Unitary Designs in Quantum Information Theory

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YITP, Kyoto University

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Outline of this talk

A journey of a thousand miles begins with a single step

The ABC's of quantum computer

A possible new paradigm of information technology

Quantum computers in a nutshell

Usual (super-)computer

Upgrade →

Quantum computer

The ABC's of quantum computer

A possible new paradigm of information technology

Quantum computers in a nutshell

Abstract components

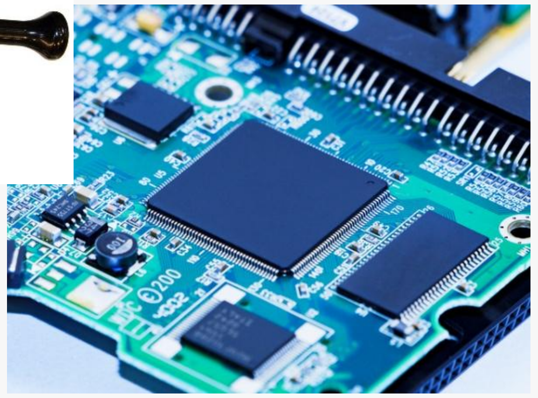
- A bit (0 or 1)
- n bits = $\{0,1\}^n$.
- Computation = Boolean function

$$f : \{0,1\}^n \to \{0,1\}^m$$

Physical components

- Mechanical
- Semiconductors

Mechanical computer



Integrated circuits

Usual (super-)computer

Upg

Physics theory behind these components is **Newton mechanics** or **Electrodynamics**.

Quantum computers in a nutshell

Abstract components

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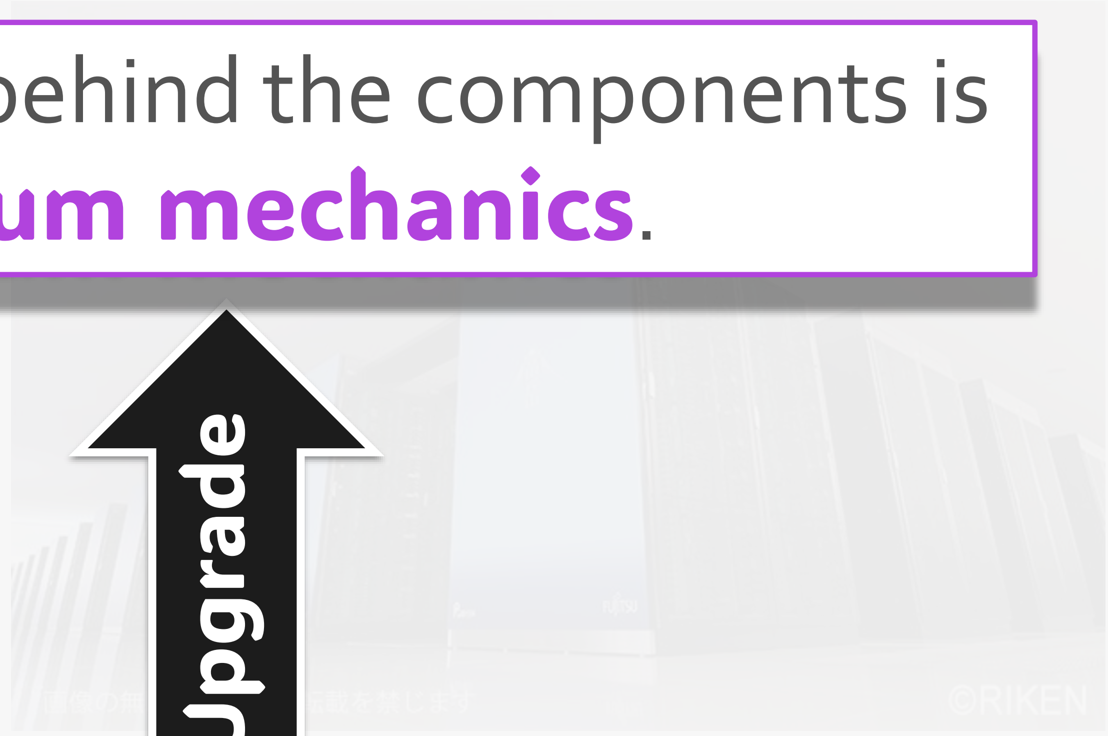
- n bits = $\{0,1\}^n$
- Computation

$f : \{0,1\}^n \to \{0,1\}$

Usual (super-)computer



Physics theory behind the components is **quantum mechanics**.

Upgrade

Mechanical computer

Physical components

- Mechanical
- Semiconductors

Integrated circuits

Physics theory behind these components is **Newton mechanics** or **Electrodynamics**.

The ABC's of quantum computer

A possible new paradigm of information technology

Quantum computers in a nutshell

Usual computer

- A bit (0 or 1)
- n bits = $\{0,1\}^n$. Cartesian product
- Computation = Boolean function $f:\{0,1\}^n \to \{0,1\}^m$

Quantum computer

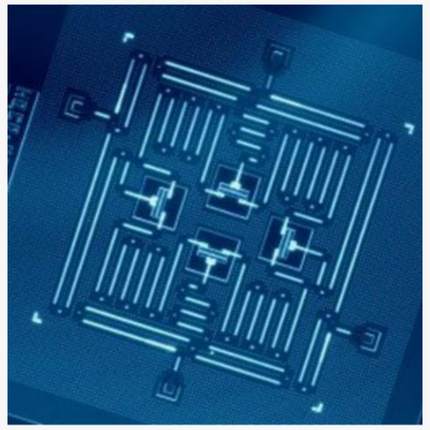
Abstract components

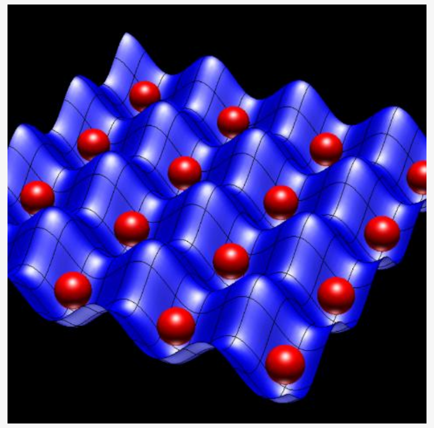
- A **qubit** $\Leftrightarrow$ 2-dimensional unit vector $\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \in \mathbb{C}^2$ $(|\alpha|^2+|\beta|^2=1)$.
- n **qubits** $\Leftrightarrow$ 2^n-dimensional unit vector $v \in (\mathbb{C}^2)^{\otimes n}$. Tensor product
- Computation = **unitary transformation** $U \in \mathbb{U}(2^n)$: $v \mapsto Uv$

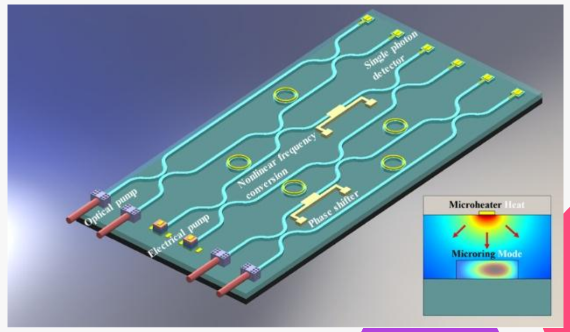
Physical components

Anything that obeys **quantum mechanics**.

e.g.) superconductors, atoms, ions, photons, etc…







Physics theory behind the components is **quantum mechanics**.

The ABC's of quantum computer

A possible new paradigm of information technology

Quantum computers in a nutshell

Usual (super-)computer

A big speed-up is expected.

Upgrade →

Quantum computer

Information is expressed by n bits $\{0,1\}^n$.
Computation by a **Boolean function**.

$$f : \{0,1\}^n \to \{0,1\}^m$$

Information is expressed by n qubits.
$\Leftrightarrow$ 2^n-dim unit vector $v \in (\mathbb{C}^2)^{\otimes n}$
Computation is by a **unitary transformation**.

$$\mathbb{U}(2^n) \ni U : v \mapsto Uv$$

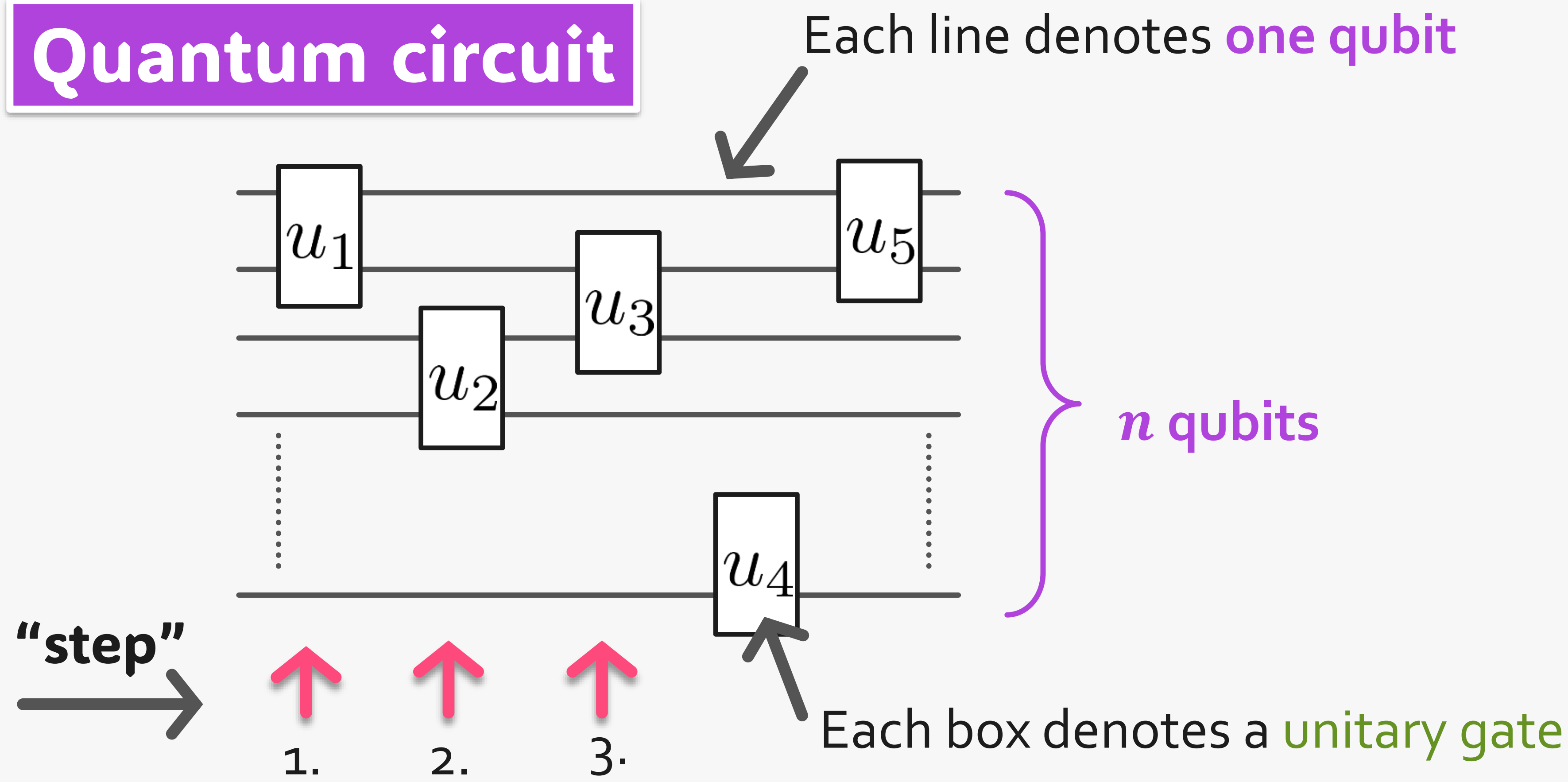
The ABC's of quantum computer

A possible new paradigm of information technology

Not all unitaries are efficient

- In a quantum computer, computation is done by a unitary transformation.
  - For a practical reason, a unitary $U \in \mathbb{U}(2^n)$ cannot be directly implemented in a quantum computer.
  - Need a quantum circuit for $U \in \mathbb{U}(2^n)$ = a decomposition of $U \in \mathbb{U}(2^n)$ to a series of unitary gates $u \in \mathbb{U}(4)$.

Quantum circuit



Unitary gates $u_j \in \mathbb{U}(4)$ are applied from left to right.

