On the growth of interfaces: dynamical scaling and beyond

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MH, **J.D. Noh** and **M. Pleimling**, Phys. Rev. **E85**, 030102(R) (2012)
MH, Nucl. Phys. **B869**, 282 (2013); MH & **S. Rouhani**, J. Phys. **A46**, 494004 (2013)
N. Allegra, **J.-Y. Fortin** and MH, J. Stat. Mech. P02018 (2014)
MH & **X. Durang**, J. Stat. Mech. P05022 (2015) *& work in progress*

Overview :

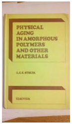
1. Physical ageing & interface growth
2. Interface growth & KPZ universality class
3. Interface growth on semi-infinite substrates
4. A spherical model of interface growth : the (first) Arcetri model
5. Linear responses and extensions of dynamical scaling
6. Form of the scaling functions & LSI
7. Conclusions

1. Physical ageing & interface growth

known & practically used since prehistoric times (metals, glasses)
systematically studied in physics since the 1970s

STRUIK '78

$\Longrightarrow$ **discovery :** ageing effects **reproducible** & **universal** !
occur in widely different systems
(structural glasses, spin glasses, polymers, simple magnets, ...)

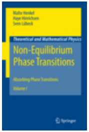


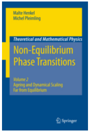
Three defining properties of **ageing** :

1. slow relaxation (non-exponential !)
2. **no** time-translation-invariance (TTI)
3. dynamical scaling

without fine-tuning of parameters

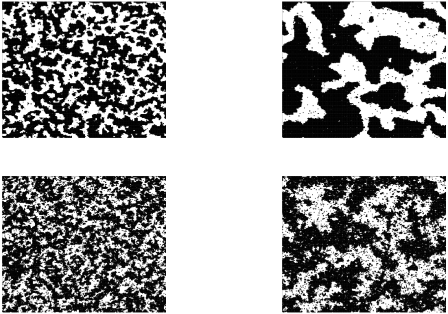
Cooperative phenomenon, far from equilibrium





Question : what can be learned about intrisically **irreversible** systems
by studying their ageing behaviour ?

$t = t_1$ $\quad$ $t = t_2 > t_1$



magnet $T < T_c$

$\longrightarrow$ ordered cluster

magnet $T = T_c$

$\longrightarrow$ correlated cluster

growth of ordered/correlated domains, of typical linear size

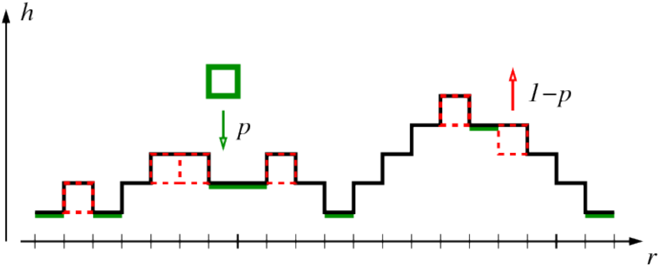
$$L(t) \sim t^{1/z}$$

dynamical exponent z : determined by equilibrium state

Interface growth

deposition (evaporation) of particles on a substrate

→ height profile $h(t, \mathbf{r})$ slope profile $\mathbf{u}(t, \mathbf{r}) = \nabla h(t, \mathbf{r})$



p = deposition prob.

$1 - p$ = evap. prob.

Questions :

* average properties of profiles & their fluctuations ?
* what about their relaxational properties ?
* are these also examples of physical ageing ?

? does dynamical scaling **always** exist ? are there extensions ?

Analogies between magnets and growing interfaces

Common properties of critical and ageing phenomena :

* **collective** behaviour,
  very **large** number of interacting degrees of freedom
* **algebraic** large-distance and/or large-time behaviour
* described in terms of **universal** critical **exponents**
* very **few** relevant scaling operators
* justifies use of extremely **simplified** **mathematical models**
  with a remarkably rich and complex behaviour
* yet of **experimental significance**

see talks by T. Sasamoto *and* K. Takeuchi *at this conference*

Magnets

thermodynamic equilibrium state

order parameter $\phi(t,\mathbf{r})$

phase transition, at critical temperature T_c

variance :

$\left\langle \left(\phi(t,\mathbf{r}) - \langle\phi(t)\rangle\right)^2 \right\rangle \sim t^{-2\beta/(\nu z)}$

relaxation, after quench to $T \leq T_c$

autocorrelator

$C(t,s) = \langle \phi(t,\mathbf{r})\phi(s,\mathbf{r})\rangle_c$

Interfaces

growth continues forever

height profile $h(t,\mathbf{r})$

same generic behaviour throughout

roughness :

$w(t)^2 = \left\langle \left(h(t,\mathbf{r}) - \overline{h}(t)\right)^2 \right\rangle \sim t^{2\beta}$

relaxation, from initial substrate :

autocorrelator $C(t,s) =$

$\left\langle \left(h(t,\mathbf{r}) - \overline{h}(t)\right) \left(h(s,\mathbf{r}) - \overline{h}(s)\right) \right\rangle$

ageing scaling behaviour :

when $t,s \to \infty$, and $y := t/s > 1$ fixed, expect, with $\left\{ \begin{array}{l} \text{waiting time } s \\ \text{observation time } t > s \end{array} \right.$

$$C(t,s) = s^{-b} f_C(t/s) \quad \text{and} \quad f_C(y) \stackrel{y\to\infty}{\sim} y^{-\lambda_C/z}$$

b, β, ν and dynamical exponent z : **universal** & related to stationary state

autocorrelation exponent λ_C : **universal** & independent of stationary exponents

Magnets

exponent value $b = \begin{cases} 0 & ; \; T < T_c \\ 2\beta/\nu z & ; \; T = T_c \end{cases}$

Interfaces

exponent value $b = -2\beta$

models :

(a) gaussian field

$\mathcal{H}[\phi] = -\frac{1}{2}\int \mathrm{d}\mathbf{r}\, (\nabla\phi)^2$

(b) **Ising model**

$\mathcal{H}[\phi] = -\frac{1}{2}\int \mathrm{d}\mathbf{r}\, \left[(\nabla\phi)^2 + \tau\phi^2 + \frac{g}{2}\phi^4\right]$

such that $\tau = 0 \leftrightarrow T = T_c$

dynamical Langevin equation (Ising) :

$$\begin{aligned} \partial_t \phi &= -D\frac{\delta\mathcal{H}[\phi]}{\delta\phi} + \eta \\ &= D\nabla^2\phi + \tau\phi + g\phi^3 + \eta \end{aligned}$$

$\eta(t,\mathbf{r})$ is the usual white noise, $\langle \eta(t,\mathbf{r})\eta(t',\mathbf{r}')\rangle = 2T\delta(t-t')\delta(\mathbf{r}-\mathbf{r}')$

phase transition exactly solved $d = 2$

relaxation exactly solved $d = 1$

Onsager '44, Glauber '63, ...

