Three Dimensional Nonlinear Sigma Models in the Wilsonian Renormalization Method

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1.Wilsonian Renormalization Group

Exact renormalization group equations

- Wilson renormalization group equation
- Wegner-Houghton equation
- Polchinski equation

are constructed by the loop correction term and rescaling part.

The WRG equation (Wegner-Houghton equation) describes the variation of effective action when energy scale Λ is changed to $\Lambda(\delta t)=\Lambda \exp[-\delta t]$.

$$
\begin{aligned}
\frac{d}{dt}S[\Omega;t] \;=\; & \frac{1}{2\delta t}\int_{p'} tr \ln\left(\frac{\delta^2 S}{\delta\Omega^i\delta\Omega^j}\right) \\
& -\frac{1}{2\delta t}\int_{p'}\int_{q'}\frac{\delta S}{\delta\Omega^i(p')}\left(\frac{\delta^2 S}{\delta\Omega^i(p')\delta\Omega^j(q')}\right)^{-1}\frac{\delta S}{\delta\Omega^j(q')} \\
& +\left[D-\sum_{\Omega_i}\int_p \hat{\Omega}_i(p)\left(d_{\Omega_i}+\gamma_{\Omega_i}+\hat{p}^\mu\frac{\partial}{\partial\hat{p}^\mu}\right)\frac{\delta}{\delta\hat{\Omega}_i(p)}\right]\hat{S}
\end{aligned}
$$

The effective action:

The Euclidean path integral is

$$Z = \int [D\Omega_i] exp[-S[\Omega]]$$

We divide all fields Ω into two groups, high frequency modes and low frequency modes. After the higher modes are integrated out, the Wilsonian effective action is obtained as

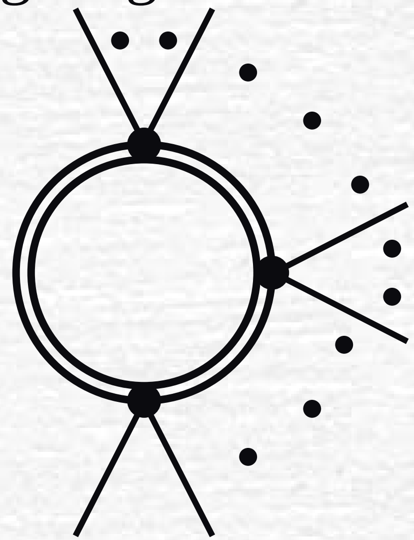
$$\begin{aligned} Z &= \int [D\Omega_i] \exp\left[-S[\Omega_i]\right] \\ &= \int [D\Omega_{i>}][D\Omega_{i<}] \exp\left[-S[\Omega_{i<}, \Omega_{i>}]\right] \\ &\equiv \int [D\Omega_{i<}] \exp\left[-S_{eff}[\Omega_{i<}]\right] . \end{aligned}$$

We assume that Z is cutoff independent:

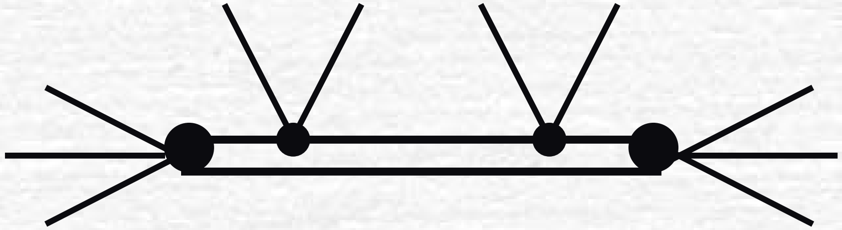
$$
\begin{aligned}
Z &= \int [D\Omega]_{\Lambda(\delta t)} [D\Omega_s] \exp\left[-S[\Omega+\Omega_s;\Lambda]\right] \\
&= \int [D\Omega]_{\Lambda(\delta t)} [D\Omega_s] \exp\left[-\left(S[\Omega;\Lambda]| + \frac{\delta S}{\delta \Omega_i}\Omega_s^i + \frac{1}{2}\Omega_s^i \frac{\delta^2 S}{\delta\Omega^i \delta\Omega^j}\Omega_s^j + O(\Omega_s^3)\right)\right] \\
&= \int [D\Omega]_{\Lambda(\delta t)} \exp\left[-\left(S[\Omega;\Lambda]| + \frac{1}{2}\int_{p'} tr \ln \left(\frac{\delta^2 S}{\delta\Omega^i \delta\Omega^j}\right)\right.\right. \\
&\qquad \left.\left. -\frac{1}{2}\int_{p'}\int_{q'} \frac{\delta S}{\delta\Omega^i}\left(\frac{\delta^2 S}{\delta\Omega^i\delta\Omega^j}\right)^{-1} \frac{\delta S}{\delta \Omega^j} + O((\delta t)^2)\right)\right] \qquad (1) \\
&\equiv \int [D\Omega]_{\Lambda(\delta t)} \exp\left[-S[\Omega;\Lambda(\delta t)]\right]. \qquad (2)
\end{aligned}
$$