1. $u_1^{(1,2)} \otimes I^{(3,4,\dots,n)}$
2. $\left(I^{(1,2)} \otimes u_2^{(3,4)} \otimes I^{(5,6,\dots,n)}\right)\left(u_1^{(1,2)} \otimes I^{(3,4,\dots,n)}\right)$
3. $\left(I^{(1)} \otimes u_3^{(2,3)} \otimes I^{(4,5,\dots,n)}\right)\left(I^{(2,3)} \otimes u_2^{(3,4)} \otimes I^{(5,6,\dots,n)}\right)\left(u_1^{(1,2)} \otimes I^{(3,4,\dots,n)}\right)$

How can we find a quantum circuit for a given $U \in \mathbb{U}(2^n)$?

The ABC's of quantum computer

A possible new paradigm of information technology

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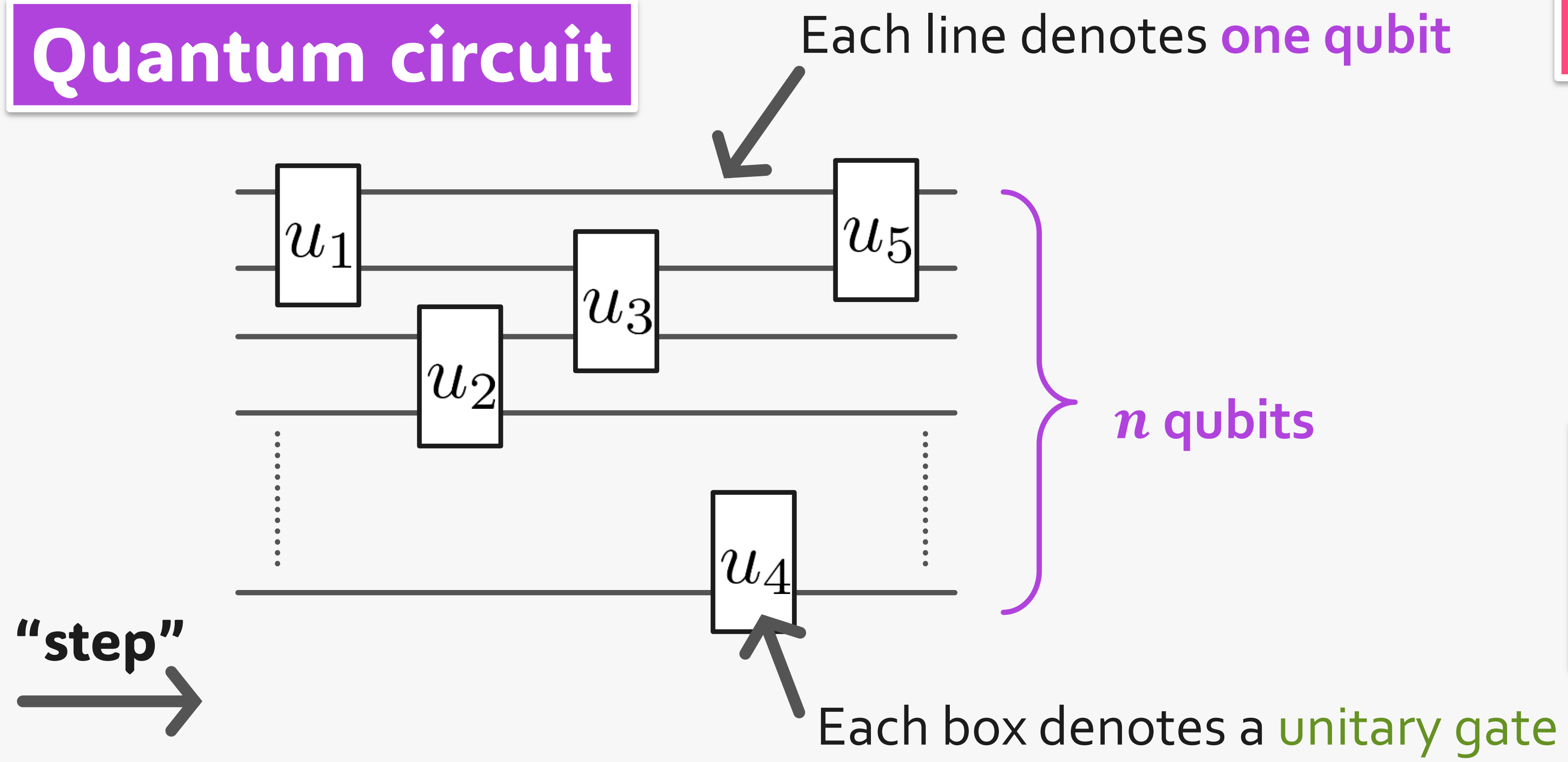
Quantum circuit



How can we find a quantum circuit for a given $U \in \mathbb{U}(2^n)$?

Brute-force method is known, ending up with $\exp(n)$ **unitary gates**.

- If # of unitary gates = $\exp(n)$, it is **INEFFICIENT** and not practical.
- If # of unitary gates = $\text{poly}(n)$, the Q circuit is **EFFICIENT**.

A challenge in quantum information
For a given $U \in \mathbb{U}(2^n)$, cleverly construct a quantum circuit with **minimal** number of unitary gates.

The ABC's of quantum computer

A possible new paradigm of information technology

All we need to remember

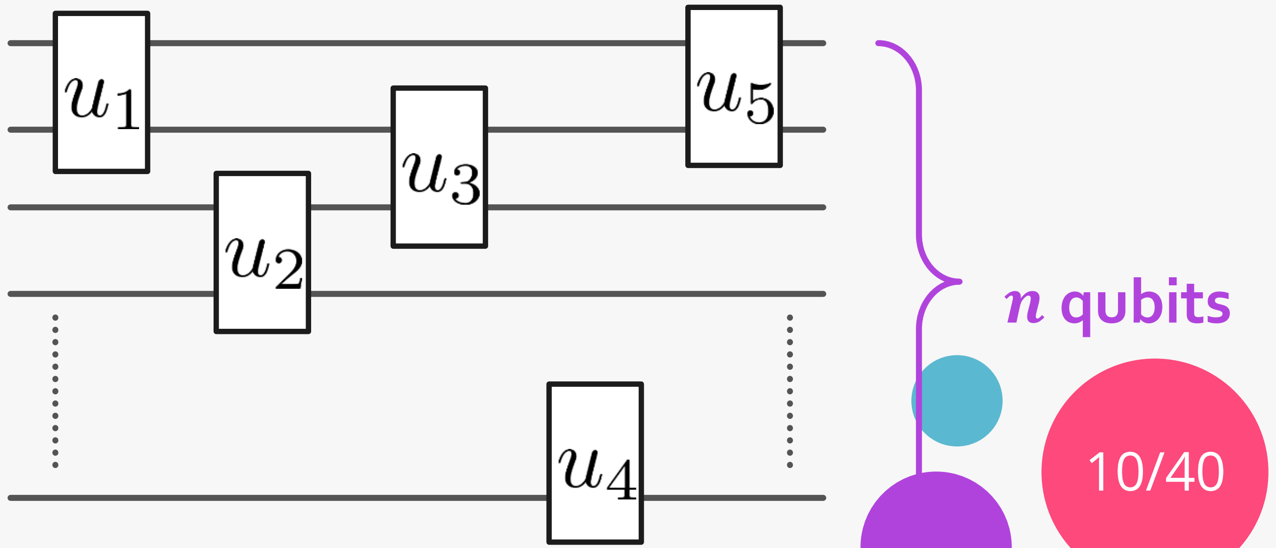
- A **quantum computer** is an upgrade of a usual computer based on quantum mechanics, in which computation is done by a unitary transformation $U_n \in \mathbb{U}(2^n)$.
  - **Quantum circuit**: decompose U_n into a series of unitary gates $u \in \mathbb{U}(4)$.
  - **Efficiency**: # of unitary gates = $\mathrm{poly}(n) \Leftrightarrow$ efficient. Otherwise, inefficient and not practical.
  - **CHALLENGE**: how can we find a quantum circuit for a given $U \in \mathbb{U}(2^n)$ with minimal number of unitary gates?

    Any algorithm to check if $U \in \mathbb{U}(2^n)$ can be implemented by a quantum circuit with $\mathrm{poly}(n)$ unitary gates?

    etc...

Upgrade

$\mathbb{U}(2^n) \ni U_n =$



Quantum circuit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Outline of this talk

A journey of a thousand miles begins with a single step

1. The ABC's of quantum computer
2. Unitary t-designs
3. Exact construction of a unitary t-design
4. Conclusion and outlook

Unitary designs

An approximation of a Haar random unitary

(normalzed) Haar measure

- $\mu_{\text{Haar}}(gS) = \mu_{\text{Haar}}(Sg) = \mu_{\text{Haar}}(S)$ ∀Borel set $S \subseteq \mathbb{U}(2^n)$, ∀g $\subseteq \mathbb{U}(2^n)$
- $\mu_{\text{Haar}}(\mathbb{U}(2^n)) = 1$

Definition of a unitary t-design

$\bar{\ }$: complex conjugate in a fixed basis

Definition [a unitary t-design on $\mathbb{U}(2^n)$]

Let t be a positive integer. A (finite) set $\{U_j\}_{j=1}^{K}$ ($U_j \in \mathbb{U}(2^n)$) is a unitary t-design if

$$\int_{\mathbb{U}(2^n)} U^{\otimes t} \otimes \bar{U}^{\otimes t} d\mu_{\text{Haar}}(U) = \frac{1}{K} \sum_{j=1}^{K} U_j^{\otimes t} \otimes \bar{U}_j^{\otimes t}.$$

- A unitary t-design is ***extremely useful*** in quantum information (many applications).
- Quantum circuit implementations for a unitary t-design?
  - Many implementations are known for *APPROXIMATE* unitary t-designs.

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- A unitary t-design is *extremely useful* in quantum information (many applications).
- Quantum circuit implementations for a unitary t-design?
  - Many implementations are known for *APPROXIMATE* unitary t-designs.
  - In applications, approximate ones are sufficient.
- **How can we implement an EXACT unitary t-design by quantum circuit?**

[E. Bannai, YN, T. Okuda, and D. Zhao, Advances in Mathematics Vol. 405, 108457 (2022).]
[YN, D. Zhao, T. Okuda, E. Bannai, and et. al., PRX Quantum **2**, 030339 (2021).]

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Exact construction of a t-design

A representation-theoretic approach

Group representation

$\bar{\cdot}$: complex conjugate in a fixed basis

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What does this look like?

Exact construction of a t-design

A representation-theoretic approach

Group representation

- How does $\int_{\mathbb{U}(2^n)} U^{\otimes t} \otimes \bar{U}^{\otimes t} d\mu_{\text{Haar}}(U)$ look?
  - As $U \in \mathbb{U}(2^n)$, $U^{\otimes t} \otimes \bar{U}^{\otimes t}$ is a $2^{2tn} \times 2^{2tn}$ matrix and may look like complicated one…

More "natural" way of thinking:
$\pi_t \colon U \mapsto U^{\otimes t} \otimes \bar{U}^{\otimes t}$ as representation of $\mathbb{U}(2^n)$.

$$
\begin{matrix} \pi_t(U) \\ \parallel \\ U^{\otimes t} \otimes \bar{U}^{\otimes t} \end{matrix} = \begin{pmatrix}
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\text{**************************} \\
\text{**************************} \\
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\text{**************************} \\
\text{**************************}
\end{pmatrix} \in \mathbb{U}(2^{2tn})
$$