(a) Edwards-Wilkinson (EW) :

$\partial_t h = \nu\nabla^2 h + \eta$

(b) **Kardar-Parisi-Zhang** (KPZ) :

$$\partial_t h = \nu\nabla^2 h + \frac{\mu}{2}(\nabla h)^2 + \eta$$

growth exactly solved $d = 1$

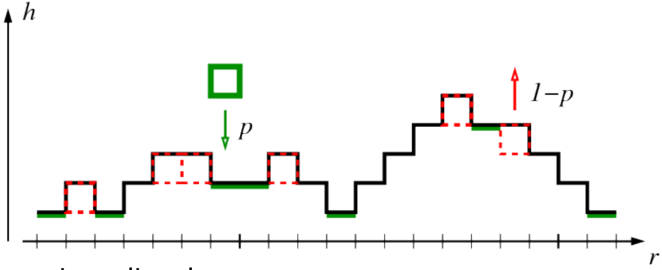
Sasamoto & Spohn '10

Calabrese & Le Doussal '11, ...

2. Interface growth & KPZ class

deposition (evaporation) of particles on a substrate $\to$ height profile $h(t, \mathbf{r})$

generic situation : RSOS (**r**estricted **s**olid-**o**n-**s**olid) model Kim & Kosterlitz 89



$p =$ deposition prob.

$1 - p =$ evap. prob.

here $p = 0.98$

some universality classes :

(a) KPZ $\partial_t h = \nu \boldsymbol{\nabla}^2 h + \frac{\mu}{2} (\boldsymbol{\nabla} h)^2 + \eta$ Kardar, Parisi, Zhang 86

(b) EW $\partial_t h = \nu \boldsymbol{\nabla}^2 h + \eta$ Edwards, Wilkinson 82

η is a gaussian white noise with $\langle \eta(t, \mathbf{r}) \eta(t', \mathbf{r}') \rangle = 2 \nu T \delta(t - t') \delta(\mathbf{r} - \mathbf{r}')$

Family-Viscek scaling on a spatial lattice of extent L^d : $\overline{h}(t) = L^{-d}\sum_j h_j(t)$

Family & Viscek 85

$$w^2(t;L) = \frac{1}{L^d}\sum_{j=1}^{L^d}\left\langle \left(h_j(t) - \overline{h}(t)\right)^2\right\rangle = L^{2\alpha} f\left(tL^{-z}\right) \sim \begin{cases} L^{2\alpha} & ; \text{ if } tL^{-z} \gg 1 \\ t^{2\beta} & ; \text{ if } tL^{-z} \ll 1 \end{cases}$$

β : growth exponent, α : roughness exponent, $\boxed{\alpha = \beta z}$

two-time correlator :

limit $L \to \infty$

$$C(t,s;\mathbf{r}) = \left\langle \left(h(t,\mathbf{r}) - \langle\overline{h}(t)\rangle\right)\left(h(s,\mathbf{0}) - \langle\overline{h}(s)\rangle\right)\right\rangle = s^{-b} F_C\left(\frac{t}{s}, \frac{\mathbf{r}}{s^{1/z}}\right)$$

with ageing exponent : $\boxed{b = -2\beta}$

Kallabis & Krug 96

expect for $y = t/s \gg 1$: $F_C(y,\mathbf{0}) \sim y^{-\lambda_C/z}$ autocorrelation exponent

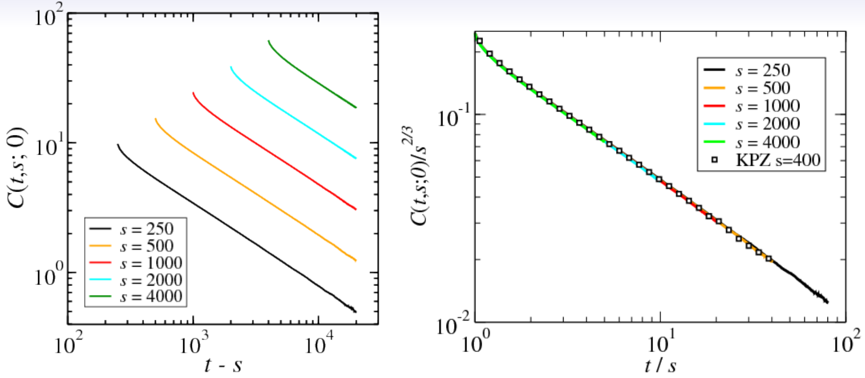
rigorous bound : $\boxed{\lambda_C \geq (d + zb)/2}$

Yeung, Rao, Desai 96 ; mh & Durang 15

KPZ class, to all orders in perturbation theory $\boxed{\lambda_C = d \text{ , if } d < 2}$

Krech 97

$1D$ relaxation dynamics, starting from an initially flat interface



observe all **3** properties of **ageing** : $\left\{ \begin{array}{l} \text{slow dynamics} \\ \text{no TTI} \\ \text{dynamical scaling} \end{array} \right.$

confirm **simple ageing** for the $1D$ KPZ universality class

confirm expected exponents $b = -2/3$, $\lambda_C/z = 2/3$ *pars pro toto*

Kallabis & Krug 96 ; Krech 97 ; Bustingorry *et al.* 07-10 ; Chou & Pleimling 10 ; D'Aquila & Täuber 11/12 ; mh, Noh, Pleimling 12 . . .

Experiment : **universality** of interface exponents, KPZ class

model/system	d	z	β	α
KPZ	1	3/2	1/3	1/2
Ag electrodeposition	1		$\approx 1/3$	$\approx 1/2$
slow paper cumbustion	1	1.44(12)	0.32(4)	0.49(4)
liquid crystal (flat)	1	1.34(14)	0.32(2)	0.43(6)
liquid crystal (circular)	1	1.44(10)	0.334(3)	0.48(5)
cell colony growth	1	1.56(10)	0.32(4)	0.50(5)
(almost) isotrope colloïds	1		0.37(4)	0.51(5)
autocatalytic reaction front	1	1.45(11)	0.34(4)	0.50(4)
KPZ	2	1.63(3)	0.2415(15)	0.393(4)
	2	1.63(2)	0.241(1)	0.393(3)
CdTe/Si(100) film	2	1.61(5)	0.24(4)	0.39(8)
EW sedimentation	2		0(log)	0(log)
/electrodispersion	2			

experimental results from **several groups**, since 1999 (**mainly** since 2010)

3. Interface growth on semi-infinite substrates

properties of growing interfaces near to a boundary ?

$\rightarrow$ crystal dislocations, face boundaries ...

Experiments : Family-Vicsek scaling not always sufficient

FERREIRA *et. al.* 11
RAMASCO *et al.* 00, 06
YIM & JONES 09, ...