This equation corresponds to the following diagrams.

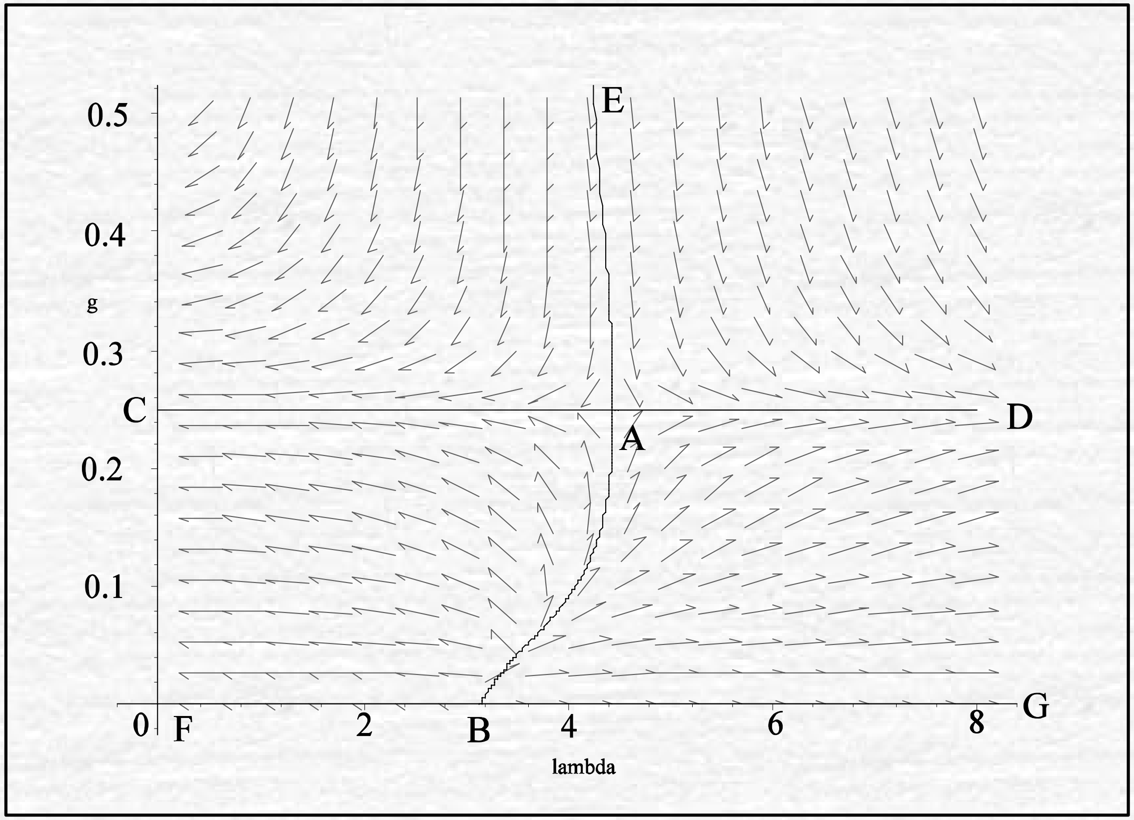
$$\int_{p'} tr \ln \left(\frac{\delta^2 S}{\delta\Omega^i \delta\Omega^j} \right)$$



$$\int_{p'} \int_{q'} \frac{\delta S}{\delta\Omega^i(p')} \left(\frac{\delta^2 S}{\delta\Omega^i(p')\delta\Omega^j(q')} \right)^{-1} \frac{\delta S}{\delta\Omega^j(q')}$$



Renormalizability and continuum limit



- critical line・・・EAB
- renormalized trajectory
- fixed point・・・A, B,

  λ=0 line

The renormalization group flow which can be extrapolated back to critical surface defines a renormalized theory.