Exact construction of a t-design

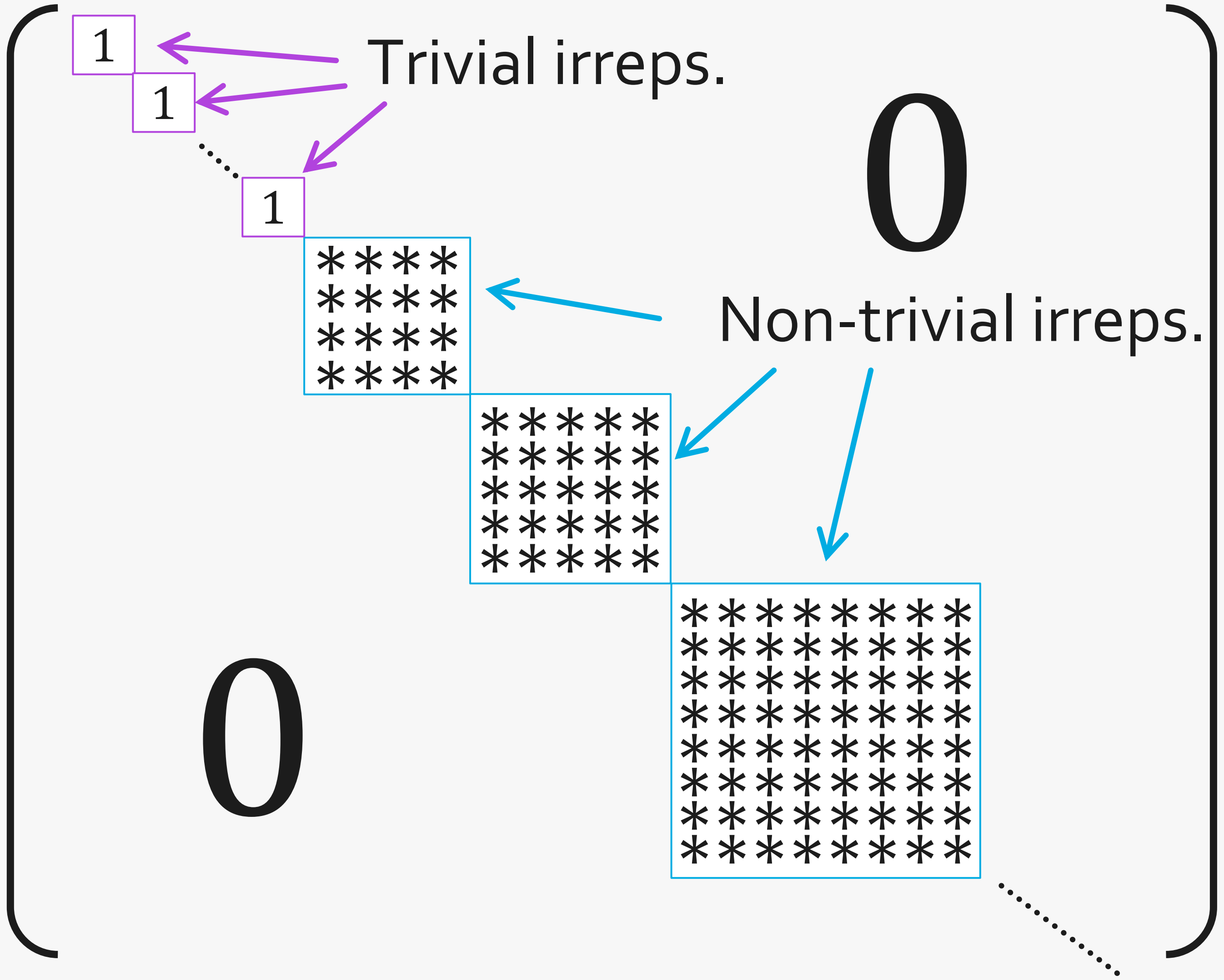
A representation-theoretic approach

Group representation

- How does $\int_{\mathbb{U}(2^n)} U^{\otimes t} \otimes \bar{U}^{\otimes t} d\mu_{\text{Haar}}(U)$ look?
  - The irreducible decomposition of π_t looks like…

$$\pi_t : \mathbb{U}(2^n) \to \mathbb{U}(2^{2tn})$$
$$U \mapsto U^{\otimes t} \otimes \bar{U}^{\otimes t}$$

$$\int_{\mathbb{U}(2^n)} \pi_t(U) d\mu_{\text{Haar}}(U)$$
$$||$$
$$\int_{\mathbb{U}(2^n)} U^{\otimes t} \otimes \bar{U}^{\otimes t} d\mu_{\text{Haar}}(U) =$$



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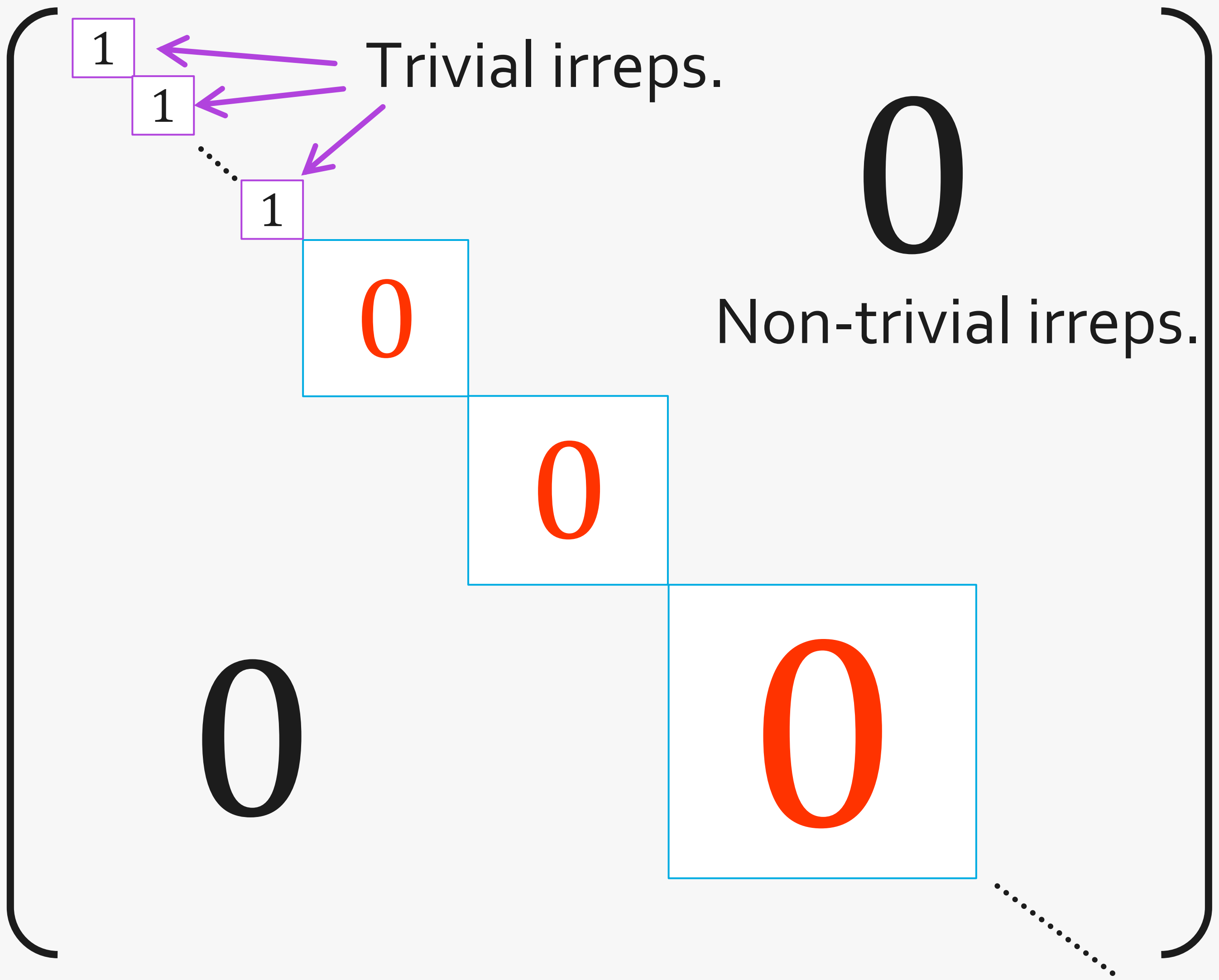

$$\in \mathbb{U}(2^{2tn})$$

…due to the Schur's orthogonality…

Exact construction of a t-design

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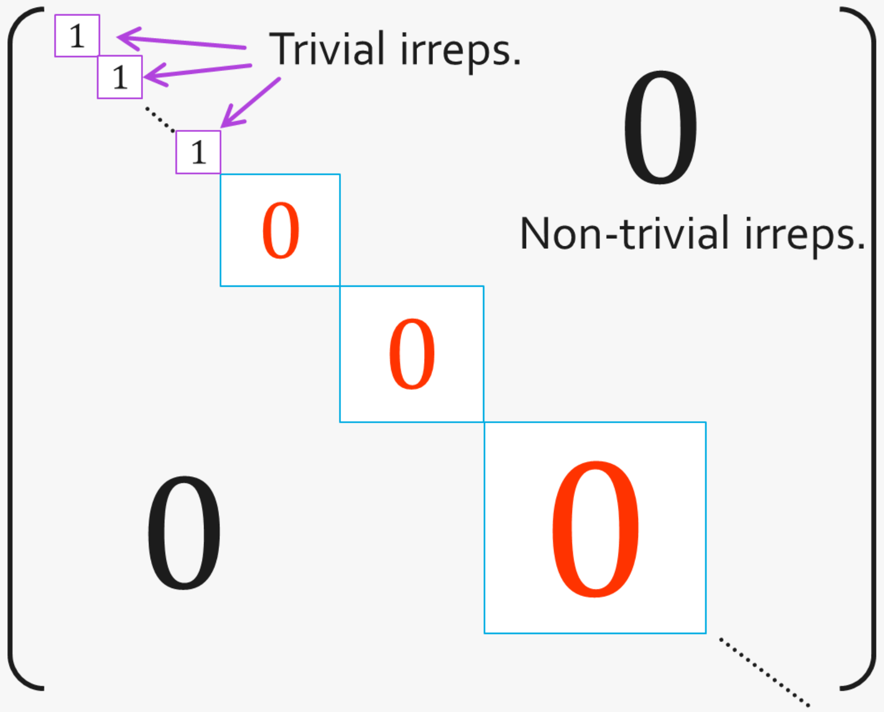
$$\int_{\mathbb{U}(2^n)} \pi_t(U) d\mu_{\text{Haar}}(U) =$$



- The LHS is a projection onto the trivial irreps of π_t.
- How can we find a finite set of unitaries that ends up with the same projection after taking the average?

Our strategy is to use a *specific* subgroup of $\mathbb{U}(2^n)$.

Gelfand pairs and **zonal spherical functions**

Exact construction of a t-design

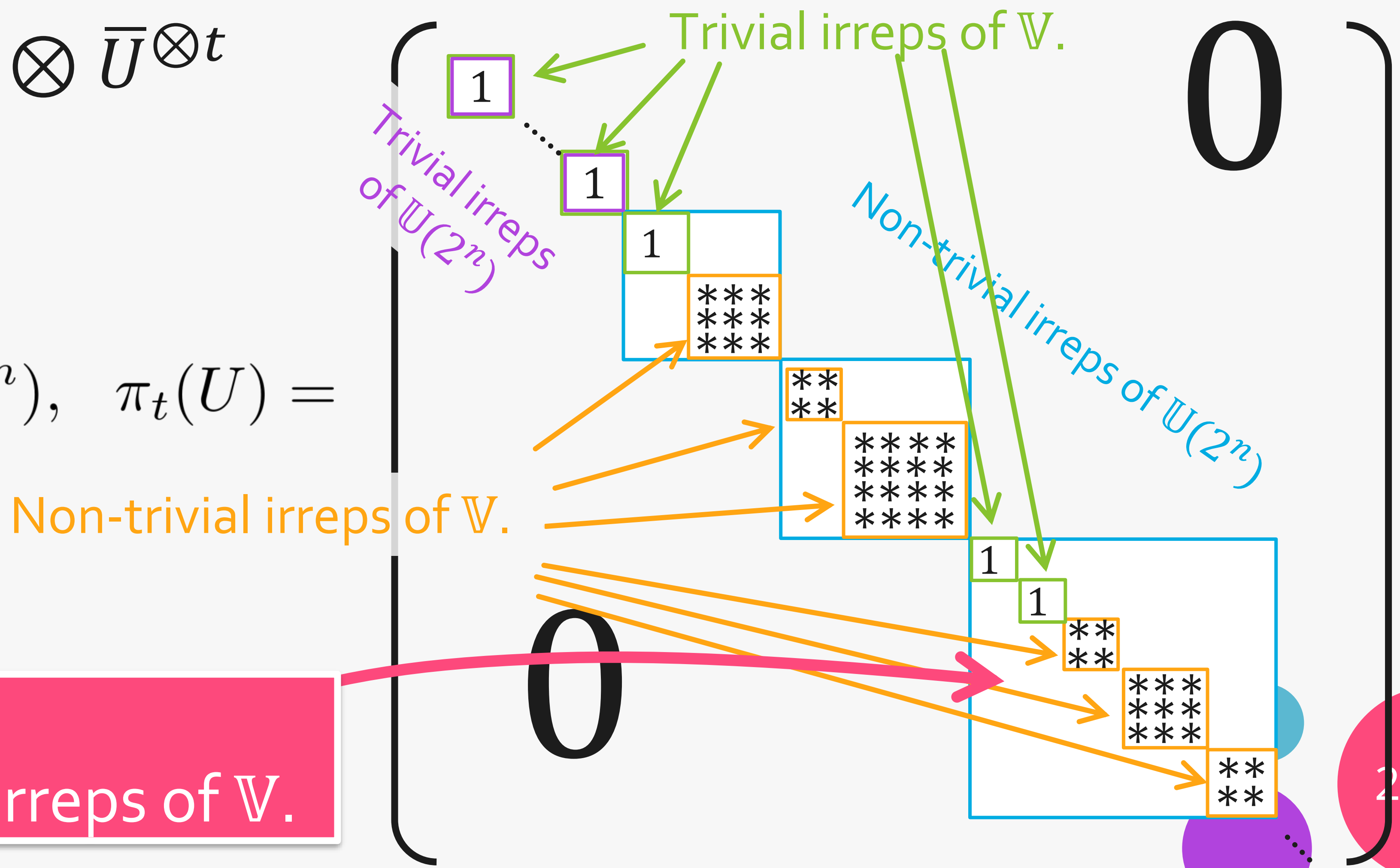
A representation-theoretic approach

$$\pi_t : \mathbb{U}(2^n) \to \mathbb{U}(2^{2tn})$$
$$U \mapsto U^{\otimes t} \otimes \bar{U}^{\otimes t}$$

