Introduction to 3+1 and BSSN formalism

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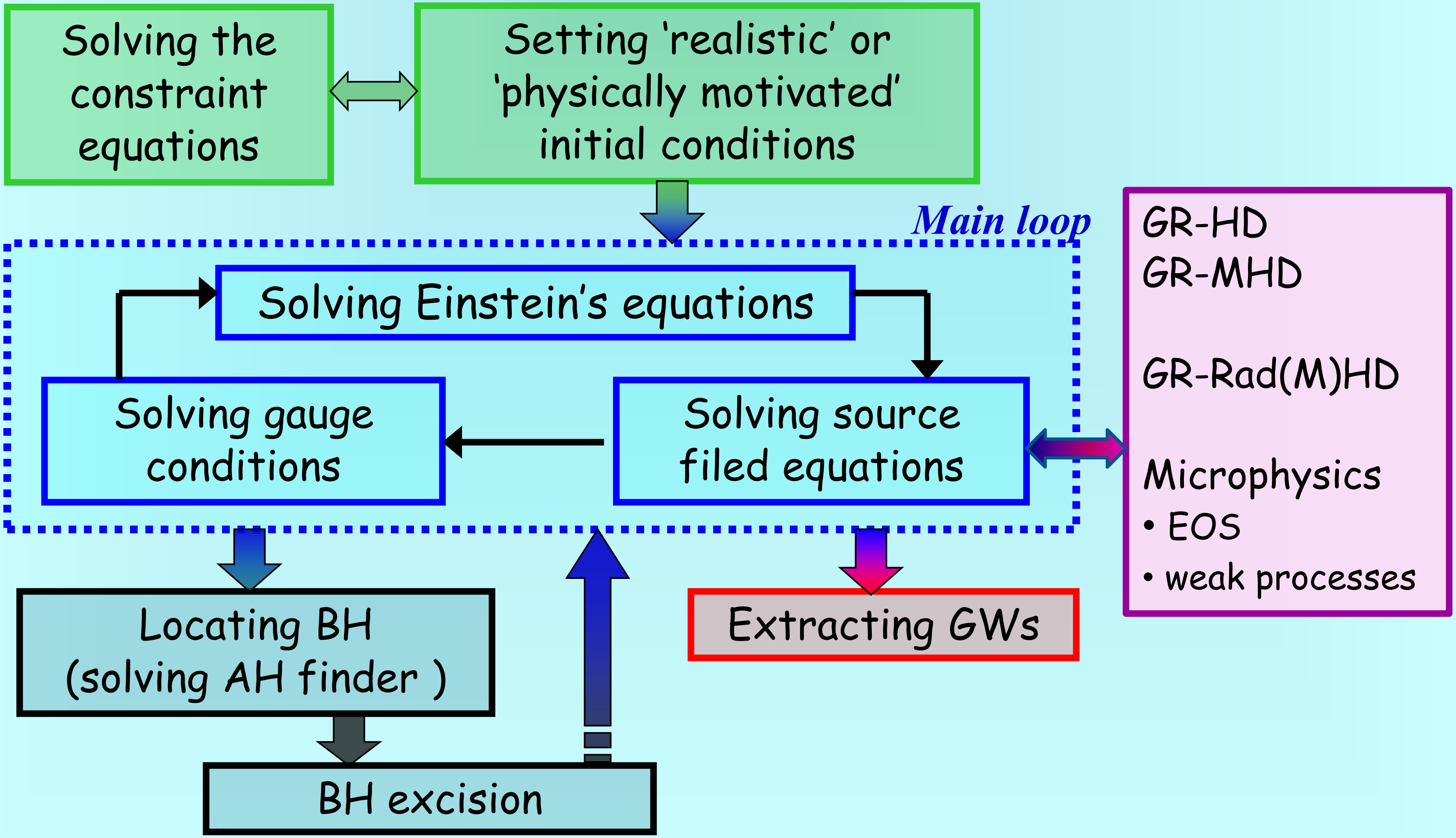
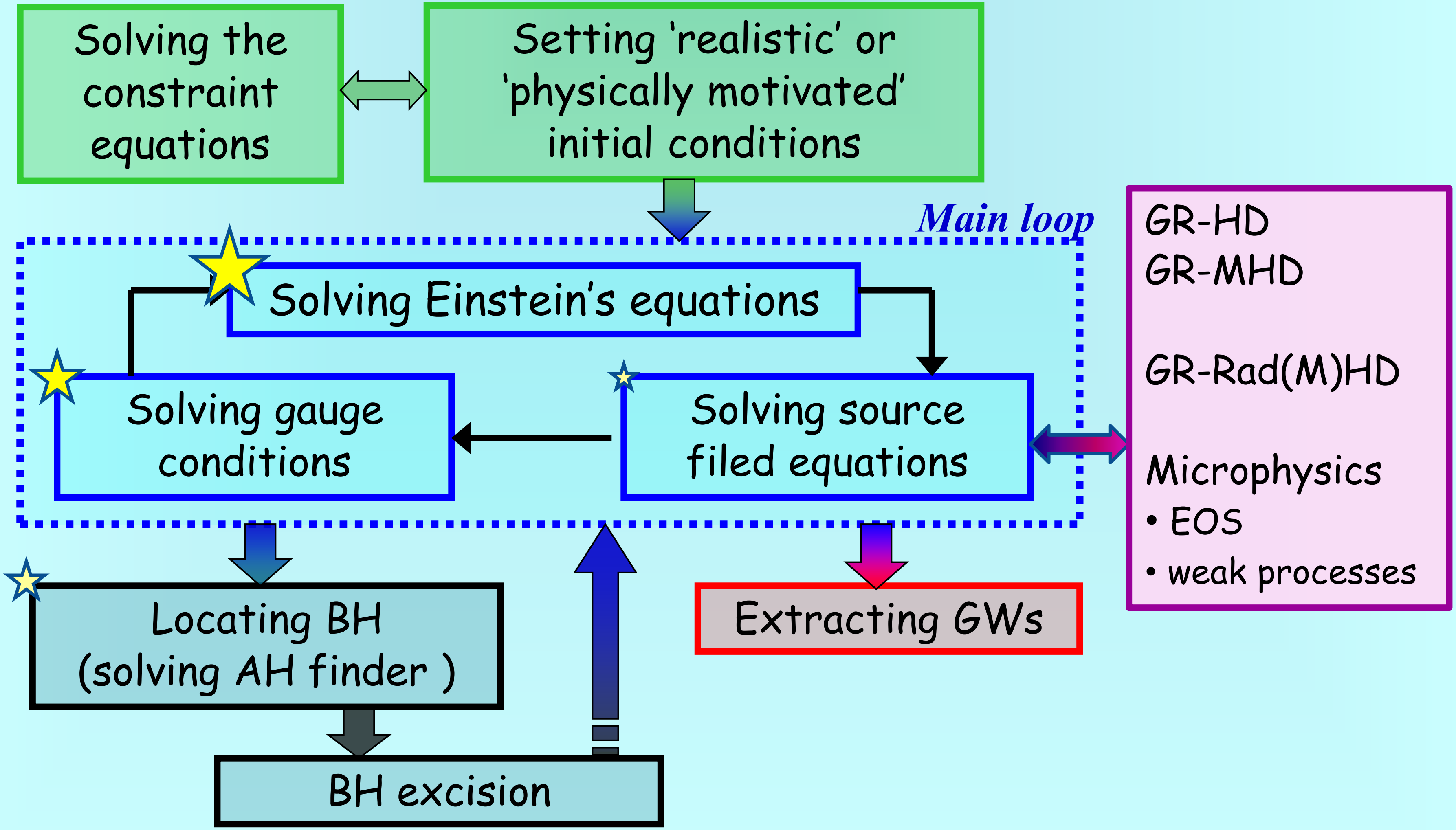
Overview of numerical relativity



