THE SYSTEM SIZE INCREASES.

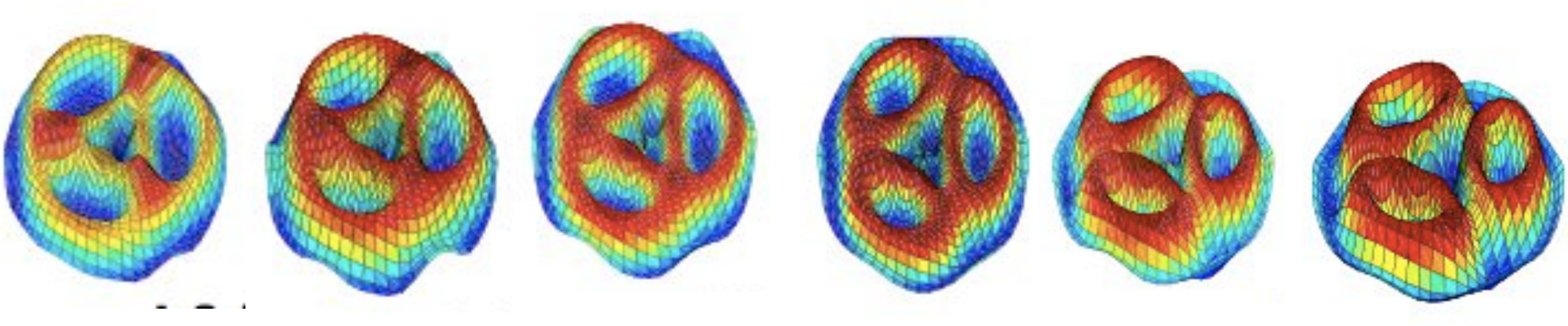
four probes, tetrahedral pattern: candidate qubit pair, 14/28



zero-point motion of vortex positions

These are made with "STM + coulomb repulsion": very close to the "exact" states!

non-Abelian Berry curvature , for increasing size (10-15 electrons)



as size increases, the (magnititude) of the non-abelian curvature field is seen to be concentrated near the quasiparticle cores, consistent with braiding. (For widely separated vortices, there should be vanishing non-abelian curvature in the regions in between the vortices, so the monodromy becomes purely topological)