$\rightarrow$ **distinct** global and **local** interface fluctuations

- **anomalous scaling**, growth exponent β larger than expected
- **grainy interface morphology**, facetting

! analyse simple models on a **semi**-infinite substrate !

frame co-moving with average interface deep in the bulk

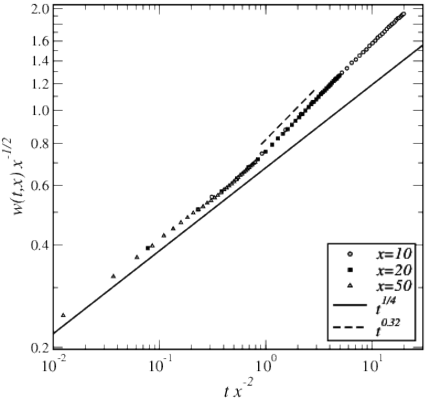
characterise interface by

$$\begin{cases} \text{height profile} & \langle h(t,\mathbf{r})\rangle \\ \text{width profile} & w(t,\mathbf{r}) = \left\langle \left[h(t,\mathbf{r}) - \langle h(t,\mathbf{r})\rangle\right]^2 \right\rangle^{1/2} \end{cases}$$

$h \to 0$ as $|\mathbf{r}| \to \infty$

specialise to $d = 1$ space dimensions ; boundary at $x = 0$, bulk $x \to \infty$

cross-over for the phenomenological growth exponent β near to boundary



EW-class ALLEGRA, FORTIN, MH 14

bulk behaviour $w \sim t^{\beta}$

'surface behaviour' $w_1 \sim t^{\beta_1}$?

cross-over, if causal interaction with boundary

experimentally observed, e.g. for semiconductor films

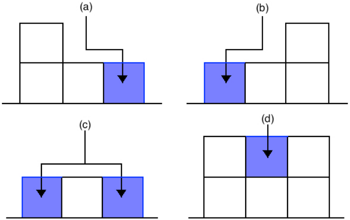
NASCIMENTO, FERREIRA, FERREIRA 11

values of growth exponents (bulk & surface) :

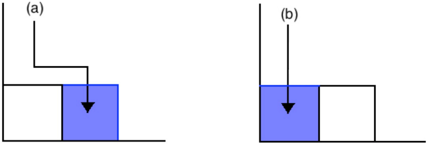
$\beta = 0.25$	$\beta_{1,\text{eff}} \simeq 0.32$	Edwards-Wilkinson class
$\beta \simeq 0.32$	$\beta_{1,\text{eff}} \simeq 0.35$	Kardar-Parisi-Zhang class

simulations of RSOS models :

well-known bulk adsorption processes (& immediate relaxation)



description of immediate relaxation if particle is adsorbed at the boundary

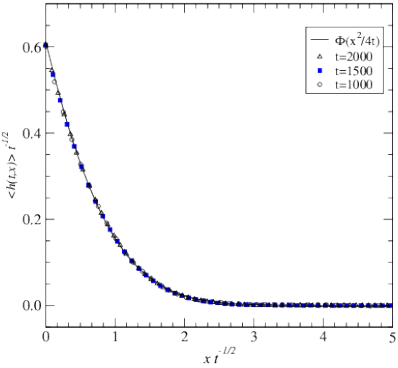


explicit boundary interactions in Langevin equation $h_1(t) = \partial_x h(t,x)|_{x=0}$

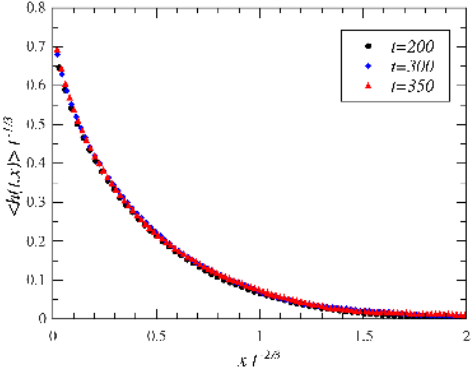
$$\left(\partial_t - \nu\partial_x^2\right) h(t,x) - \frac{\mu}{2}\left(\partial_x h(t,x)\right)^2 - \eta(t,x) = \nu\left(\kappa_1 + \kappa_2 h_1(t)\right)\delta(x)$$

height profile $\langle h(t,x)\rangle = t^{1/\gamma}\Phi\left(xt^{-1/z}\right)$, $\gamma = \frac{z}{z-1} = \frac{\alpha}{\alpha-\beta}$

EW & exact solution, $h(t,0) \sim \sqrt{t}$ self-consistently



KPZ



Scaling of the width profile :

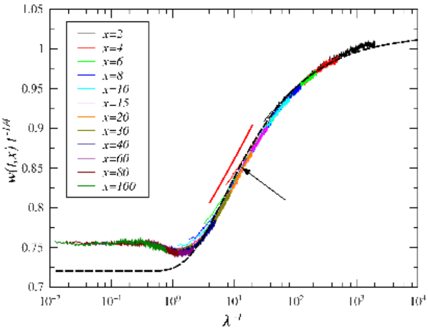
Allegra, Fortin, mh 14

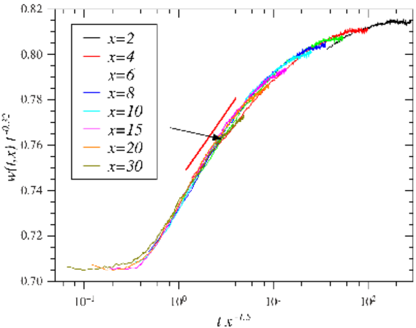
EW & exact solution $\lambda^{-1} = 4tx^{-2}$

KPZ

bulk

boundary





same growth scaling exponents in the bulk and near to the boundary

large **intermediate scaling regime** with effective exponent (**slopes**)

agreement with RG for non-disordered, local interactions   Lopéz, Castro, Gallego 05

? ageing behaviour near to a boundary ?

4. A spherical model of interface growth : the Arcetri model

? KPZ $\longrightarrow$ **intermediate model** $\longrightarrow$ EW ?

preferentially exactly solvable, and this in $d \geq 1$ dimensions

inspiration : mean **spherical model** of a ferromagnet

Berlin & Kac 52
Lewis & Wannier 52

Ising spins $\sigma_i = \pm 1$ obey $\sum_i \sigma_i^2 = \mathcal{N} = \#$ sites

spherical spins $S_i \in \mathbb{R}$ spherical constraint $\left\langle \sum_i S_i^2 \right\rangle = \mathcal{N}$

hamiltonian $\mathcal{H} = -J \sum_{(i,j)} S_i S_j - \lambda \sum_i S_i^2$ Lagrange multiplier λ

exponents **non**-mean-field for $2 < d < 4$ and $T_c > 0$ for $d > 2$

kinetics from Langevin equation

$$\partial_t \phi = -D \frac{\delta \mathcal{H}[\phi]}{\delta \phi} + \mathfrak{z}(t)\phi + \eta$$

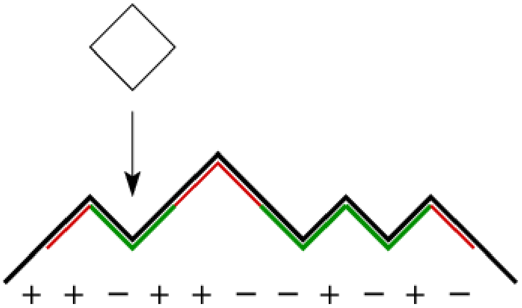
time-dependent Lagrange multiplier $\mathfrak{z}(t)$ fixed from spherical constraint

all equilibrium and ageing exponents exactly known, for $T < T_c$ and $T = T_c$

Ronca 78, Coniglio & Zannetti 89, Cugliandolo, Kurchan, Parisi 94, Godrèche & Luck '00,
Corberi, Lippiello, Fusco, Gonnella & Zannetti 02-14 ...

consider **RSOS**/**ASEP**-adsorption process :

rigorous : continuum limit gives KPZ — BERTINI & GIACOMIN 97



use **not** the heights $h_n(t) \in \mathbb{N}$ on a discrete lattice,

but rather the **slopes** $u_n(t) = \frac{1}{2}\left(h_{n+1}(t) - h_{n-1}(t)\right) = \pm 1$ **RSOS**

? let $u_n(t) \in \mathbb{R}$, & impose a spherical constraint $\boxed{\sum_n \langle u_n(t)^2 \rangle \stackrel{!}{=} \mathcal{N}}$?