Approximation method:

Symmetry and Derivative expansion

Consider a single real scalar field theory that is invariant under

$$\varphi \to -\varphi \ (Z_2 \text{ symmetry})$$

We expand the most generic action as

$$S[\varphi] = \int d^D x V[\varphi] + \frac{1}{2}K[\varphi](\partial_\mu \varphi)^2 + H_1[\varphi](\partial_\mu \varphi)^4 + H_2[\varphi](\partial_\mu \partial^\mu \varphi)^2 + \cdots$$

In this work, we expand the action up to second order in derivative and constraint it $\mathcal{N}$=2 supersymmetry.

2. Introduction

- We consider the Wilsonian effective action which has derivative interactions.

  In bosonic theory, such action corresponds to non-linear sigma models.

$$S = \int d^3x V[\varphi] + g_{i\bar{j}}[\varphi]\partial_\mu \varphi^i \partial^\mu \varphi^{\bar{j}} .$$

- Three dimensional nonlinear sigma model is unrenormalizable in perturbation theory, and we have to use nonperturbative methods.

Large-N expansion, WRG equation etc

D=3 $\mathcal{N}$ =2 supersymmetric non linear sigma model

$$S=\int d^3x d^2\theta d^2\bar{\theta} K[\Phi^i, \Phi^{\dagger\bar{i}}]$$

***i*=1～N：N is the dimensions of target spaces**

Where K is Kaehler potential and Φ is chiral superfield.

$$\begin{aligned}
\Phi^i(y) &= \varphi^i(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\varphi^i(x) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^\mu\partial_\mu\varphi^i(x) \\
&\quad +\sqrt{2}\theta\psi^i(x) - \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi^i(x)\sigma^\mu\bar{\theta} + \theta\theta F^i(x) \\
&\equiv \varphi^i(x) + \delta\Phi^i(x)
\end{aligned}$$

We expand the action around the scalar fields.

$$S=\int d^3x\Big[g_{n\bar{m}}\left(\partial^{\mu}\varphi^{n}\partial_{\mu}\varphi^{*\bar{m}}+i\bar{\psi}^{\bar{m}}\sigma^{\mu}(D_{\mu}\psi)^{n}+\bar{F}^{\bar{m}}F^{n}\right)$$

$$-\frac{1}{2}K_{,nm\bar{l}}\bar{F}^{\bar{l}}\psi^{n}\psi^{m}-\frac{1}{2}K_{,n\bar{m}\bar{l}}F^{n}\bar{\psi}^{\bar{m}}\bar{\psi}^{\bar{l}}+\frac{1}{4}K_{,nm\bar{k}\bar{l}}(\bar{\psi}^{\bar{k}}\bar{\psi}^{\bar{l}})(\psi^{n}\psi^{m})\Big]$$

where

$$K_{,i}\equiv\frac{\delta K}{\delta\varphi^{i}}$$

$$g_{i\bar{j}}=K_{,i\bar{j}}$$: the metric of target spaces

From equation of motion, the auxiliary filed F can be vanished.

$$F^{n}=\frac{1}{2}g^{n\bar{m}}K_{,kl\bar{m}}\psi^{k}\psi^{l}$$

3.WRG equation for non linear sigma model

$$\frac{d}{dt}S[\Omega;t] = \frac{1}{2\delta t}\int_{p'} tr \ln\left(\frac{\delta^2 S}{\delta\Omega^i \delta\Omega^j}\right) - \frac{1}{2\delta t}\int_{p'}\int_{q'} \frac{\delta S}{\delta\Omega^i(p')}\left(\frac{\delta^2 S}{\delta\Omega^i(p')\delta\Omega^j(q')}\right)^{-1}\frac{\delta S}{\delta\Omega^j(q')} + \left[D - \sum_{\Omega_i}\int_p \hat{\Omega}_i(p)\left(d_{\Omega_i} + \gamma_{\Omega_i} + \hat{p}^\mu \frac{\partial}{\partial \hat{p}^\mu}\right)\frac{\delta}{\delta\hat{\Omega}_i(p)}\right]\hat{S}$$

Consider the bosonic part of the action.