Scope of this lecture



The Goal

- The main goal that we are aiming at:

  "To Derive Einstein's equations in BSSN formalism"

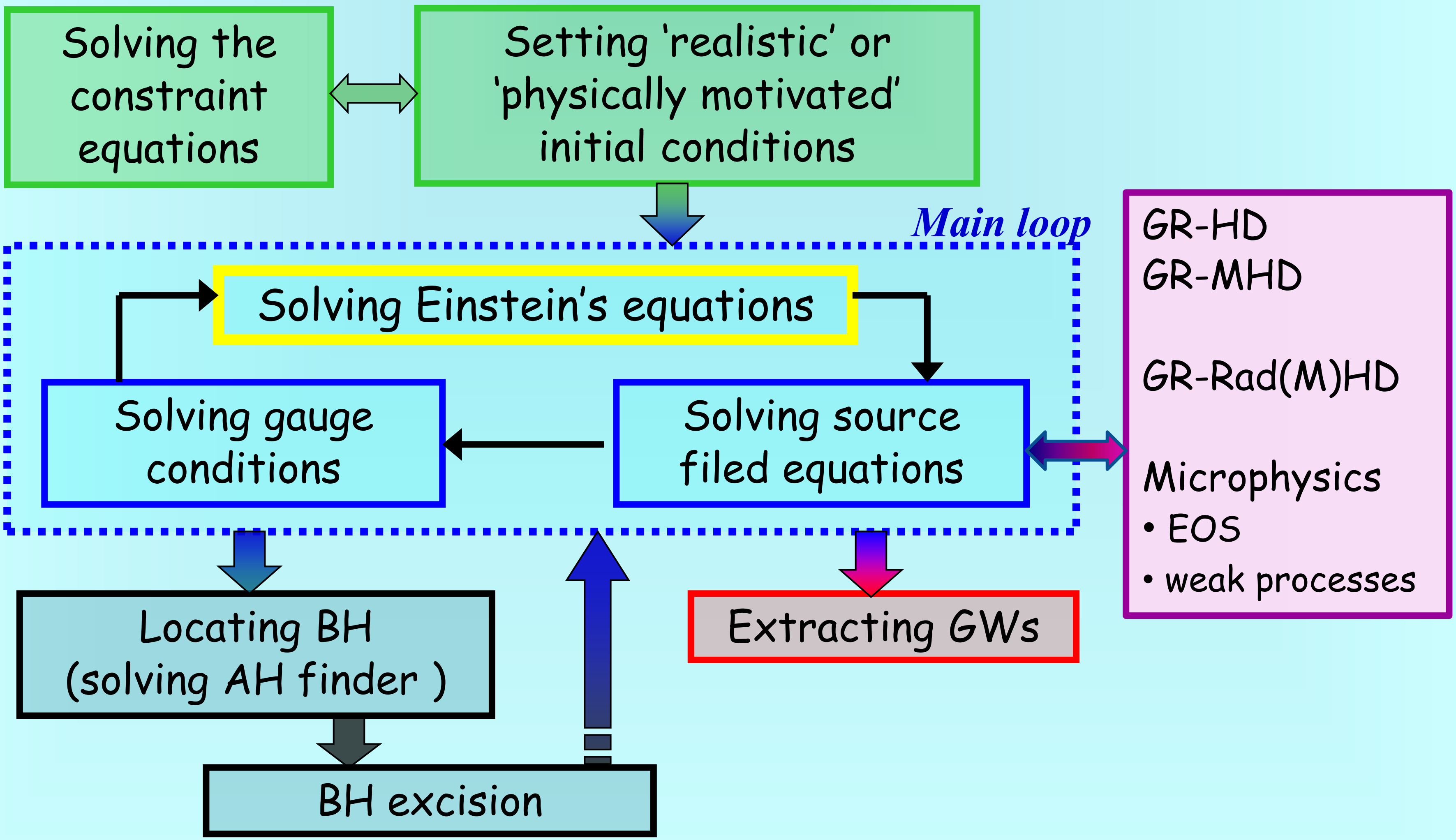
Notation and Convention

- The signature of the metric: $(-+++)$
- (We will use the abstract index notation)
  - e.g. *Wald (1984);* see also *Penrose, R. and Rindler, W. Spinors and spacetime vol.1, Cambridge Univ. Press (1987)*
- Geometrical unit $c=G=1$
- symmetric and anti-symmetric notations