? consequences of the 'hardening' of a soft EW-interface by a 'spherical constraint' on the u_n ?

Arcetri model : precise formulation & simple ageing

slope $u(t,x) = \partial_x h(t,x)$ obeys Burgers' equation, MH & DURANG 15

replace its non-linearity by a mean spherical condition $\Longrightarrow$

$$\begin{aligned}\partial_t u_n(t) &= \nu\left(u_{n+1}(t) + u_{n-1}(t) - 2u_n(t)\right) + \mathfrak{z}(t) u_n(t) \\ &\quad + \frac{1}{2}\left(\eta_{n+1}(t) - \eta_{n-1}(t)\right) \\ \sum_n \left\langle u_n(t)^2 \right\rangle &= N\end{aligned}$$

$\langle \eta_n(t)\eta_m(s)\rangle = 2T\nu\delta(t-s)\delta_{n,m}$

Extension to $d \geq 1$ dimensions : $\mathfrak{z}(t)$ Lagrange multiplier

define gradient fields $u_a(t,\mathbf{r}) := \nabla_a h(t,\mathbf{r})$, $a = 1,\ldots,d$:

$$\begin{aligned}\partial_t u_a(t,\mathbf{r}) &= \nu \nabla_{\mathbf{r}} \cdot \nabla_{\mathbf{r}} u_a(t,\mathbf{r}) + \mathfrak{z}(t) u_a(t,\mathbf{r}) + \nabla_a \eta(t,\mathbf{r}) \\ \sum_{\mathbf{r}} \sum_{a=1}^{d} \left\langle u_a(t,\mathbf{r})^2 \right\rangle &= dN^d\end{aligned}$$

interface height : $\widehat{u}_a(t,\mathbf{q}) = \mathrm{i} \sin q_a \, \widehat{h}(t,\mathbf{q})$; $\mathbf{q} \neq \mathbf{0}$ in Fourier space exact solution :

$\omega(\mathbf{q}) = \sum_{a=1}^{d}(1 - \cos q_a)$, $\mathbf{q} \neq \mathbf{0}$

$$\widehat{h}(t,\mathbf{q}) = \widehat{h}(0,\mathbf{q})e^{-2t\omega(\mathbf{q})}\sqrt{\frac{1}{g(t)}} + \int_0^t \mathrm{d}\tau\; \widehat{\eta}(\tau,\mathbf{q})\sqrt{\frac{g(\tau)}{g(t)}}\, e^{-2(t-\tau)\omega(\mathbf{q})}$$

in terms of the auxiliary function $g(t) = \exp\left(-2\int_0^t \mathrm{d}\tau\, \mathfrak{z}(\tau)\right)$,
which satisfies Volterra equation

$$g(t) = f(t) + 2T\int_0^t \mathrm{d}\tau\, g(\tau) f(t-\tau) \;\;,\;\; f(t) := d\,\frac{e^{-4t} I_1(4t)}{4t}\left(e^{-4t} I_0(4t)\right)^{d-1}$$

* for $d=1$, identical to 'spherical spin glass', with $T = 2T_{\rm SG}$:
hamiltonian $\mathcal{H} = -\frac{1}{2}\sum_{i,j} J_{ij} S_i S_j$; J_{ij} random matrix, its eigenvalues distributed according to Wigner's semi-circle law — Cugliandolo & Dean 95

* also related to distribution of first gap of random matrices — Perret & Schehr 15/16

* for $2 < d < 4$, scaling functions identical to the ones of the critical **b**osonic **p**air-**c**ontact **p**rocess with **d**iffusion, with rates
$\Gamma[2A \to (2+k)A] = \Gamma[2A \to (2-k)A] = \mu$ $\qquad k = 1, 2$

Howard & Täuber 97 ; Houchmandzadeh 02 ; Paessens & Schütz 04 ; Baumann, mh, Pleimling, Richert 05

phase transition : long-range correlated surface growth for $T \leq T_c$

$$\frac{1}{T_c(d)} = \frac{1}{2}\int_0^\infty \mathrm{d}t\; e^{-dt} t^{-1} I_1(t) I_0(t)^{d-1} \quad ; \quad T_c(1) = 2,\; T_c(2) = \frac{2\pi}{\pi - 2}$$

Some results : always simple ageing   upper critical dimension $d^* = 2$

1. $T = T_c$, $d < 2$:

rough interface, width $w(t) = t^{(2-d)/4} \Longrightarrow \beta = \frac{2-d}{4} > 0$

ageing exponents $a = b = \frac{d}{2} - 1$, $\lambda_R = \lambda_C = \frac{3d}{2} - 1$; $z = 2$

exponents z, β, a, b same as EW, but exponent $\lambda_C = \lambda_R$ different

2. $T = T_c$, $d > 2$:

smooth interface, width $w(t) = \text{cste.} \Longrightarrow \beta = 0$

ageing exponents $a = b = \frac{d}{2} - 1$, $\lambda_R = \lambda_C = d$; $z = 2$

same asymptotic exponents as EW, but scaling functions are distinct

3. $T < T_c$:

rough interface, width $w^2(t) = (1 - T/T_c)t \Longrightarrow \beta = \frac{1}{2}$

ageing exponents $a = \frac{d}{2} - 1$, $b = -1$, $\lambda_R = \lambda_C = \frac{d-2}{2}$; $z = 2$

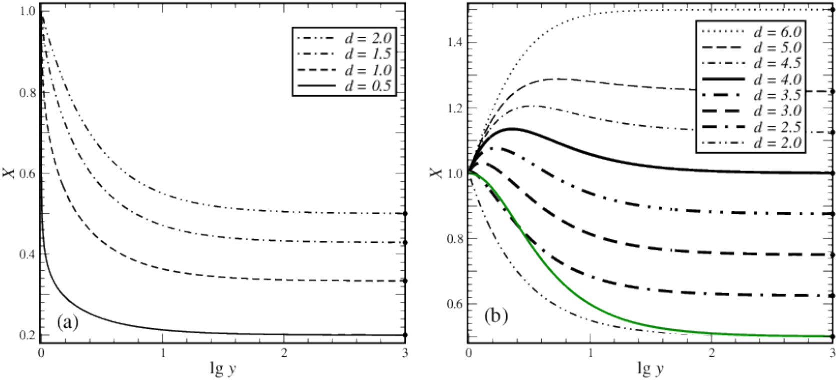
Illustration : Shape of the height **F**luctuation-**D**issipation **R**atio, $T = T_c$

Cugliandolo, Kurchan, Parisi 94

$$X(t,s) := TR(t,s) \Bigg/ \frac{\partial C(t,s)}{\partial s} = X\left(\frac{t}{s}\right) \stackrel{t/s\to\infty}{\longrightarrow} X_\infty = \begin{cases} d/(d+2) & ; \ 0 < d < 2 \\ d/4 & ; \ 2 < d \end{cases}$$

limit FDR X_∞ is **universal**

Godrèche & Luck 00



distinct from $X_{\mathrm{EW},\infty} = 1/2$ for all $d > 0$

green line : X_{EW} for $d = 4$

Summary of results in the (first) Arcetri model :

Captures at least some qualitative properites of growing interfaces.