The second term of the right hand side vanishes in this approximation $O(\partial^2)$.

The first term of the right hand side

$$\frac{1}{2\delta t}\int_{p'} tr \ln\left(\frac{\delta^2 S}{\delta\Omega^i \delta\Omega^{\bar{j}}}\right)$$

From the bosonic part of the action using KNC

$$\sim \frac{1}{4\pi^2}\ln\det g_{i\bar{j}} + \frac{1}{2\pi^2}R_{i\bar{j}}(\partial_\mu\varphi)^i(\partial^\mu\varphi^*)^{\bar{j}}$$

From the fermionic kinetic term

$$\sim -\frac{1}{4\pi^2}\ln\det g_{i\bar{j}}$$

Non derivative term is cancelled.

Finally, we obtain the WRG eq. for bosonic part of the action as follow:

$$
\begin{aligned}
&\frac{d}{dt}\int d^3x g_{i\bar{j}}(\partial_\mu\varphi)^i(\partial^\mu\varphi^*)^{\bar{j}} \\
= &\int d^3x\Big[-\frac{1}{2\pi^2}R_{i\bar{j}} - \gamma[\varphi^k g_{i\bar{j},k} + \varphi^{*\bar{k}} g_{i\bar{j},\bar{k}} + 2g_{i\bar{j}}] \\
&-\frac{1}{2}[\varphi^k g_{i\bar{j},k} + \varphi^{*\bar{k}} g_{i\bar{j},\bar{k}}]\Big](\partial_\mu\varphi)^i(\partial^\mu\varphi^*)^{\bar{j}}.
\end{aligned}
$$

The β function for the Kaehler metric is

$$
\begin{aligned}
\frac{d}{dt}g_{i\bar{j}} &= -\frac{1}{2\pi^2}R_{i\bar{j}} - \gamma[\varphi^k g_{i\bar{j},k} + \varphi^{*\bar{k}} g_{i\bar{j},\bar{k}} + 2g_{i\bar{j}}] \\
&\quad -\frac{1}{2}[\varphi^k g_{i\bar{j},k} + \varphi^{*\bar{k}} g_{i\bar{j},\bar{k}}] \\
&\equiv -\beta(g_{i\bar{j}}).
\end{aligned}
$$

4.Renormalization Group Flow

In 3-dimension, the β function for Kaehler metric is written:

$$\beta = \frac{1}{2\pi^2} R_{i\bar{j}} + \gamma[\varphi^k g_{i\bar{j},k} + \varphi^{*\bar{k}} g_{i\bar{j},\bar{k}} + 2g_{i\bar{j}}] + \frac{1}{2}[\varphi^k g_{i\bar{j},k} + \varphi^{*\bar{k}} g_{i\bar{j},\bar{k}}].$$

The C*P*^{N} model :SU(N+1)/[SU(N) ×U(1)]

$$K[\Phi, \Phi^\dagger] = \frac{1}{\lambda^2} \ln(1 + \vec{\Phi}\vec{\Phi}^\dagger),$$

From this Kaehler potential, we derive the metric and Ricci tensor as follow:

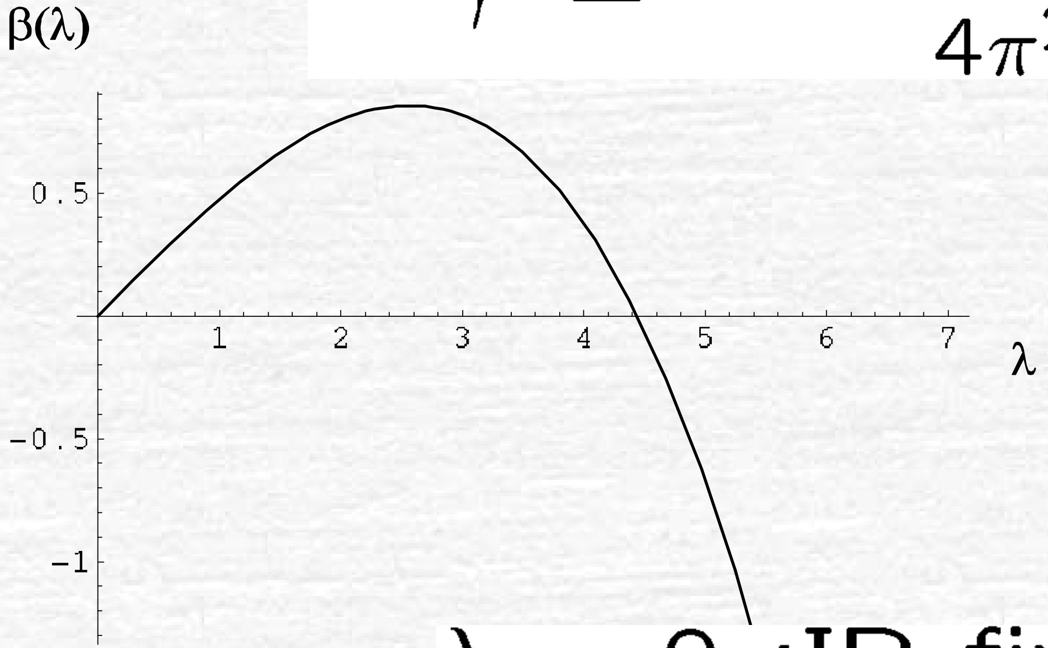
$$g_{i\bar{j}} = \frac{\delta_{i\bar{j}}}{1 + \lambda^2 \varphi\varphi^*} - \frac{\lambda^2 \varphi_i^* \varphi_{\bar{j}}}{(1 + \lambda^2 \varphi\varphi^*)}$$

$$R_{i\bar{j}} = (N+1)\lambda^2 g_{i\bar{j}}$$

The β function and anomalous dimension of scalar field are given by

$$\beta(\lambda) = -\frac{(N+1)\lambda^3}{4\pi^2} + \frac{1}{2}\lambda,$$

$$\gamma = -\frac{(N+1)\lambda^2}{4\pi^2}.$$



There are two fixed points:

$$\lambda = 0 \text{ :IR fixed point}$$

$$\lambda^2 = \frac{2\pi^2}{N+1} \text{ :UV fixed point}$$

Einstein-Kaehler manifolds

The Einstein-Kaehler manifolds satisfy the condition

$$R_{i\bar{j}} = h\lambda^2 g_{i\bar{j}}.$$

If *h* is positive, the manifold is compact.

$$\begin{aligned}
-\beta(g_{i\bar{j}}) &= \frac{\partial}{\partial t}\tilde{g}_{i\bar{j}}(\lambda\tilde{\varphi}, \lambda\tilde{\varphi}^*) \\
&= -\frac{1}{2\pi^2}\tilde{R}_{i\bar{j}} - \gamma[\tilde{\varphi}^k \tilde{g}_{i\bar{j},k} + \tilde{\varphi}^{*\bar{k}} \tilde{g}_{i\bar{j},\bar{k}} + 2\tilde{g}_{i\bar{j}}] \\
&\quad -\frac{1}{2}[\tilde{\varphi}^k \tilde{g}_{i\bar{j},k} + \tilde{\varphi}^{*\bar{k}} \tilde{g}_{i\bar{j},\bar{k}}].
\end{aligned}$$

The value of *h* for hermitian symmetric spaces.

G/H	Dimensions(complex)	***h***
SU(N+1)/[SU(N)×U(1)]	N	N+1
SU(N)/SU(N-M)×U(M)	M(N-M)	N
SO(N+2)/SO(N)×U(1)	N	N
Sp(N)/U(N)	N(N+1)/2	N+1
SO(2N)/U(N)	N(N+1)/2	N-1
E_6/[SO(10) ×U(1)]	16	12
E_7/[E_6×U(1)]	27	18