Gelfand pair and zonal spherical function

- Let $\mathbb{G}$ be a compact group, and $\mathbb{K} \subseteq \mathbb{G}$ be a subgroup.
  - $(\mathbb{G}, \mathbb{K})$ is a Gelfand pair $\Leftrightarrow \forall$irreps ρ of $\mathbb{G}$, $\dim \mathrm{span}\{v \in V_\rho : \rho(k)v = v, \ \forall k \in \mathbb{K}\} = 0, 1$
  - Example: $\left(\mathbb{U}(2^n), \mathbb{W} := \mathbb{U}(2^{n-1}) \times \mathbb{U}(2^{n-1})\right)$
- Interested in $\pi_t : \mathbb{U}(2^t) \to \mathbb{U}(2^{tn}), U \mapsto U^{\otimes t} \otimes \bar{U}^{\otimes t}$

$$\text{For } U \in \mathbb{V} \subseteq \mathbb{U}(2^n), \quad \pi_t(U) =$$



If $(\mathbb{U}, \mathbb{V})$ is not a Gelfand pair,
some irreps of $\mathbb{U}(2^n)$ contains multiple trivial irreps of $\mathbb{V}$.

Exact construction of a t-design

A representation-theoretic approach

$$\pi_t : \mathbb{U}(2^n) \to \mathbb{U}(2^{2tn})$$
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Gelfand pair and zonal spherical function

- Let $\mathbb{G}$ be a compact group, and $\mathbb{K} \subseteq \mathbb{G}$ be a subgroup.
  - $(\mathbb{G}, \mathbb{K})$ is a Gelfand pair $\Leftrightarrow \forall \text{irreps } \rho \text{ of } \mathbb{G}, \ \dim \operatorname{span}\{v \in V_\rho : \rho(k)v = v, \ \forall k \in \mathbb{K}\} = 0, 1$
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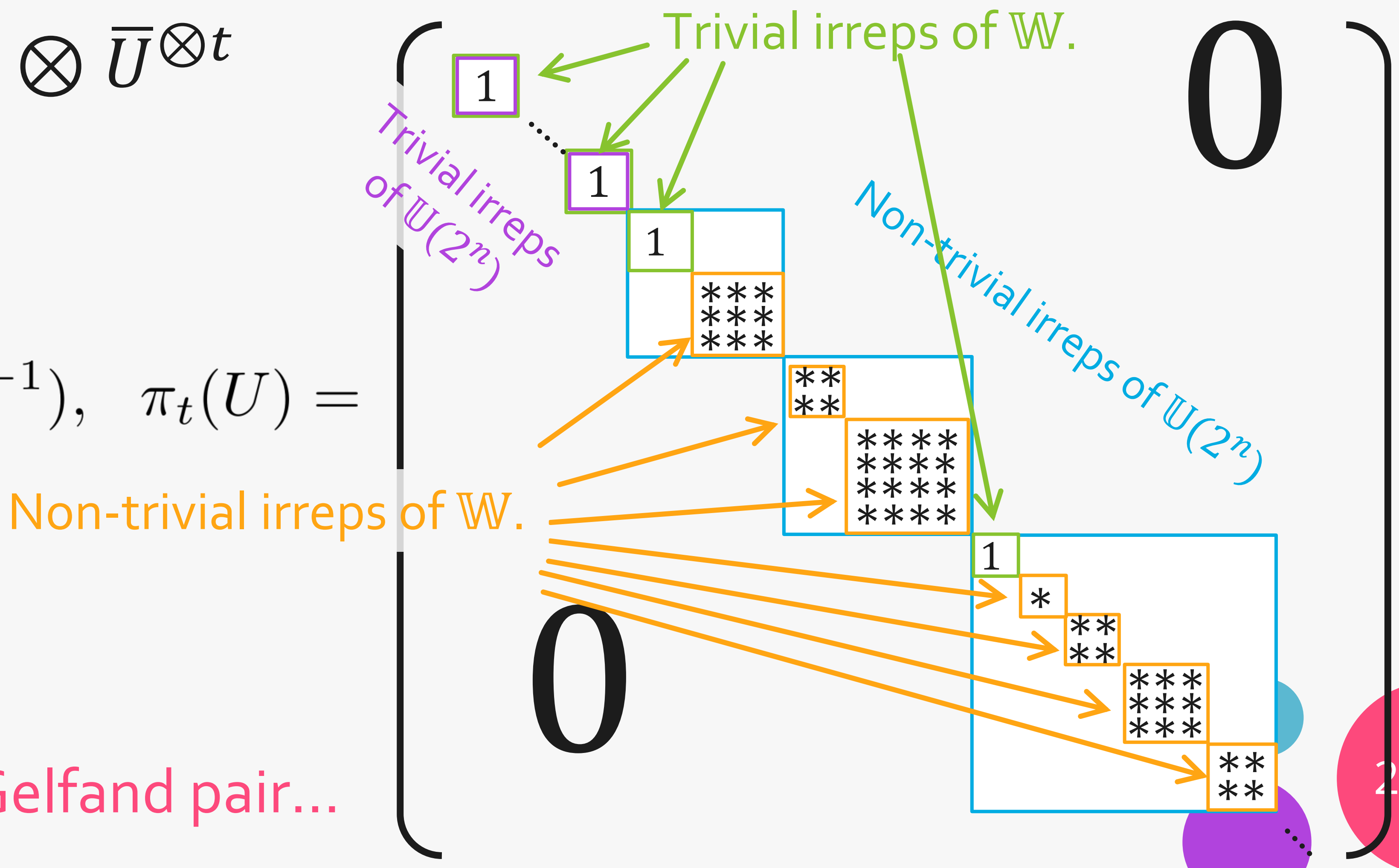
For $U \in \mathbb{W} = \mathbb{U}(2^{n-1}) \times \mathbb{U}(2^{n-1}), \quad \pi_t(U) =$



...because $(\mathbb{U}(2^n), \mathbb{W})$ is a Gelfand pair...

Exact construction of a t-design

A representation-theoretic approach

Gelfand pair and zonal spherical function

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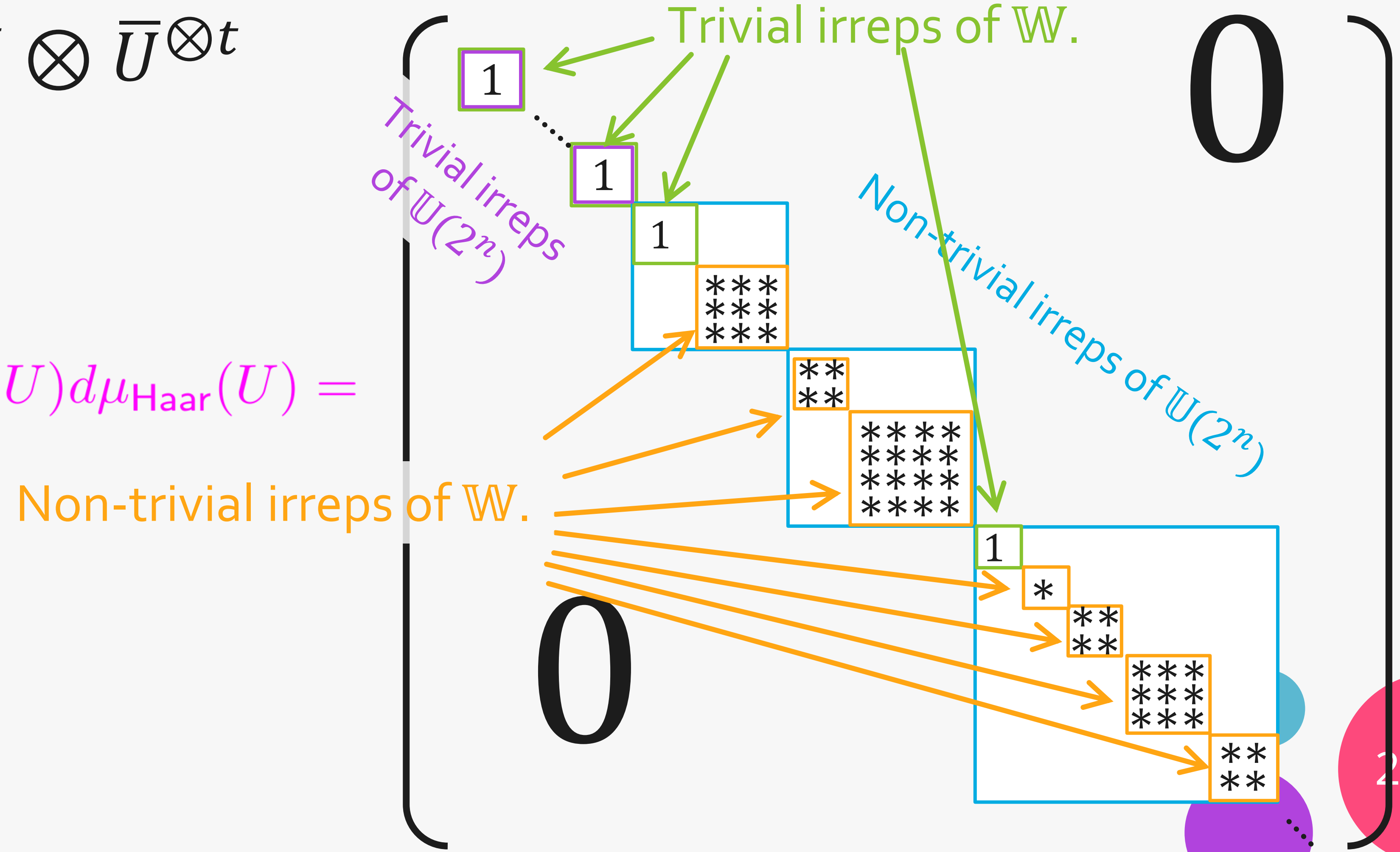
$$\int_{\mathbb{W}} \pi_t(U) d\mu_{\text{Haar}}(U) =$$