* phenomenology of relaxation analogous to domain growth in simple magnets $\Longrightarrow$ **dynamical scaling form of simple ageing**
* existence of a critical point $T_c(d) > 0$ for all $d > 0$ **as a magnet**
* at $T = T_c$, rough interface for $d < 2$, smooth interface for $d > 2$; upper critical dimension $d^* = 2$
* at $T = T_c$, $d < 2$, the stationary exponents (β, z) are those of EW, but the non-stationary ageing exponents are different
  explicit example for expectation from field-theory renormalisation group in domain growth of independent exponents $\lambda_{C,R}$
  different from EW and KPZ classes, where $\lambda_C = d$ for all $d < 2$ KRECH 97
* at $T = T_c$, $d > 2$, **distinct from** EW, although all exponents agree
* for $d = 1$, equivalent to $p = 2$ spherical spin glass
* at $T = T_c$ and $2 < d < 4$, same ageing behaviour as at the multicritical point of the bosonic pair-contact process with diffusion (BPCPD)
* for $T < T_c$, **distinct universality class**

5. Linear responses and extensions of dynamical scaling

extend Family-Viscek scaling to two-time responses :

analogue : TRM integrated response in magnetic systems

two-time integrated response : MH, NOH, PLEIMLING 12

* sample **A** with deposition rates $p_i = p \pm \epsilon_i$, up to time s,
* sample **B** with $p_i = p$ up to time s;

then switch to common dynamics $p_i = p$ for all times $t > s$

$$\chi(t, s; \mathbf{r}) = \int_0^s \mathrm{d}u\, R(t, u; \mathbf{r}) = \frac{1}{L} \sum_{j=1}^{L} \left\langle \frac{h_{j+r}^{(\mathbf{A})}(t; s) - h_{j+r}^{(\mathbf{B})}(t)}{\epsilon_j} \right\rangle = s^{-a} F_\chi\left(\frac{t}{s}, \frac{|\mathbf{r}|^z}{s}\right)$$

with a : ageing exponent

expect for $y = t/s \gg 1$: $F_R(y, \mathbf{0}) \sim y^{-\lambda_R/z}$ autoresponse exponent

? Values of these exponents **?**

Effective action of the KPZ equation :

$$\mathcal{J}[\phi,\widetilde{\phi}] = \int \mathrm{d}t\mathrm{d}\mathbf{r}\ \left[\widetilde{\phi}\left(\partial_t\phi - \nu\nabla^2\phi - \frac{\mu}{2}(\nabla\phi)^2\right) - \nu T\widetilde{\phi}^2\right]$$

$\Longrightarrow$ Very special properties of KPZ in $d = 1$ spatial dimension !

Exact critical exponents $\beta = 1/3$, $\alpha = 1/2$, $z = 3/2$, $\lambda_C = 1$ — KPZ 86 ; Krech 97

related to precise symmetry properties :

A) tilt-invariance (Galilei-invariance) — Forster, Nelson, Stephen 77

kept under renormalisation **!** — Medina, Hwa, Kardar, Zhang 89

$\Rightarrow$ exponent relation $\alpha + z = 2$ (holds for any dimension d)

B) time-reversal invariance — Lvov, Lebedev, Paton, Procaccia 93; Frey, Täuber, Hwa 96

special property in $1D$, where also $\alpha = \frac{1}{2}$

Special KPZ symmetry in $1D$: let $v = \frac{\partial \phi}{\partial r}$, $\widetilde{\phi} = \frac{\partial}{\partial r}\left(\widetilde{p} + \frac{v}{2T}\right)$

$$\mathcal{J} = \int \mathrm{d}t\mathrm{d}r \left[\widetilde{p}\partial_t v - \frac{\nu}{4T}\left(\partial_r v\right)^2 - \frac{\mu}{2}v^2\partial_r\widetilde{p} + \nu T\left(\partial_r\widetilde{p}\right)^2\right]$$

is invariant under **time-reversal**

$$t \mapsto -t \ , \ v(t,r) \mapsto -v(-t,r) \ , \ \widetilde{p} \mapsto +\widetilde{p}(-t,r)$$

$\Rightarrow$ **fluctuation-dissipation relation** for $t \gg s$

$$\boxed{TR(t,s;r) = -\partial_r^2 C(t,s;r)}$$

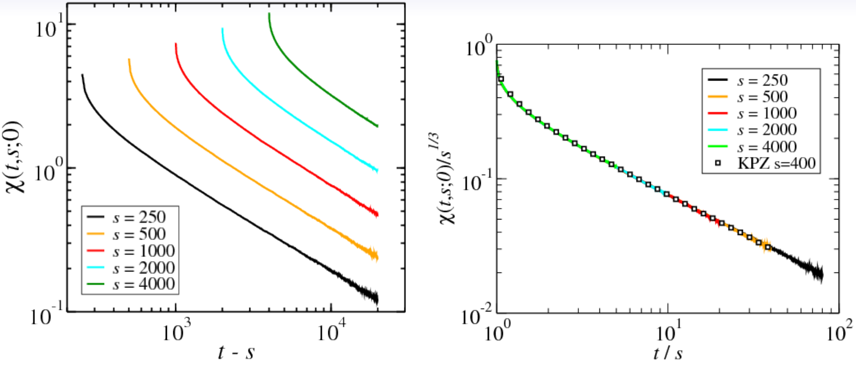
distinct from the **equilibrium** FDT $TR(t-s) = \partial_s C(t-s)$ KUBO

Combination with ageing scaling, gives the ageing exponents :

$\boxed{\lambda_R = \lambda_C = 1}$ and $\boxed{1 + a = b + \frac{2}{z}}$

KALLABIS, KRUG 96 MH, NOH, PLEIMLING 12

relaxation of the integrated response, $1D$

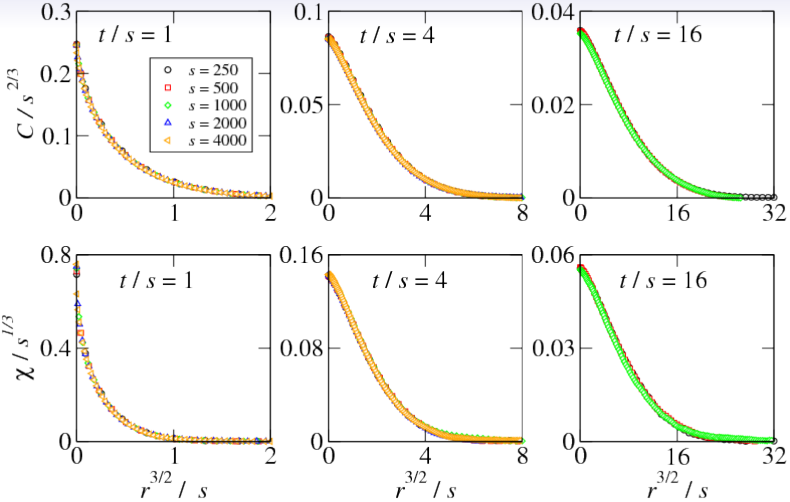


observe all **3** properties of **ageing** : $\left\{ \begin{array}{l} \text{slow dynamics} \\ \text{no TTI} \\ \text{dynamical scaling} \end{array} \right.$