Because only λ depends on *t*, the WRG eq. can be rewritten

$$\frac{\dot{\lambda}}{\lambda}\tilde{\varphi}^k \tilde{g}_{i\bar{j},k} + \frac{\dot{\lambda}}{\lambda}\tilde{\varphi}^{*\bar{k}}\tilde{g}_{i\bar{j},\bar{k}}$$
$$= -\left(\frac{h\lambda^2}{2\pi} + 2\gamma\right)\tilde{g}_{i\bar{j}} - (\gamma + \frac{1}{2})[\tilde{\varphi}^k \tilde{g}_{i\bar{j},k} + \tilde{\varphi}^{*\bar{k}}\tilde{g}_{i\bar{j},\bar{k}}].$$

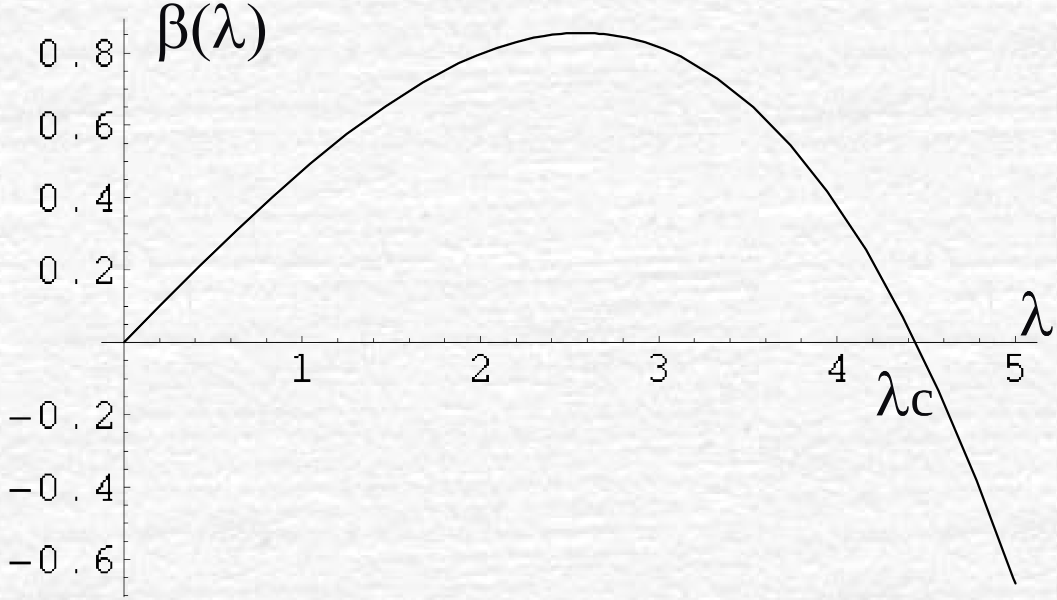
We obtain the anomalous dimension and β function of λ:

$$\gamma = -\frac{h\lambda^2}{4\pi^2}$$
$$\beta(\lambda) \equiv -\frac{d\lambda}{dt} = -\frac{h}{4\pi^2}\lambda^3 + \frac{1}{2}\lambda.$$