Exact construction of a t-design

A representation-theoretic approach

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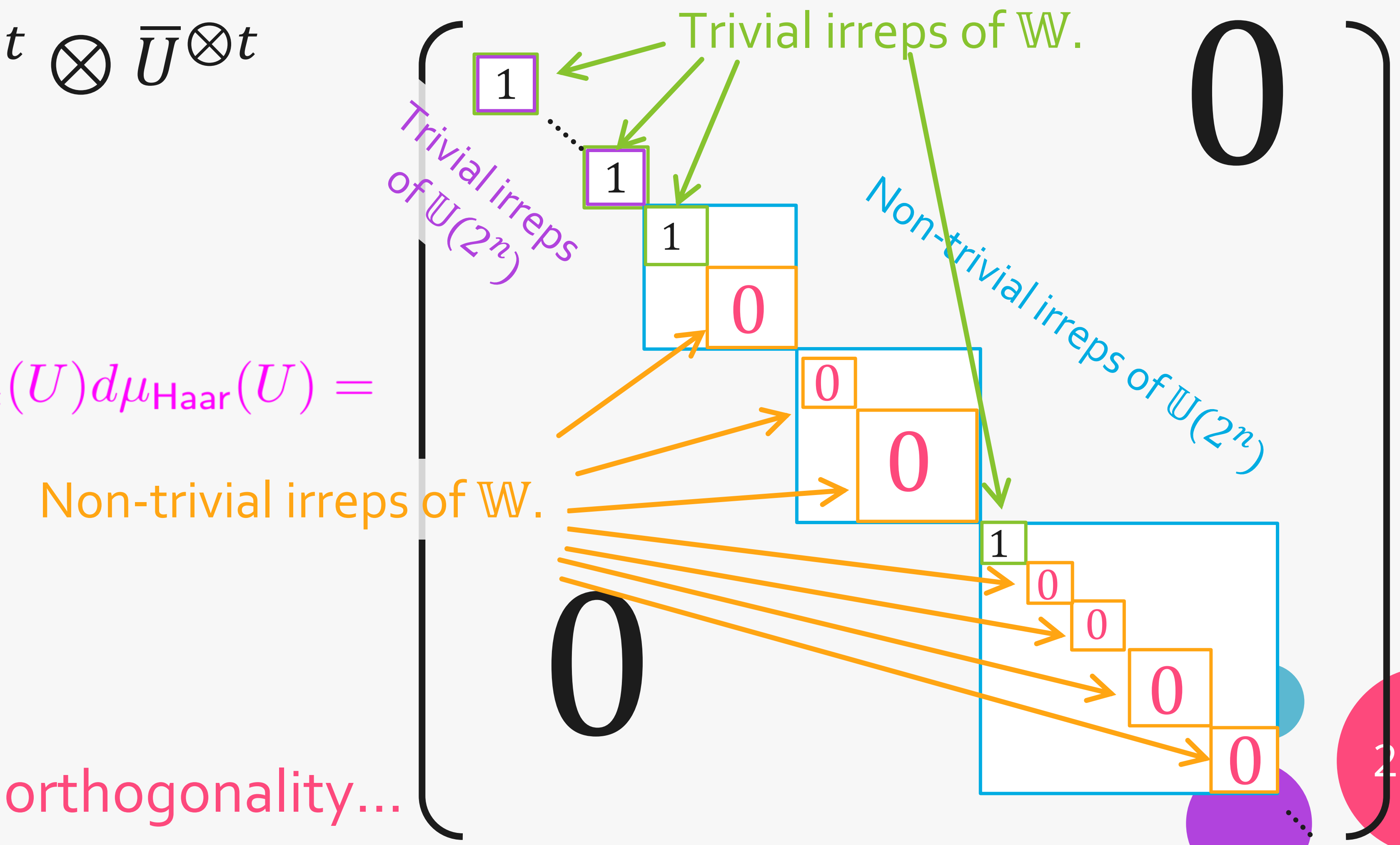

...due to the Schur's orthogonality...

Exact construction of a t-design

A representation-theoretic approach

$$\pi_t : \mathbb{U}(2^n) \to \mathbb{U}(2^{2tn})$$
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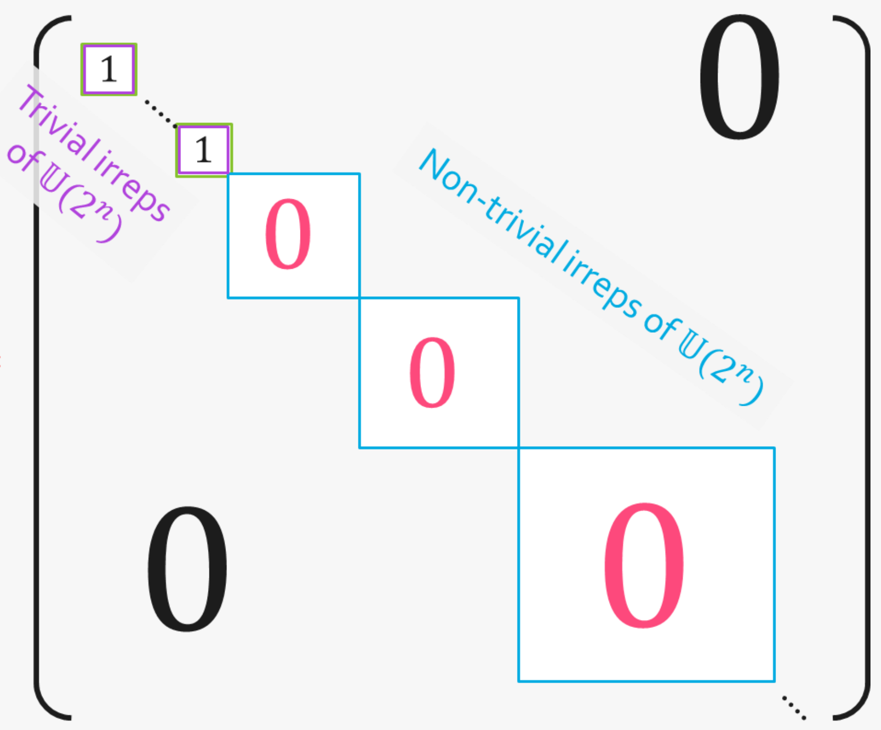
Gelfand pair and zonal spherical function

- For a Gelfand pair $(\mathbb{U}(2^n), \mathbb{W})$, the integral over $\mathbb{W}$ resembles the Haar integral.
  - All irreps of $\mathbb{U}$ that contains a single irrep of $\mathbb{W}$ have been already specified!
  - A t-design is obtained if we can "erase" undesired trivial irreps of $\mathbb{W}$.

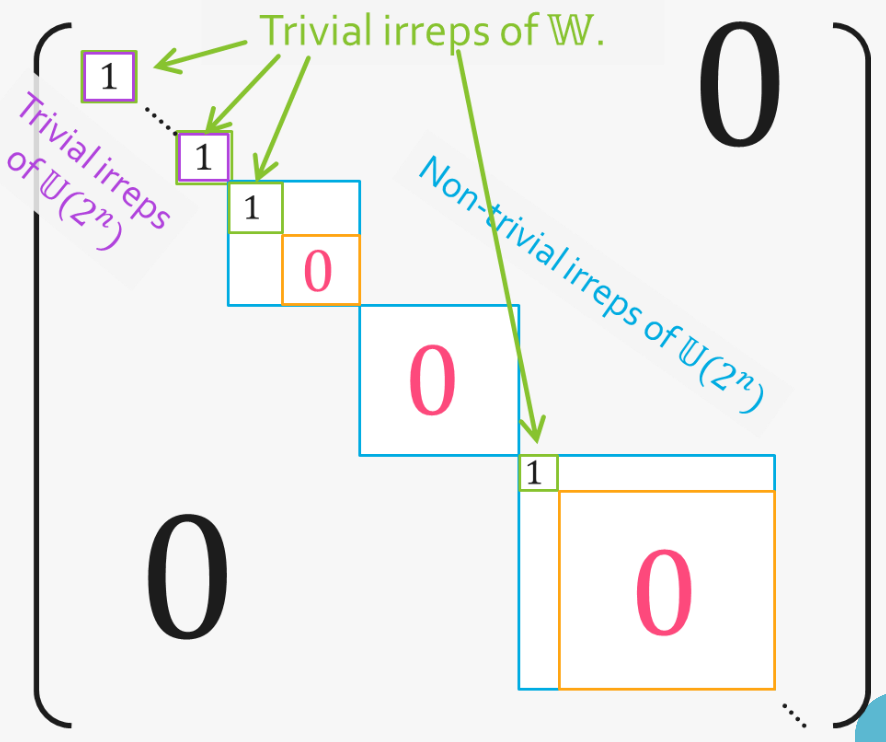
→ **zeros of zonal spherical functions**

$$\int_{\mathbb{U}(2^n)} \pi_t(U) d\mu_{\text{Haar}}(U) =$$



Integral over $\mathbb{U}(2^n)$

$$\int_{\mathbb{W}} \pi_t(U) d\mu_{\text{Haar}}(U) =$$



Integral over $\mathbb{W} = \mathbb{U}(2^{n-1}) \times \mathbb{U}(2^{n-1})$

Exact construction of a t-design

A representation-theoretic approach

Gelfand pair and zonal spherical function

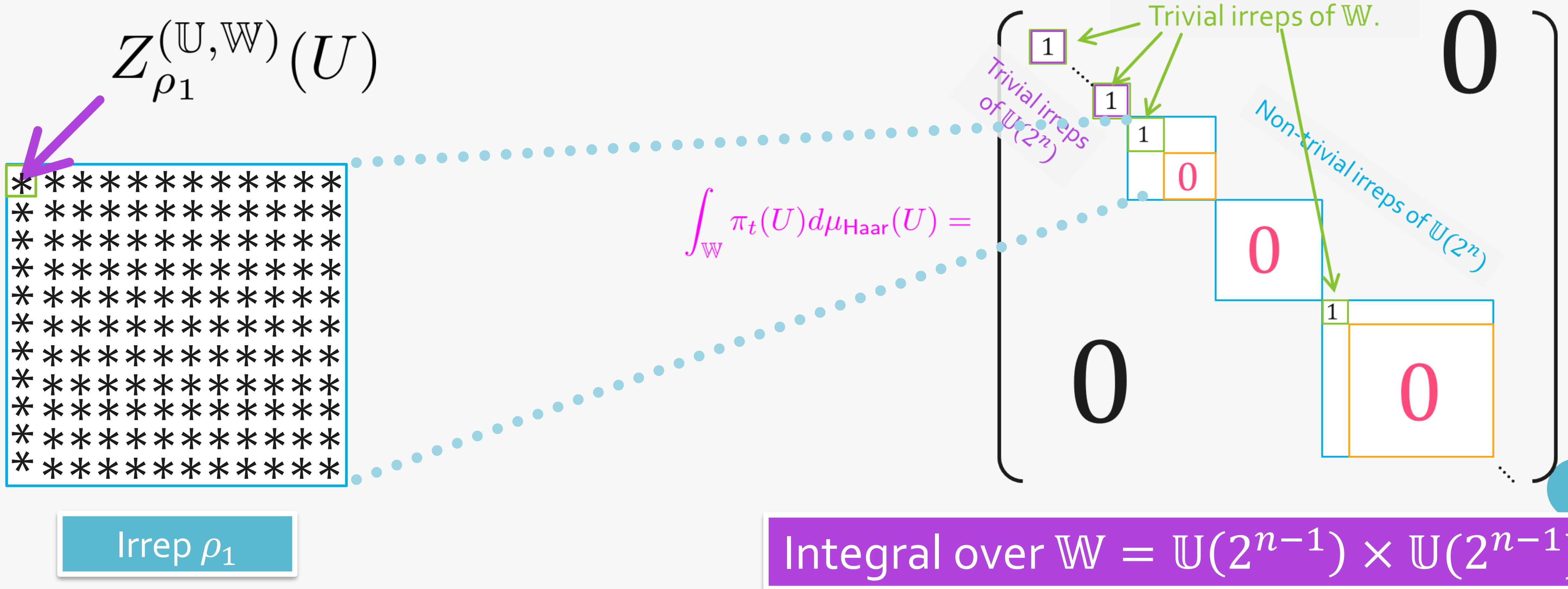
$$\pi_t : \mathbb{U}(2^n) \to \mathbb{U}(2^{2tn})$$
$$U \mapsto U^{\otimes t} \otimes \bar{U}^{\otimes t}$$

- **Zonal spherical functions** $Z_\rho^{(\mathbb{G},\mathbb{K})} : \mathbb{G} \to \mathbb{C}$ for a Gelfand pair $(\mathbb{G}, \mathbb{K})$ and an irrep ρ of $\mathbb{G}$.
  - Let $v \in V_\rho$ be a $\mathbb{K}$-invariant vector. Then, $Z_\rho^{(\mathbb{G},\mathbb{K})}(g) := \langle v, \rho(g) v \rangle$

We can write down the explicit form!