exponents $a = -1/3$, $\lambda_R/z = 2/3$, as expected from FDR

N.B. : numerical tests for 2 models in KPZ class

Simple ageing is also seen in space-time observables



correlator $C(t,s;r) = s^{2/3} F_C\left(\frac{t}{s}, \frac{r^{3/2}}{s}\right)$

integrated response $\chi(t,s;r) = s^{1/3} F_\chi\left(\frac{t}{s}, \frac{r^{3/2}}{s}\right)$

$\Bigg\}$ confirm $z = 3/2$

6. Form of the scaling functions & LSI

Question : ? Are there model-independent results on the **form** of universal scaling functions ?

'Natural' starting point : try to draw analogies with conformal invariance at equilibrium

$\Longrightarrow$ 'normally' works for sufficiently 'local' theories

What about **time**-dependent critical phenomena ?

Cardy 85, mh 93

Theorem : *Consideration of the* **'deterministic part'** *of the Janssen-de Dominicis action permits to reconstruct the full time-dependent responses and correlators, from the dynamical symmetries of the 'deterministic part'.*

Picone & mh 04

essential tool : Bargman superselection rule of 'deterministic part'

Time-dependent critical phenomena & ageing

Characterised by **dynamical exponent** z : $t \mapsto tb^{-z}$, $\mathbf{r} \mapsto \mathbf{r}b^{-1}$

? Can one extend to **local** dynamical scaling, with $z \neq 1$ **?**

For $z = 2$, example of the **Schrödinger group** : JACOBI 1842, LIE 1881

$$t \mapsto \frac{\alpha t + \beta}{\gamma t + \delta} \;,\; \mathbf{r} \mapsto \frac{\mathcal{D}\mathbf{r} + \mathbf{v}t + \mathbf{a}}{\gamma t + \delta} \;;\; \alpha\delta - \beta\gamma = 1$$

$\Rightarrow$ study **ageing** phenomena as paradigmatic example

essential : (i) **absence** of TTI **&** (ii) **Galilei**-invariance

Transformation $t \mapsto t'$ with $\beta(0) = 0$ and $\dot{\beta}(t') \geq 0$ and

$$t = \beta(t') \;,\; \phi(t) = \left(\frac{\mathrm{d}\beta(t')}{\mathrm{d}t'}\right)^{-x/z} \left(\frac{\mathrm{d}\ln\beta(t')}{\mathrm{d}t'}\right)^{-2\xi/z} \phi'(t')$$

out of equilibrium, have **2 distinct** scaling dimensions, x and ξ.

mean-field for **magnets** : expect $\begin{cases} \xi = 0 \text{ in ordered phase } T < T_c \\ \xi \neq 0 \text{ at criticality } T = T_c \end{cases}$

NB : if TTI (equilibrium criticality), then $\xi = 0$.

Dynamical symmetry I : Schrödinger algebra $\mathfrak{sch}(d)$

dynamical symmetries of Langevin equation (deterministic part **!**)

Schrödinger operator in d space dimensions : $\mathcal{S} = 2\mathcal{M}\partial_t - \partial_{\mathbf{r}} \cdot \partial_{\mathbf{r}}$

$$\left.\begin{array}{l}\text{(free) Schrödinger/heat equation} \\ \text{(noiseless) Edwards-Wilkinson equation}\end{array}\right\} : \qquad \boxed{\mathcal{S}\phi = 0}$$

$$\begin{aligned}
[\mathcal{S}, \mathbf{Y}_{\pm 1/2}] &= [\mathcal{S}, M_0] = [\mathcal{S}, X_{-1}] = [\mathcal{S}, \mathcal{R}] = 0 \\
[\mathcal{S}, X_0] &= -\mathcal{S} \\
[\mathcal{S}, X_1] &= -2t\mathcal{S} + 2\mathcal{M}\left(x - \frac{d}{2}\right)
\end{aligned}$$

infinitesimal change : $\delta\phi = \varepsilon\mathcal{X}\phi$, $\qquad \mathcal{X} \in \mathfrak{sch}(d), |\varepsilon| \ll 1$

Lemma : If $\mathcal{S}\phi = 0$ and $x = x_\phi = \frac{d}{2}$, then $\mathcal{S}(\mathcal{X}\phi) = 0$. LIE 1881, NIEDERER '72

$\mathfrak{sch}(d)$ maps solutions of $\mathcal{S}\phi = 0$ onto solutions.

Dynamical symmetry II : ageing algebra $\mathfrak{age}(d)$

$1D$ Schrödinger operator : $\mathcal{S} = 2\mathcal{M}\partial_t - \partial_r^2 + 2\mathcal{M}\left(x + \xi - \frac{1}{2}\right)t^{-1}$

generalised 'Schrödinger equation' : $\boxed{\mathcal{S}\phi = 0}$

extra potential term arises in several models, **without** time-translations

(e.g. $1D$ Glauber-Ising, spherical & Arcetri models)

if time-translations ($X_{-1} = -\partial_t$) are included, then $\xi = 0$

$$
\begin{aligned}
[\mathcal{S}, Y_{\pm 1/2}] &= [\mathcal{S}, M_0] = 0 \\
[\mathcal{S}, X_0] &= -\mathcal{S} \\
[\mathcal{S}, X_1] &= -2t\mathcal{S}
\end{aligned}
$$

infinitesimal change : $\delta\phi = \varepsilon\mathcal{X}\phi$, $\mathcal{X} \in \mathfrak{age}(d), |\varepsilon| \ll 1$

Lemma : If $\mathcal{S}\phi = 0$, then $\mathcal{S}(\mathcal{X}\phi) = 0$. NIEDERER '74 ; MH & STOIMENOV '11

$\boxed{\mathfrak{age}(d) \text{ maps solutions of } \mathcal{S}\phi = 0 \text{ onto solutions}}$.