● The constant h is positive (compact E-K) **Renormalizable**

We have an IR fixed point at $\lambda=0$ and a UV fixed point at

$$\lambda^2 = \frac{2\pi^2}{h} \equiv \lambda_c^2$$



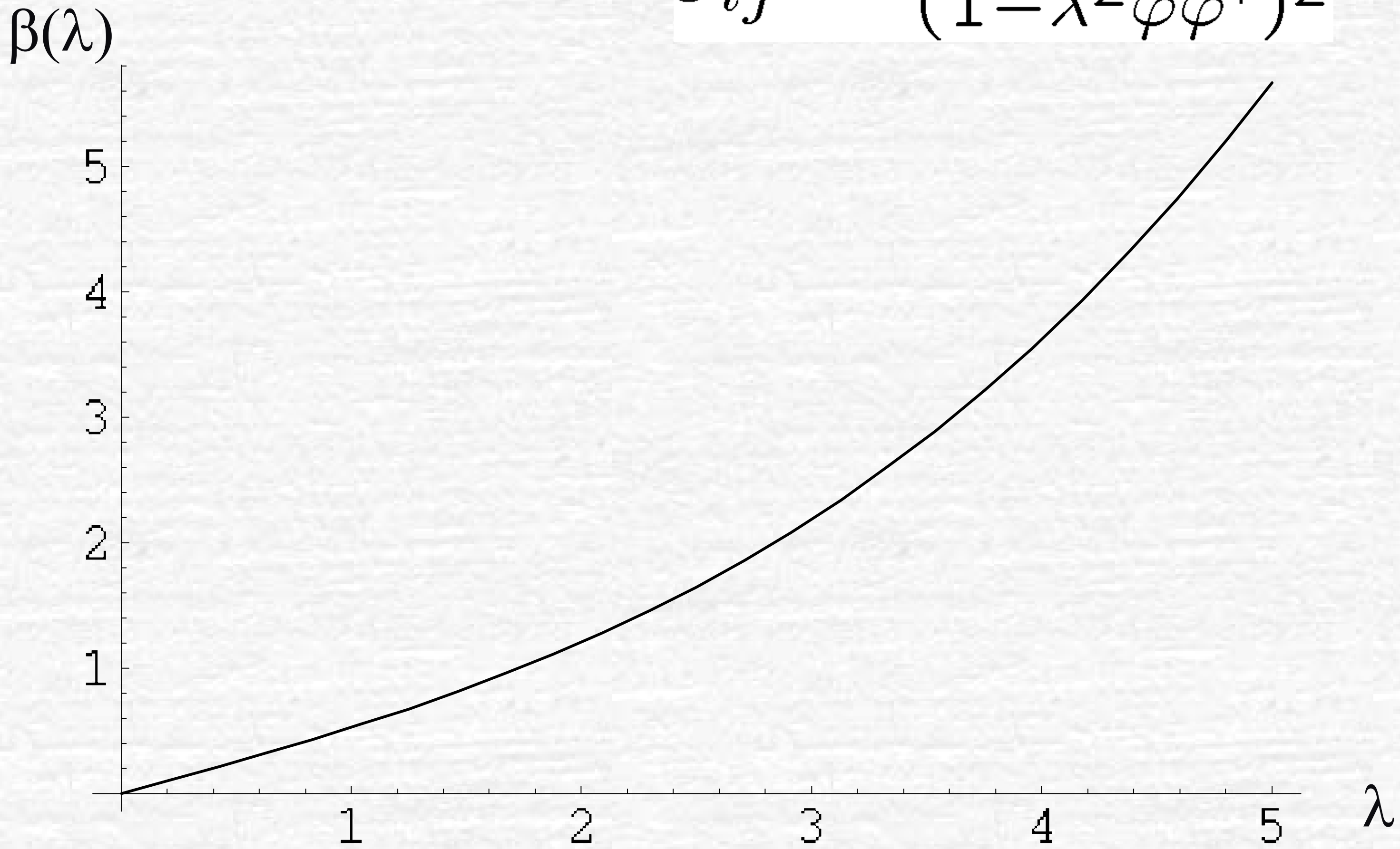
If the constant h is positive, it is possible to take the continuum limit by choosing the cutoff dependence of the bare coupling constant as

$$\lambda(\Lambda) \to \lambda_c - \frac{M}{\Lambda}.$$

M is a finite mass scale.

●The constant *h* is negative (example Disc with Poincare metric)

$$g_{i\bar{j}} = \frac{\delta_{i\bar{j}}}{(1-\lambda^2\varphi\varphi^*)^2} \quad i,j=1$$



We have only IR fixed point at λ=0.

5.SU(N) symmetric solution of WRG equatrion

We derive the action of the conformal field theory corresponding to the fixed point of the β function.

$$\begin{aligned}\beta(g_{i\bar{j}}) &= \frac{1}{2\pi^2} R_{i\bar{j}} \\ &+ \gamma[\varphi^k g_{i\bar{j},k} + \varphi^{*\bar{k}} g_{i\bar{j},\bar{k}} + 2g_{i\bar{j}}] \\ &+ \frac{1}{2}[\varphi^k g_{i\bar{j},k} + \varphi^{*\bar{k}} g_{i\bar{j},\bar{k}}] \\ &= 0.\end{aligned}$$

To simplify, we assume SU(N) symmetry for Kaehler potential.

$$K[\Phi, \Phi^\dagger] = \sum_{n=1}^{\infty} g_n x^n \equiv f(x)$$

Where,

$$x \equiv \vec{\Phi} \cdot \vec{\Phi}^\dagger$$

The function *f(x)* have infinite number of coupling constants.