Gelfand pair $(\mathbb{U}(2^n), \mathbb{W})$

For $U \in \mathbb{U}(2^n)$, $\rho_1(U) =$



Integral over $\mathbb{W} = \mathbb{U}(2^{n-1}) \times \mathbb{U}(2^{n-1})$

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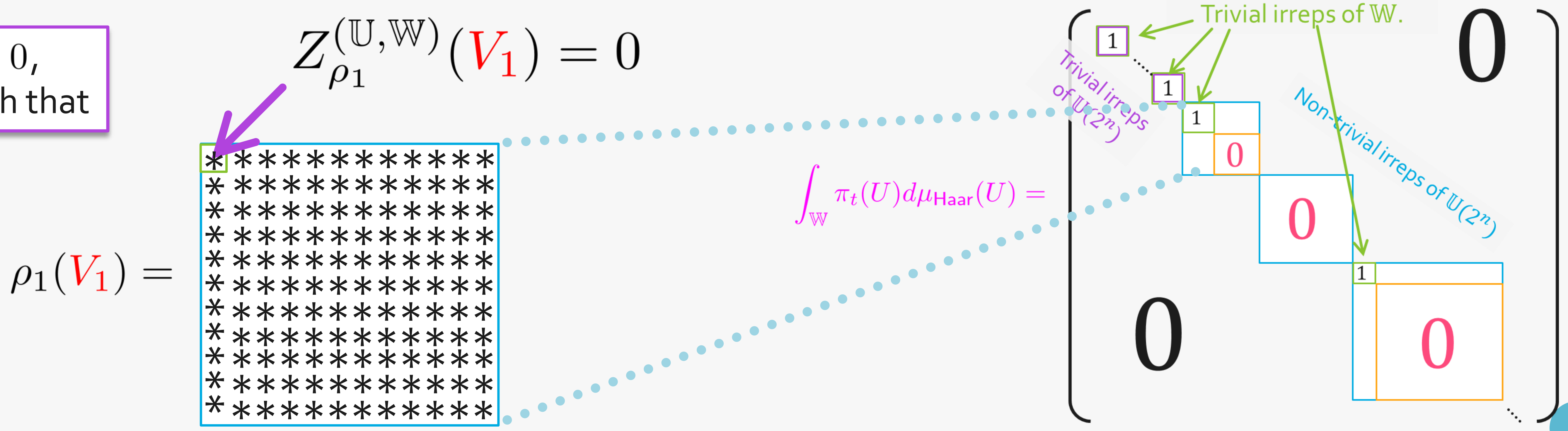
We can write down the explicit form!

Gelfand pair $(\mathbb{U}(2^n), \mathbb{W})$

By solving $Z_{\rho_1}^{(\mathbb{U},\mathbb{W})}(U) = 0$,
we can find $V_1 \in \mathbb{U}(2^n)$ such that

$$Z_{\rho_1}^{(\mathbb{U},\mathbb{W})}(V_1) = 0$$



Irrep ρ_1

Integral over $\mathbb{W} = \mathbb{U}(2^{n-1}) \times \mathbb{U}(2^{n-1})$

Exact construction of a t-design

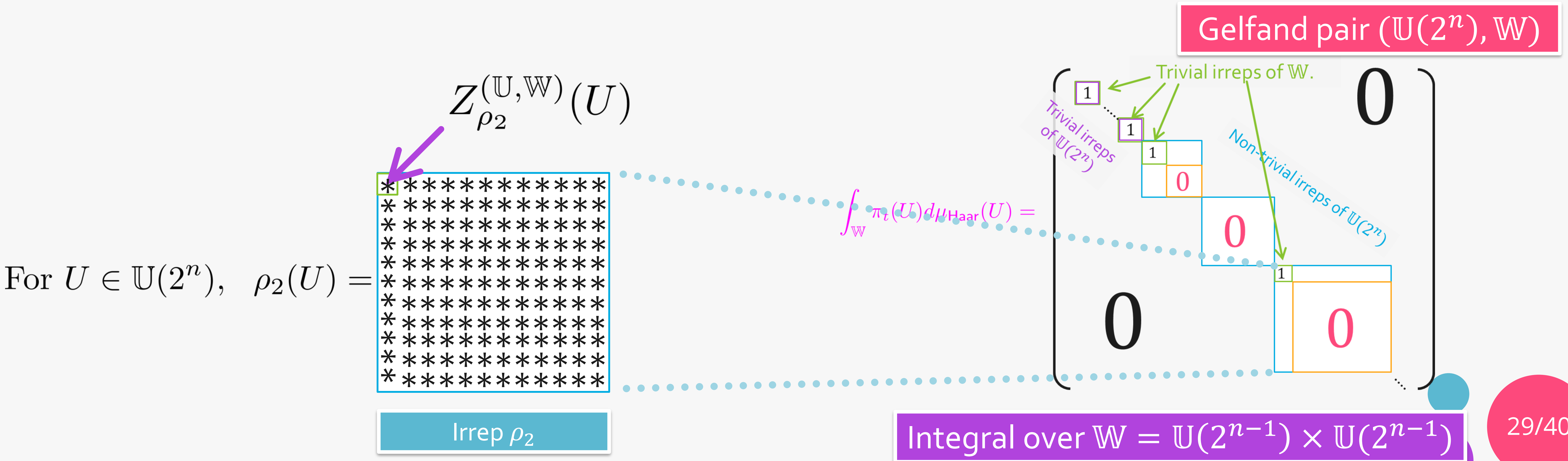
A representation-theoretic approach

Gelfand pair and zonal spherical function

$\pi_t : \mathbb{U}(2^n) \to \mathbb{U}(2^{2tn})$
$U \mapsto U^{\otimes t} \otimes \bar{U}^{\otimes t}$

- **Zonal spherical functions** $Z_\rho^{(\mathbb{G},\mathbb{K})} : \mathbb{G} \to \mathbb{C}$ for a Gelfand pair $(\mathbb{G}, \mathbb{K})$ and an irrep ρ of $\mathbb{G}$.
  - Let $v \in V_\rho$ be a $\mathbb{K}$-invariant vector. Then, $Z_\rho^{(\mathbb{G},\mathbb{K})}(g) := \langle v, \rho(g) v \rangle$

We can write down the explicit form!



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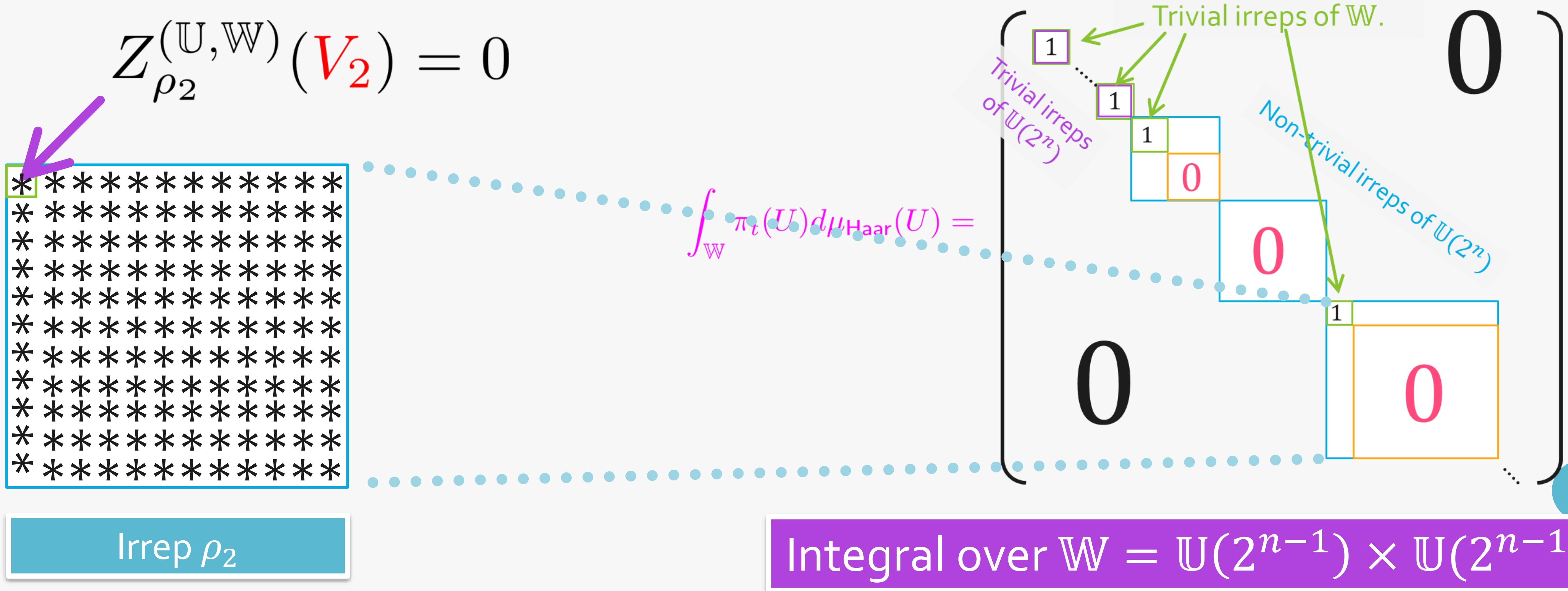
We can write down the explicit form!

By solving $Z_{\rho_2}^{(\mathbb{U},\mathbb{W})}(U = 0$,
we can find $V_2 \in \mathbb{U}(2^n)$ such that

$$Z_{\rho_2}^{(\mathbb{U},\mathbb{W})}(V_2) = 0$$

$\rho_2(V_2) =$



Integral over $\mathbb{W} = \mathbb{U}(2^{n-1}) \times \mathbb{U}(2^{n-1})$

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Gelfand pair and zonal spherical function