Example for the t^{-1}-term in Langevin eq. : Arcetri model

continuous slopes $u_{\mathbf{i}} \in \mathbb{R}^d$, constraint $\sum_{\mathbf{i}\in\Lambda} \mathbf{u}_{\mathbf{i}}^2 = d\mathcal{N}$

for $d > 0$ phase transition $T_c(d) > 0$, exponents not mean-field if $d < 2$

spherical constraint : $\left\langle \sum_{\mathbf{i}\in\Lambda} u_{\mathbf{i}}^2 \right\rangle = d\mathcal{N}$

Langevin equation, with Lagrange multiplier $\mathfrak{z}(t)$ & centered gaussian noise $\eta_{\mathbf{i}}(t)$

$$\frac{\partial u_a(t,\mathbf{r})}{\partial t} = \nu\Delta u_a(t,\mathbf{r}) + \mathfrak{z}(t)u_a(t,\mathbf{r}) + \partial_a\eta(t,\mathbf{r}) \;\;,\;\; \langle\eta(t,\mathbf{r})\eta(s,\mathbf{r}')\rangle = 2\nu T\delta(t-s)\delta(\mathbf{r}-\mathbf{r}')$$

set $g(t) := \exp\left(2\int_0^t \mathrm{d}t'\,\mathfrak{z}(t')\right)$, spherical constraint gives Volterra eq.

$$g(t) = f(t) + 2T\int_0^t \mathrm{d}\tau\, f(t-\tau)g(\tau) \;\;,\;\; f(t) = \frac{de^{-4t}I_1(4t)}{4t}\left(e^{-4t}I_0(4t)\right)^{d-1}$$

find for $T \leq T_c$: $g(t) \overset{t\to\infty}{\sim} t^{-F} \Leftrightarrow \boxed{\mathfrak{z}(t) \sim \frac{F}{2}t^{-1}}$

quite analogous to spherical model of a ferromagnet

Godrèche & Luck 00
Picone & mh 04

Schrödinger- & ageing-covariant two-point functions

two-point function $R = R(t,s;\mathbf{r}_1-\mathbf{r}_2) := \langle \phi_1(t,\mathbf{r}_1)\widetilde{\phi}_2(s,\mathbf{r}_2)\rangle$

Each ϕ_i characterized by (i) scaling dimensions x_i, ξ_i (ii) mass $\mathcal{M}_i$

* from Schrödinger-invariance

$$R(t,s,\mathbf{r}) = r_0\delta_{x_1,x_2}\, s^{-1-a}\left(\frac{t}{s}-1\right)^{-1-a}\exp\left[-\frac{\mathcal{M}_1}{2}\frac{\mathbf{r}^2}{t-s}\right]$$

* from ageing-invariance

$$R(t,s;\mathbf{r}) = r_0 s^{-1-a}\left(\frac{t}{s}\right)^{1+a'-\lambda_R/2}\left(\frac{t}{s}-1\right)^{-1-a'}\exp\left(-\frac{\mathcal{M}_1}{2}\frac{\mathbf{r}^2}{t-s}\right)$$

with $1+a=\frac{x_1+x_2}{2}$, $a'-a=\xi_1+\xi_2$, $\lambda_R=2(x_1+\xi_1)$, $\underbrace{\mathcal{M}_1+\mathcal{M}_2=0}_{\text{Bargman rule}}$

can derive **causality condition** $t>s$ MH & Unterberger 03, MH 14

$\Rightarrow$ R is physically a **response function**.

$1D$ **KPZ** : find $R(t,s) = \left\langle \psi(t)\widetilde{\psi}(s) \right\rangle$ from 'logarithmic partner' of order parameter (ψ, ϕ) MH 13

scaling dimensions become Jordan matrices $\begin{pmatrix} x & x' \\ 0 & x \end{pmatrix}$, $\begin{pmatrix} \xi & \xi' \\ 0 & \xi \end{pmatrix}$ and similarly for response fields

* good collapse $\Rightarrow$ **no** logarithmic corrections $\Rightarrow$ $\boxed{x' = \widetilde{x}' = 0}$

* **no** logarithmic factors for $y \gg 1$ $\Rightarrow$ $\boxed{\xi' = 0}$

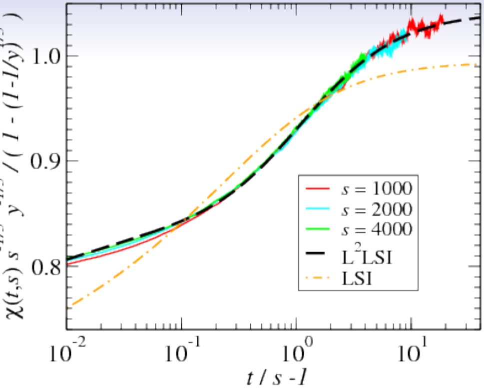
$\Rightarrow$ only $\widetilde{\xi}' = 1$ remains

$$f_R(y) = y^{-\lambda_R/z} \left(1 - \frac{1}{y}\right)^{-1-a'} \left[h_0 - g_0 \ln\left(1 - \frac{1}{y}\right) - \frac{1}{2} f_0 \ln^2\left(1 - \frac{1}{y}\right) \right]$$

find integrated autoresponse $\chi(t,s) = \int_0^s \mathrm{d}u \, R(t,u) = s^{1/3} f_\chi(t/s)$

$$f_\chi(y) = y^{1/3} \left\{ A_0 \left[1 - \left(1 - \frac{1}{y}\right)^{-a'} \right] + \left(1 - \frac{1}{y}\right)^{-a'} \left[A_1 \ln\left(1 - \frac{1}{y}\right) + A_2 \ln^2\left(1 - \frac{1}{y}\right) \right] \right\}$$

with free parameters A_0, A_1, A_2 and a' — for the $1D$ KPZ class, use $\frac{\lambda_R}{z} - a = 1$



MH, NOH, PLEIMLING 12

non-log LSI with $a = a'$:
deviations $\approx 20\%$

non-log LSI with $a \neq a'$:
works up to $\approx 5\%$

log LSI : works **better than** $\approx 0.1\%$

R	a'	A_0	A_1	A_2
$\langle\phi\widetilde{\phi}\rangle$ – LSI	-0.500	0.662	0	0
$\langle\phi\widetilde{\psi}\rangle$ – L^{1}LSI	-0.500	0.663	$-6\cdot 10^{-4}$	0
$\langle\psi\widetilde{\psi}\rangle$ – L^{2}LSI	-0.8206	0.7187	0.2424	-0.09087

logarithmic LSI fits data at least down to $y \simeq 1.01$, with $a' - a \approx -0.4873$ (can we make a conjecture ?)

7. Conclusions

* long-time dynamics of growing interfaces naturally evolves towards dynamical scaling & ageing
* phenomenology very similar to ageing phenomena in simple magnets
* subtleties in the precise scaling forms & space-dependent profiles
* shape of two-time response functions compatible with extended forms of dynamical scaling, according to LSI
* in certain cases logarithmic contributions in the scaling functions (but **without** logarithmic corrections to scaling) :

$\Longrightarrow$ implications for interpretation of numerical data for the $2D$ KPZ, where $\lambda_{C,\mathrm{eff}} \neq \lambda_{R,\mathrm{eff}} \neq 2$ **?** HALPIN-HEALY *et al.* 14, ÓDOR *et al.* 14

proving dynamical symmetries can remain a delicate affair !