$$f(x) = x + g_2 x^2 + g_3 x^3 + \cdots .$$

The Kaehler potential gives the Kaehler metric and tensor as follows:

$$
\begin{aligned}
g_{i\bar{j}} &\equiv \partial_i \partial_{\bar{j}} K[\varphi, \varphi^\dagger] = f' \delta_{i\bar{j}} + f'' \varphi_i^* \varphi_{\bar{j}}, \\
R_{i\bar{j}} &\equiv -\partial_i \partial_{\bar{j}} tr \ln g_{i\bar{j}} \\
&= -[(N-1)\frac{f''}{f'} + \frac{2f'' + f''' x}{f' + f'' x}]\delta_{i\bar{j}} \\
&\quad -[(N-1)(\frac{f^{(3)}}{f''} - \frac{(f'')^2}{(f')^2}) \\
&\quad + \frac{3f^{(3)} + f^{(4)} x}{f' + f'' x} - \frac{(2f'' + f''' x)^2}{(f' + f'' x)^2}]\varphi_i^* \varphi_{\bar{j}},
\end{aligned}
$$

$$
f' = \frac{df(x)}{dx}.
$$

We substitute this metric and Ricci tensor into the β function and compare the coefficients of $\delta_{i\bar{j}}$ and $\varphi^i\varphi^{*\bar{j}}$.

$$
\begin{aligned}
\frac{\partial}{\partial t}f' &= \frac{1}{2\pi}[(N-1)\frac{f''}{f'} + \frac{2f''+f'''x}{f'+f''x}] - 2\gamma(f'+f''x) - f''x, \\
\frac{\partial}{\partial t}f'' &= \frac{1}{2\pi}[(N-1)(\frac{f^{(3)}}{f''} - \frac{(f'')^2}{(f')^2}) + \frac{3f^{(3)}+f^{(4)}x}{f'+f''x} - \frac{(2f''+f'''x)^2}{(f'+f''x)^2}] \\
&\quad -2\gamma(2f''+f'''x) - (f'''x+f'').
\end{aligned}
$$

We can derive an infinite number of coupled differential equations relating the coupling constants g_n .

To obtain the Lagrangian of the scale invariant field theory, we have to solve the differential equation:

$$
\begin{aligned}
\frac{\partial}{\partial t}f' &= \frac{1}{2\pi}[(N-1)\frac{f''}{f'} + \frac{2f''+f'''x}{f'+f''x}] - 2\gamma(f'+f''x) - f''x \\
&= 0.
\end{aligned}
$$

We can fix all coupling constant g_n using g_2 order by order.

The following function satisfies β=0 for any values of parameter

$$f' = 1 + 2g_2 x + [\frac{2(3N+5)}{N+2} g_2^2 + \frac{2\pi^2}{N+2} g_2] x^2 + \frac{4}{3(N+2)(N+3)} [(16N^2 + 66N + 62) g_2^3 + 2\pi^2 (6N+14) g_2^2 + 2\pi^4 g_2] x^3 + \cdots .$$

If we fix the value of g_2, we obtain a conformal field theory.

We take the specific values of the parameter, the function takes simple form.

- $g_2 = 0$

$$f(x) = x$$

This theory is equal to IR fixed point of CP^N model

- $g_2 = -\frac{1}{2} \cdot \frac{2\pi^2}{N+1} \equiv -\frac{1}{2} b$

$$f(x) = \frac{1}{b} \ln(1 + bx)$$

This theory is equal to UV fixed point of CP^N model.

Then the parameter describes a marginal deformation from the IR to UV fixed points of the CP^N model in the theory spaces.

6. Summary and Discussions

In this work, we argue that some $\mathcal{N}=2$ supersymmetric nonlinear sigma models are renormalizable in three dimensions.

When the target space is an Einstein-Kaehler manifold with positive scalar curvature, there are nontrivial ultra violet fixed point, which can be used to define the nontrivial continuum theory.

Finally, we construct a class of conformal field theories with SU(N) symmetry, defined at the fixed point of the nonperturbative β function. These conformal field theories have a free parameter, and this parameter describe a marginal deformation from the IR to UV fixed point CP^N model in the theory spaces.