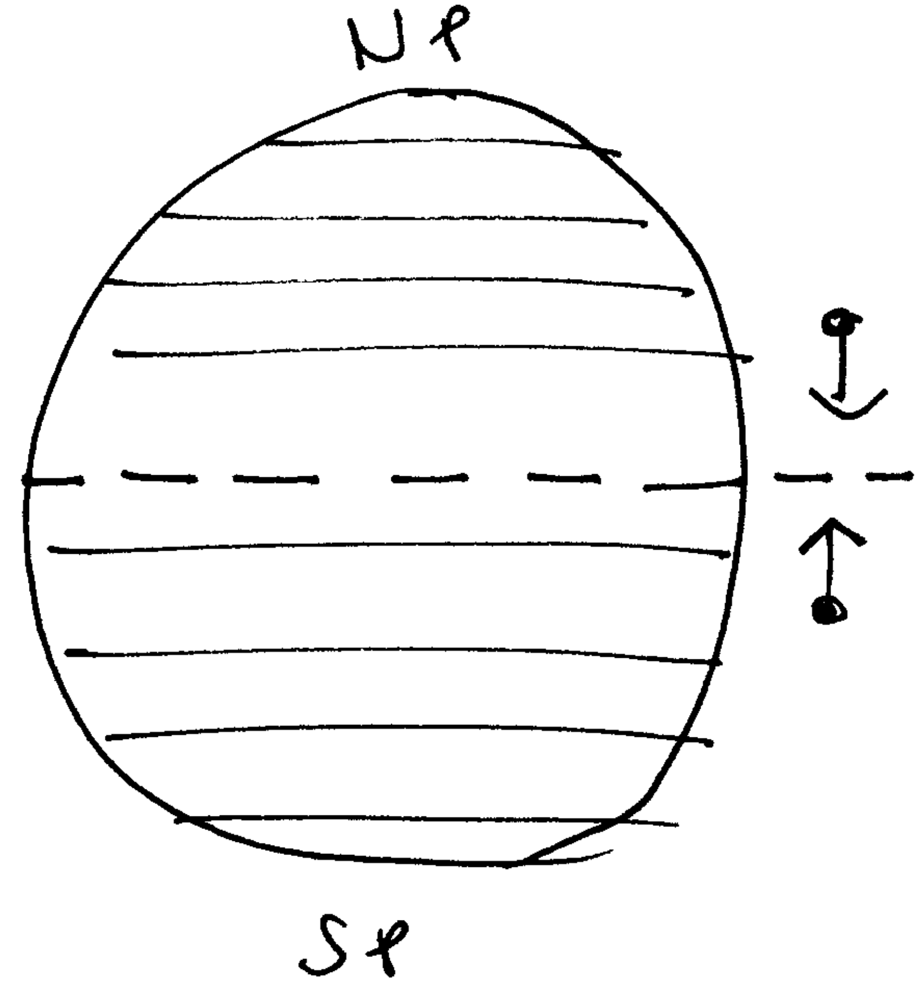
10/20

11/22

12/24

Entanglement spectra and “dominance”

- Schmidt decomposition of Fock space into N and S hemispheres.
- Classify states by Lz and N in northern hemisphere, relative to dominant configuration. Lz always decreases relative to this (squeezing)



Represent bipartite Schmidt decomposition like an excitation spectrum (with Hui Li)

$$|\Psi\rangle = \sum_{\alpha} e^{-\beta_\alpha/2} |\Psi_{N\alpha}\rangle \otimes |\Psi_{S\alpha}\rangle$$

- like CFT of edge states.
- A lot more information than single number (entropy)