Arcetri model, exact solution :

$\omega(\mathbf{q}) = \sum_{a=1}^{d}(1-\cos q_a)$, $\mathbf{q} \neq \mathbf{0}$

$$\widehat{h}(t,\mathbf{q}) = \widehat{h}(0,\mathbf{q})e^{-2t\omega(\mathbf{q})}\sqrt{\frac{1}{g(t)}} + \int_0^t \mathrm{d}\tau\; \widehat{\eta}(\tau,\mathbf{q})\sqrt{\frac{g(\tau)}{g(t)}}\, e^{-2(t-\tau)\omega(\mathbf{q})}$$

in terms of the auxiliary function $g(t) = \exp\left(-2\int_0^t \mathrm{d}\tau\, \mathfrak{z}(\tau)\right)$,
which satisfies Volterra equation

$$g(t) = f(t) + 2T\int_0^t \mathrm{d}\tau\, g(\tau)f(t-\tau) \;,\quad f(t) := d\,\frac{e^{-4t}I_1(4t)}{4t}\left(e^{-4t}I_0(4t)\right)^{d-1}$$

* for $d=1$, identical to 'spherical spin glass', with $T = 2T_{\rm SG}$:
  hamiltonian $\mathcal{H} = -\frac{1}{2}\sum_{i,j} J_{ij}S_iS_j$; J_{ij} random matrix, its eigenvalues distributed according to Wigner's semi-circle law   Cugliandolo & Dean 95
* also related to distribution of first gap of random matrices Perret & Schehr 15/16

a further auxiliary function : $F_{\mathbf{r}}(t) := \prod_{a=1}^{d} e^{-2t} I_{r_a}(2t)$ I_n : modified Bessel function
for initially uncorrelated heights and initially flat interface

height autocorrelator :

$$C(t,s) = \langle h(t,\mathbf{r})h(s,\mathbf{r})\rangle_c = \frac{2F_{\mathbf{0}}(t+s)}{\sqrt{g(t)g(s)}} + \frac{2T}{\sqrt{g(t)g(s)}}\int_0^s \mathrm{d}\tau\, g(\tau)F_{\mathbf{0}}(t+s-2\tau)$$

interface width : $w^2(t) = C(t,t) = \frac{2F_{\mathbf{0}}(2t)}{g(t)} + \frac{2T}{g(t)}\int_0^t \mathrm{d}\tau\, g(\tau)F_{\mathbf{0}}(2t-2\tau)$

slope autocorrelator :

$$A(t,s) = \sum_{a=1}^{d} \langle u_a(t,\mathbf{r})u_a(s,\mathbf{r})\rangle_c = \frac{2f((t+s)/2)}{\sqrt{g(t)g(s)}} + \int_0^s \mathrm{d}\tau\, \frac{2Tg(\tau)}{\sqrt{g(t)g(s)}} f((t+s)/2-\tau)$$

height response : $R(t,s;\mathbf{r}) = \left.\frac{\delta\langle h(t,\mathbf{r})\rangle}{\delta j(s,\mathbf{0})}\right|_{j=0} = \Theta(t-s)\sqrt{\frac{g(s)}{g(t)}}\; F_{\mathbf{r}}(t-s)$

slope autoresponse : $Q(t,s;\mathbf{0}) = \Theta(t-s)\sqrt{\frac{g(s)}{g(t)}}\; f((t-s)/2)$

* **correspondence of $1D$ $\mathrm{A}I$ model with spherical spin glass** : spins $S_i \leftrightarrow$ slopes u_n

spin glass autocorrelator $C_{\mathrm{SG}}(t,s) = \frac{1}{\mathcal{N}}\sum_{i=1}^{\mathcal{N}} \overline{\langle S_i(t)S_i(s)\rangle} = A(t,s)$

spin glass response $R_{\mathrm{SG}}(t,s) = \sum_{i=1}^{\mathcal{N}} \left.\frac{\delta\overline{\langle S_i(t)\rangle}}{\delta h_i(s)}\right|_{h=0} = 2Q(t,s)$

* kinetics of heights $h_n(t)$ in model $\mathrm{A}I$ driven by phase-ordering of the spherical spin glass $\equiv 3D$ kinetic spherical model

Relationship with the critical diffusive bosonic pair-contact process (BPCPD)

Howard & Täuber 97 ; Houchmandzadeh 02 ; Paessens & Schütz 04 ; Baumann, mh, Pleimling, Richert 05

* each site of a hypercubic lattice is occupied by $n_i \in \mathbb{N}_0$ particles
* single particles hop to a nearest-neighbour site with diffusion rate D
* on-site reactions, with rates $\Gamma[2A \to (2+k)A] = \Gamma[2A \to (2-k)A] = \mu$

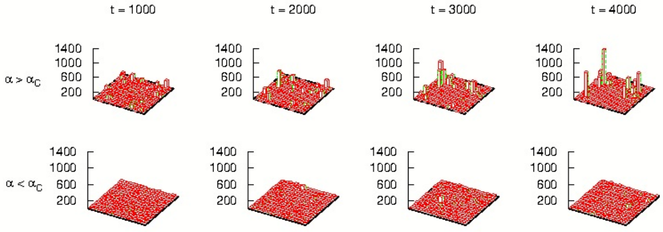
  k is either 1 or 2

* control parameter $\alpha := k^2\mu/D$

$\Longrightarrow$ for $d > 2$, particles cluster on a few sites only, if $\alpha > \alpha_C$ BHPR 05

Figure : $2D$ section of BPCPD in $d = 3$; height of columns $\sim$ particle number Baumann 07

$\Longrightarrow$ fluctuations grow with t when $\alpha > \alpha_C$ & are bounded for $\alpha < \alpha_C$



bosonic creation operator $a^\dagger(t,\mathbf{r})$, commutator $[a(t,\mathbf{r}),a^\dagger(t,\mathbf{r}')]=\delta(\mathbf{r}-\mathbf{r}')$

$\Longrightarrow$ average particle number is constant !

$$n(t,\mathbf{r})=\langle a^\dagger(t,\mathbf{r})a(t,\mathbf{r})\rangle=\langle a(t,\mathbf{r})\rangle=\rho_0=\mathrm{cste.}$$

clustering transition at $\alpha=\alpha_C$, caracterised by changes in the variance.

$$\bar{C}(t,s) := \left\langle a^\dagger(t,\mathbf{r})a(s,\mathbf{r})\right\rangle-\rho_0^2 \stackrel{t,s\to\infty}{\simeq} \langle n(t,\mathbf{r})n(s,\mathbf{r})\rangle-\rho_0^2=s^{-b}f_C(t/s)$$

$$\bar{R}(t,s) := \left.\frac{\delta\langle a(t,\mathbf{r})\rangle}{\delta j(s,\mathbf{r})}\right|_{j=0}=s^{1-a}f_R(t/s)$$

obey simple ageing for $\alpha\leq\alpha_C$. Precisely **at** the clustering transition $\alpha=\alpha_C$, for $2<d<4$, the scaling functions are **identical** :

$\mathrm{BPCPD}: \; b+1=a=d/2-1$ $\quad$ $\mathrm{Arcetri}: \; b=a=d/2-1$

$$f_{R,\mathrm{BPCPD}}(y) = (y-1)^{d-2} = f_{R,\mathrm{Arc}}(y)$$

$$f_{C,\mathrm{BPCPD}}(y) = (y+1)^{-d/2}{}_2F_1\left(\frac{d}{2},\frac{d}{2};\frac{d}{2}+1;\frac{2}{1+y}\right) = f_{C,\mathrm{Arc}}(y)$$

N.B. : for $d>4$, Arcetri $\neq$ BPCPD $\neq$ EW, although all exponents, up to b, agree.