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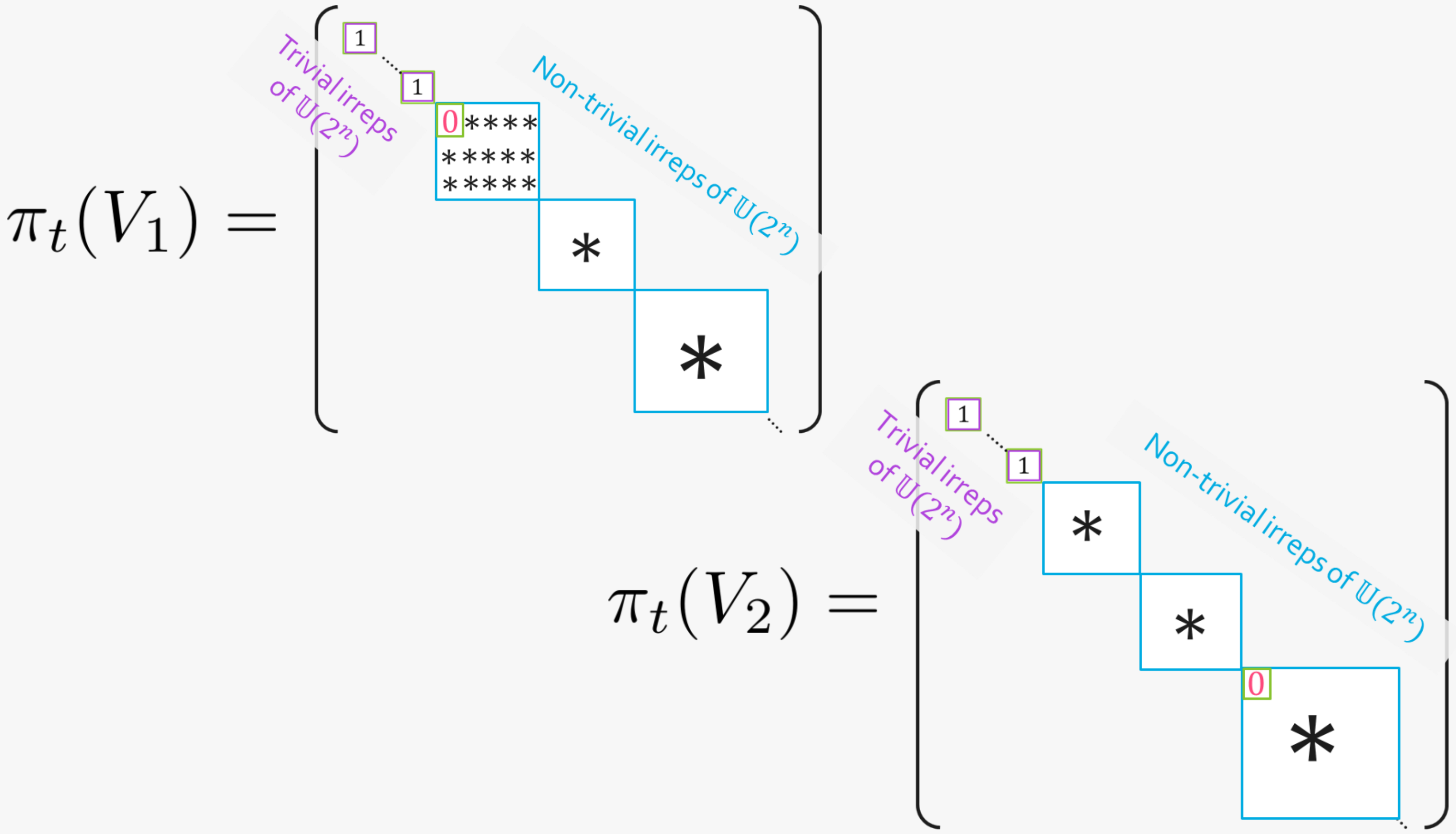
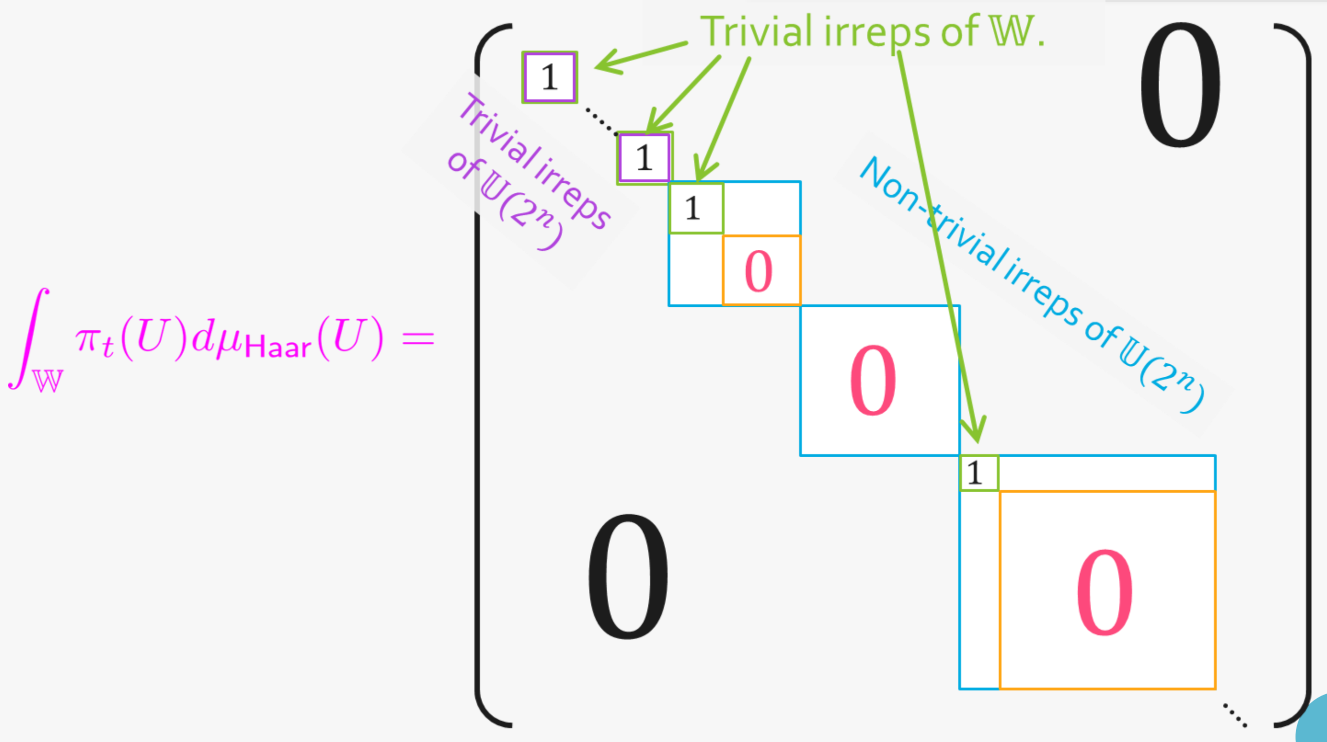
  - Let $v \in V_\rho$ be a $\mathbb{K}$-invariant vector. Then, $Z_\rho^{(\mathbb{G},\mathbb{K})}(g) := \langle v, \rho(g)v \rangle$
  - From the zeros of zonal spherical functions, we obtain $\{V_1, V_2, \ldots, V_m\}$ $(V_j \in \mathbb{U}(2^n))$



etc...

Gelfand pair $(\mathbb{U}(2^n), \mathbb{W})$



Integral over $\mathbb{W} = \mathbb{U}(2^{n-1}) \times \mathbb{U}(2^{n-1})$

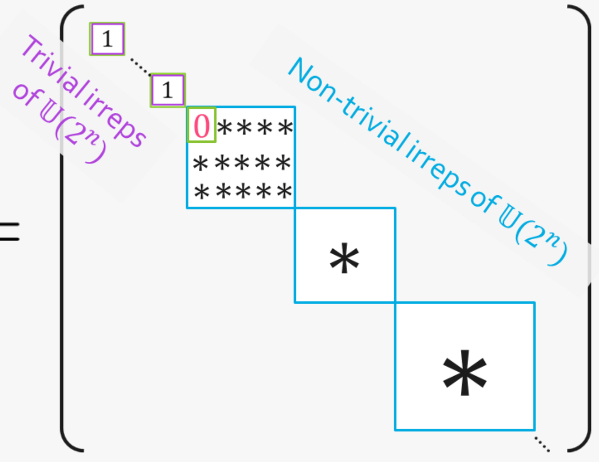
Exact construction of a t-design

A representation-theoretic approach

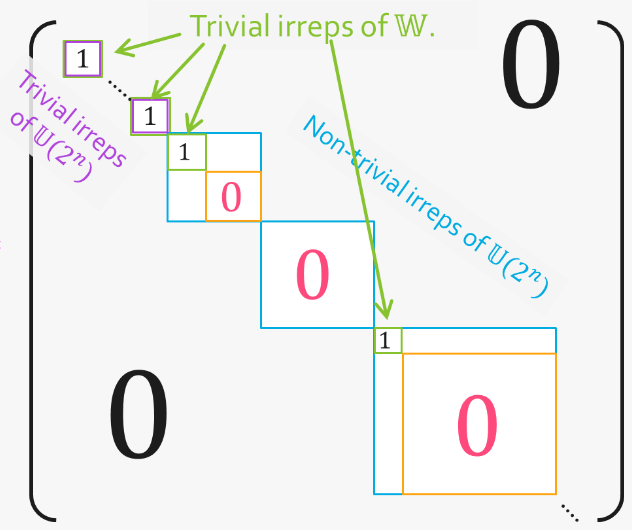
Design from Gelfand pair and zonal spherical function

- **Inductive** construction of a unitary t-design on $\mathbb{U}(2^n)$
  - Suppose the Haar measure (or t-design) on $\mathbb{U}(2^{n-1})$ is available. ➡ A unitary t-design on $\mathbb{W} := \mathbb{U}(2^{n-1}) \times \mathbb{U}(2^{n-1})$
  - Consider $\left(\int_{\mathbb{W}} \pi_t(U) d\mu_{\text{Haar}}(U)\right) \pi_t(V_1) \left(\int_{\mathbb{W}} \pi_t(U) d\mu_{\text{Haar}}(U)\right)$

$\pi_t(V_1) =$



$\int_{\mathbb{W}} \pi_t(U) d\mu_{\text{Haar}}(U) =$



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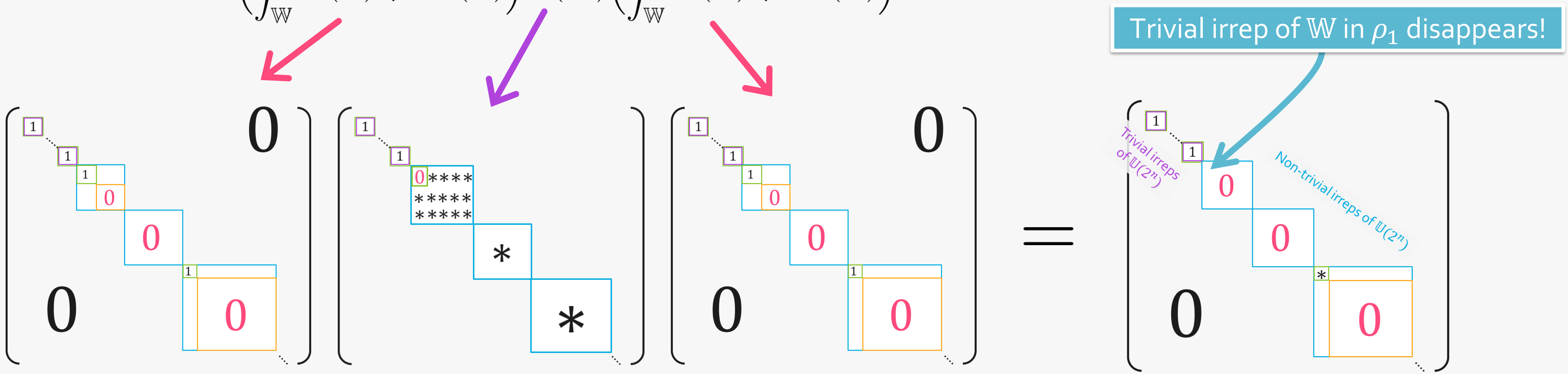

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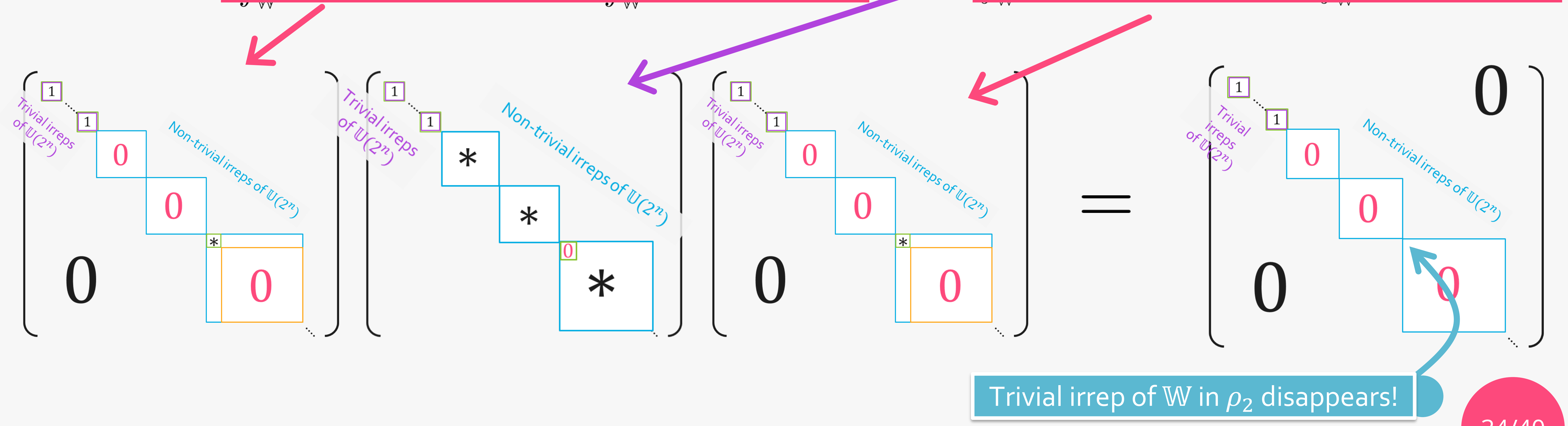