- many zero eigenvalues

$$e^{-\beta_\alpha} = 0$$

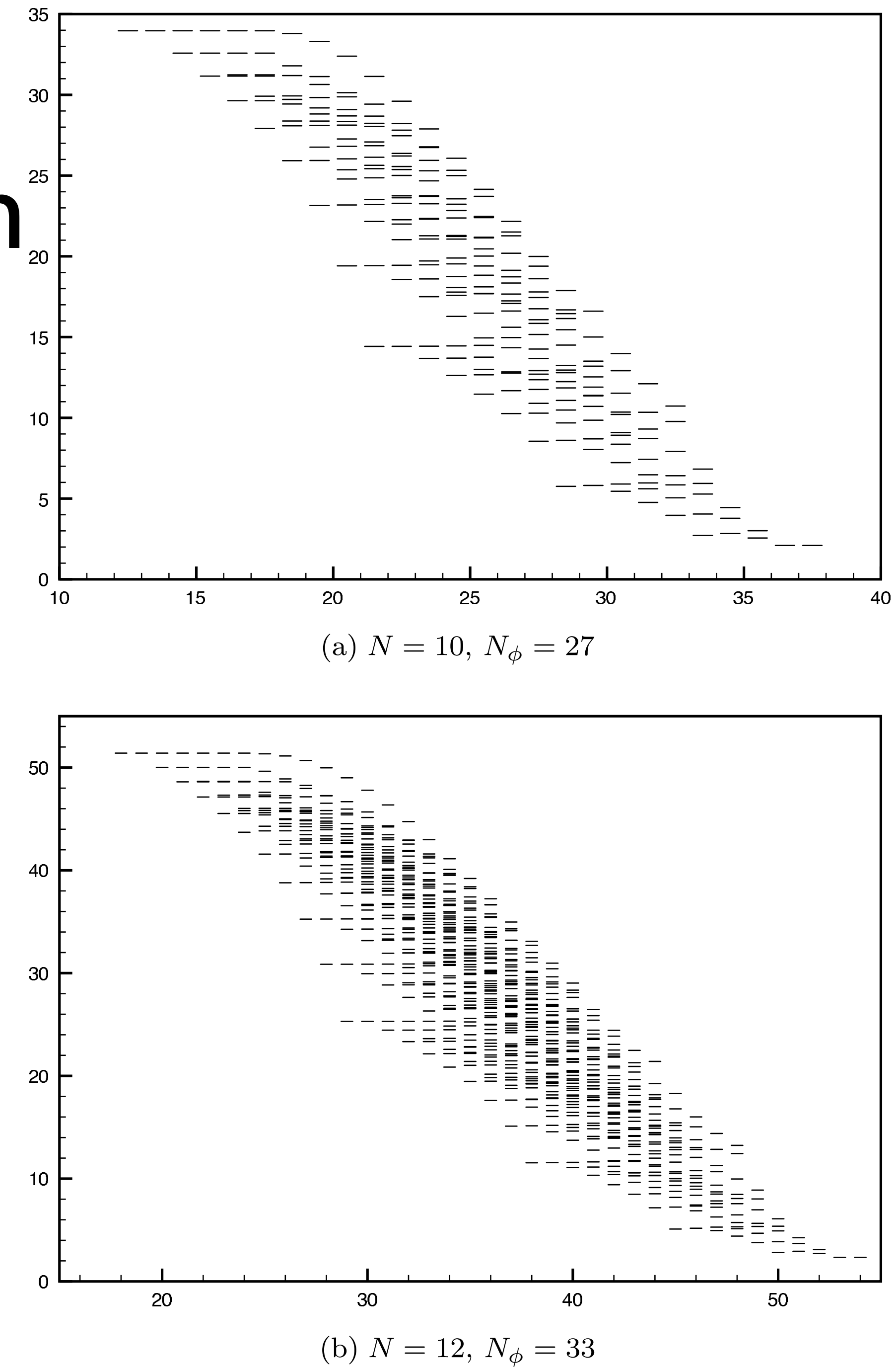
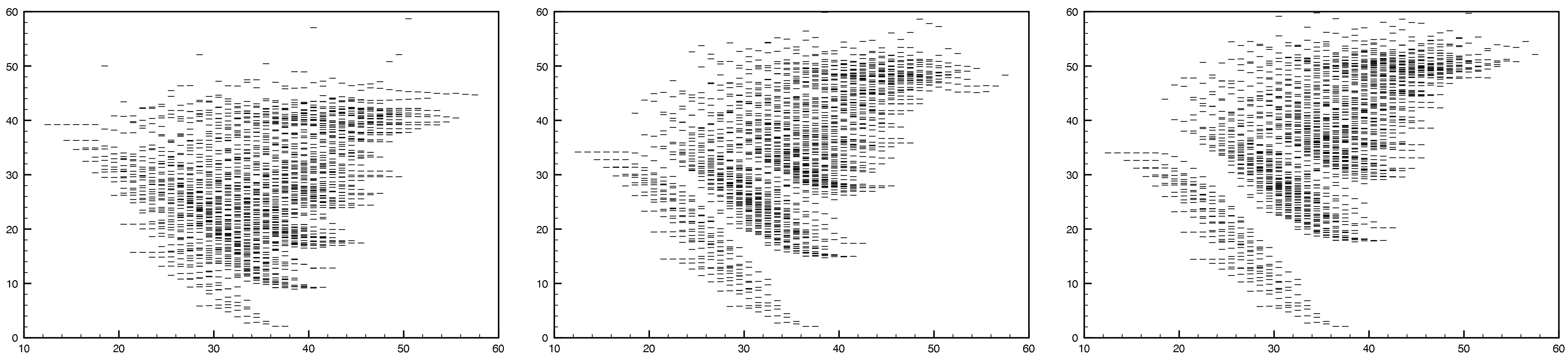


FIG. 1: Entanglement spectrum for the 1/3-filling Laughlin states, for $N = 10, m = 3, N_\phi = 27$ and $N = 12, m = 3, N_\phi = 33$. Only sectors of $N_A = N_B = N/2$ are shown.

Look at difference between Laughlin state,entanglement spectrum and state that interpolates to Coulomb ground state.

3



(a) $x = 1$ (b) $x = 1/3$ (c) $x = 1/10$

FIG. 2: Entanglement spectrum for the ground state, for a system of $N = 10$ electrons in the lowest Landau level on a sphere enclosing $N_\phi = 27$ flux quanta, of the Hamiltonian in Eq. (12) for various values of x.

$$H = xH_c + (1 - x)V_1$$

x=0 is pure Laughlin

Can we identify topological order in “physical as opposed to model wavefunctions from low-energy entanglement spectra?

Summary

- Generalized Pauli-like "admissibility" criterion gives counting of Laughlin/Moore-Read/Read-Rezayi states with quasi-holes, and specifies "dominant" configurations.
- Gives a simplified basis for practical calculations (analog of projection into the lowest Landau level, now into Read-Rezayi zero mode space)
  - (Generalizes to describe quasiPARTICLES too, -with A. Bernevig)
  - Entanglement spectra.

- Finally, non-Unitary cft's seem to be well-behaved and regularized when derived from polynomials: expect heff instead of h for primary field propagators!
- Idea runs counter to Read's idea that non-Unitary means a gapless bulk theory inside the droplet,