Exact construction of a t-design

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- **Inductive** construction of a unitary t-design on $\mathbb{U}(2^n)$
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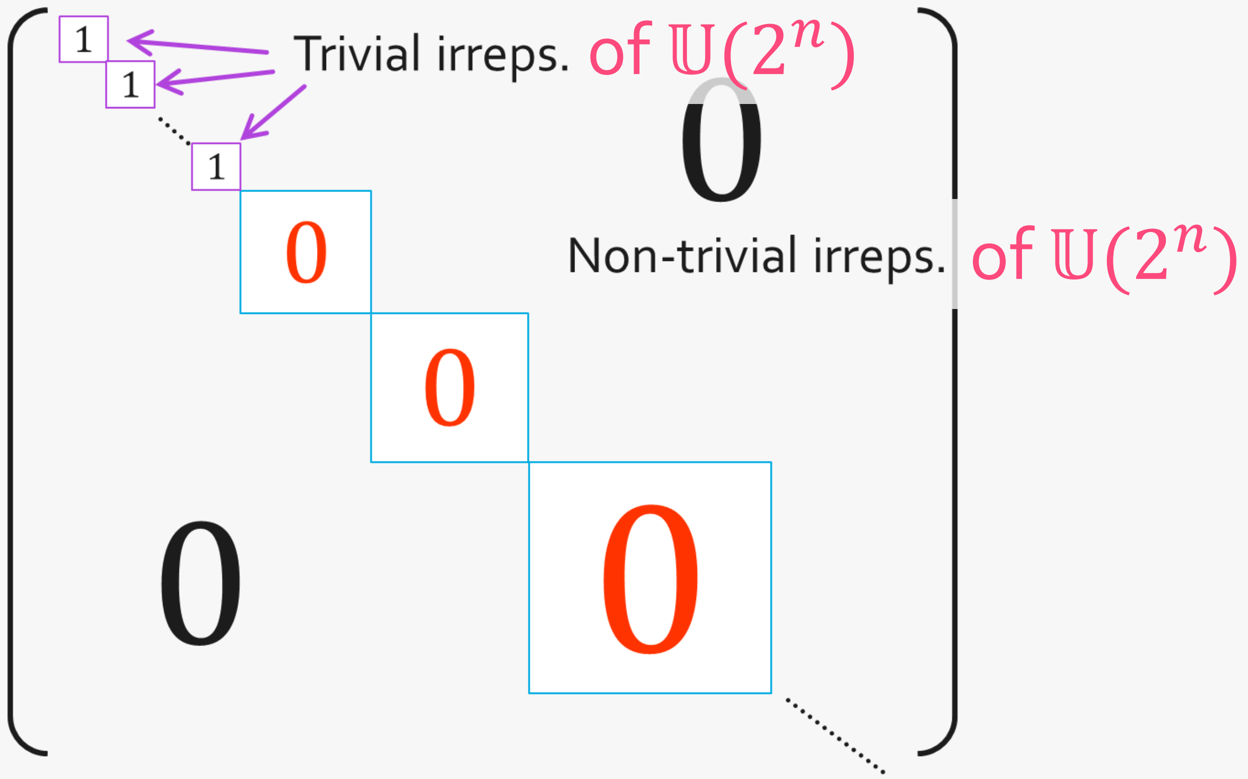
  - By repeating this for all $\{V_1, V_2, \ldots, V_m\}$, all (undesired) trivial irreps. of $\mathbb{W}$ vanish, and we obtain



$$= \int_{\mathbb{U}(2^n)} \pi_t(U) d\mu_{\text{Haar}}(U)$$

Exact construction of a t-design

A representation-theoretic approach

Design from Gelfand pair and zonal spherical function

- **Inductive** construction of a unitary t-design on $\mathbb{U}(2^n)$
  - A unitary t-design on $\mathbb{U}(2^n)$ from a t-design on $\mathbb{U}(2^{n-1})$.

Induction down to a t-design on $\mathbb{U}(1)$, which is easy to construct by hand.

$$\left\{W_1 V_1 W_2 V_2 \ldots W_m V_m W_{m+1}\right\}_{W_j \in \mathbb{W}}$$ is a unitary t-design.

- W_j are from a t-design on $\mathbb{W} = \mathbb{U}(2^{n-1}) \times \mathbb{U}(2^{n-1})$.
- V_j are **zeros of zonal spherical functions**.

Exact construction of a t-design

A representation-theoretic approach

Design from Gelfand pair and zonal spherical function

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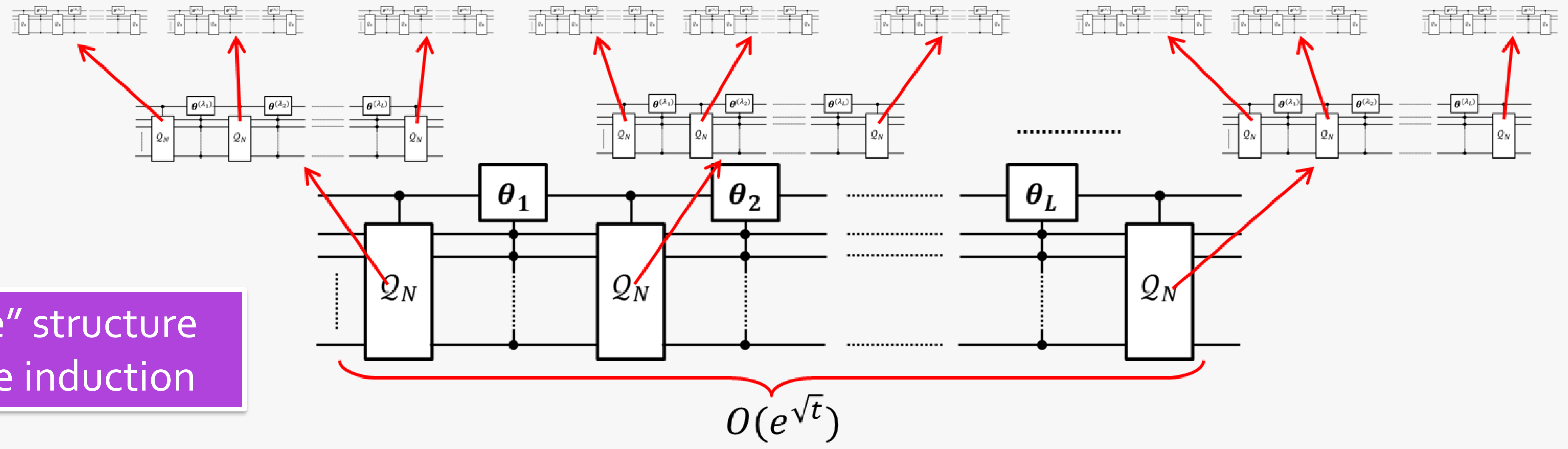
Induction down to a t-design on $\mathbb{U}(1)$, which is easy to construct by hand.

  - A unitary t-design on $\mathbb{U}(2^n)$ can be explicitly constructed from that on $\mathbb{U}(1)$.
  - This construction can be easily translated to a quantum circuit.



"Tree-like" structure due to the induction

Exact construction of a t-design

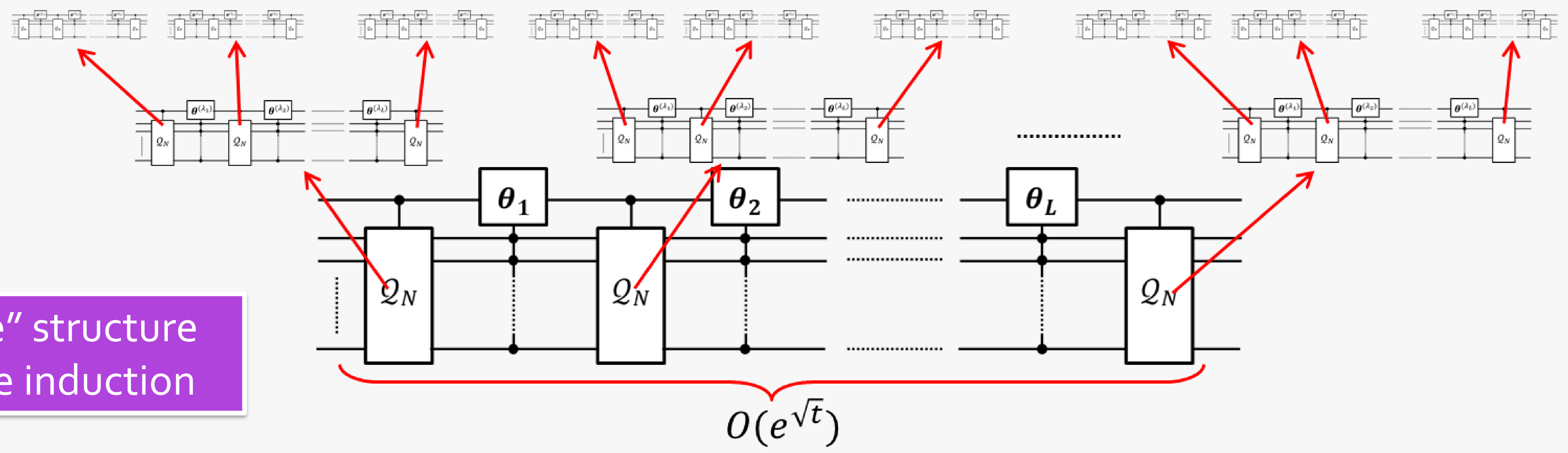
A representation-theoretic approach

Design from Gelfand pair and zonal spherical function

- Inductive construction of a unitary t-design on $\mathbb{U}(2^n)$
  - The first-ever EXPLICIT construction of an exact unitary t-design!
  - However, the # of unitary gates = $O(2^{n\sqrt{t}})$ due to the induction.

Inefficient and unpractical…

Open: can we improve this? I.e., quantum circuit for a unitary t-design with $\mathrm{poly}(n,t)$ unitary gates?



"Tree-like" structure due to the induction

Outline of this talk

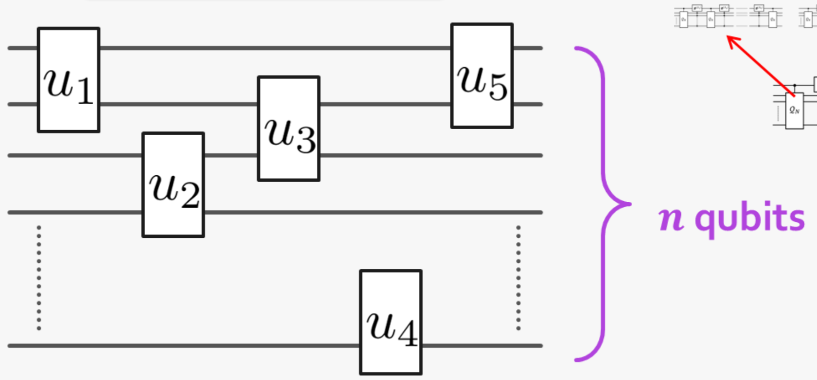
A journey of a thousand miles begins with a single step

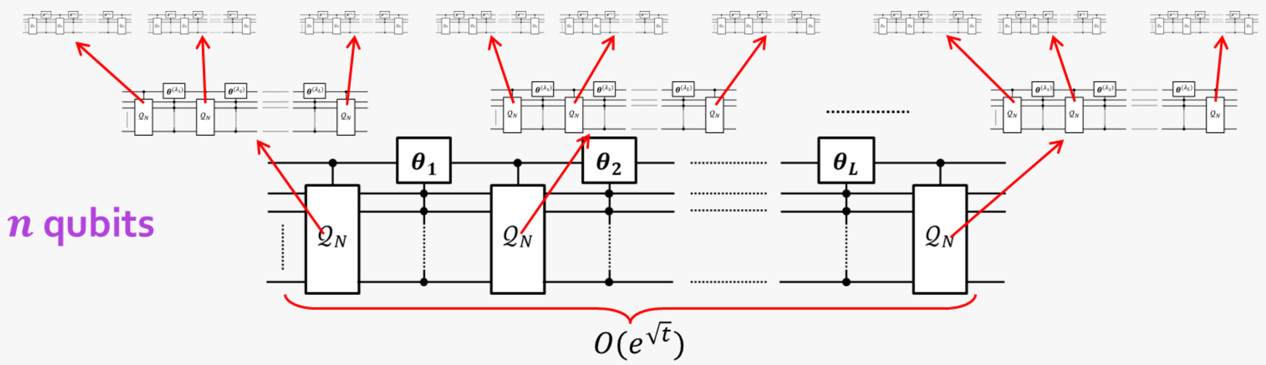
1. The ABC's of quantum computer
2. Unitary t-designs
3. Exact construction of a unitary t-design
4. **Conclusion and outlook**

Outlook

What's next?

Quantum circuit





Q. Circuit for unitary t-design

Conclusion and outlook

- **Quantum computation** is by a **unitary transformation** $U \in \mathbb{U}(2^n)$.
  - **Quantum circuit**: decomposition of a unitary $U \in \mathbb{U}(2^n)$ into a series of **unitary gates**.
  - **Challenge**: Given $U \in \mathbb{U}(2^n)$, find a quantum circuit with minimal number of unitary gates!
- **A unitary t-design**
  - Unitary t-designs are extremely useful in quantum information.
  - We constructed a **quantum circuit for an exact unitary t-design**.
    - Group representation approach (**Gelfand pair** and **zonal spherical functions**).
  - Due to the inductive construction, it requires $O(2^{n\sqrt{t}})$ **unitary gates**, which is **inefficient**.
  - **Challenge**: quantum circuits with poly(n, t) unitary gates? or prove that this is impossible!
- **Final remark**
  - **Group representation** is everywhere in quantum information! Welcome to join!!

for your attention

[E. Bannai, YN, T. Okuda, and D. Zhao, Advances in Mathematics Vol. 405, 108457 (2022).]
[YN, D. Zhao, T. Okuda, E. Bannai, and et. al., PRX Quantum **2**, 030339 (2021).]