$$T_{(a_1\ldots a_n)} \equiv \frac{1}{n!}\sum_{\pi} T_{a_{\pi(1)}\ldots a_{\pi(n)}}, \qquad T_{[a_1\ldots a_n]} \equiv \frac{1}{n!}\sum_{\pi} \mathrm{sgn}(\pi) T_{a_{\pi(1)}\ldots a_{\pi(n)}}$$

$$T_{(ab)} = \frac{1}{2}(T_{ab}+T_{ba}), \qquad T_{[ab]} = \frac{1}{2}(T_{ab}-T_{ba})$$

Overview of numerical relativity



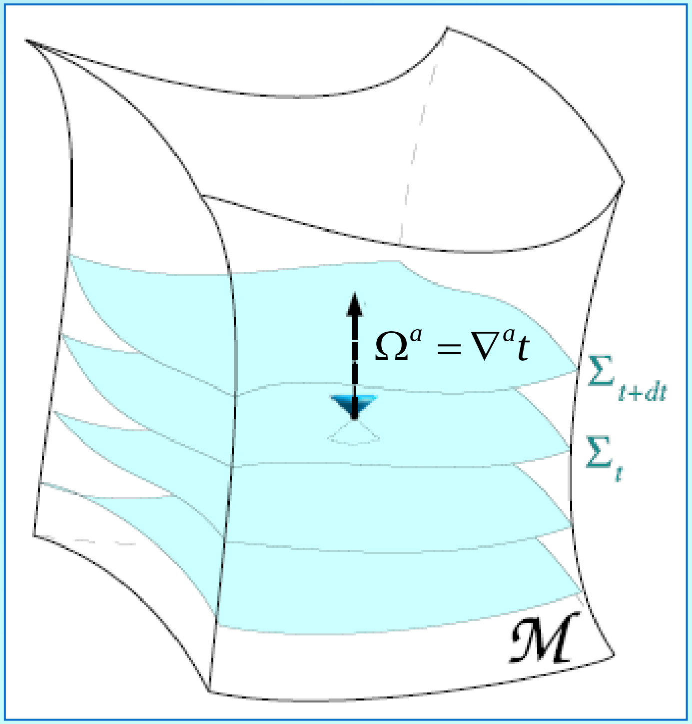
Solving Einstein's equations on computers

- Einstein's equations in full covariant form are a set of coupled partial differential equations
  - The solution, metric g_{ab}, is not a dynamical object and represents the full geometry of the spacetime just as the metric of a two-sphere does
  - To reveal the dynamical nature of Einstein's equations, we must break the 4D covariance and exploit the special nature of time
- One method is **3+1 decomposition** in which spacetime manifold and its geometry (graviational fields) are divided into a sequence of 'instants' of time
- Then, Einstein's equations are posed as a Cauchy problem which can be solved numerically on computers

3+1 decomposition of spacetime manifold

- Let us start to introduce foliation or slicing in the spacetime manifold M
- **Foliation $\{\Sigma\}$** of M is a family of **slices** (spacelike hypersurfaces) which do not intersect each other and fill the whole of M
  - In a globally hyperbolic spacetime, each Σ is a Cauchy surface which is parameterized by a global time function, t , as Σ_t
- Foliation is characterized by the gradient one-form

$$\Omega_a = \nabla_a t, \quad \nabla_{[a}\Omega_{b]} = 0$$



The lapse function

- The norm of Ω_a is related to a function called "**lapse function**", $\alpha(x^a)$, as :

$$g^{ab}\Omega_a\Omega_b = g^{ab}\nabla_a t\nabla_b t = -\frac{1}{\alpha^2}$$

  - *As we shall see later, the lapse function characterize the proper time between the slices*
- Also let us introduce the normalized one-form :

$$n_a = -\alpha\Omega_a, \quad g^{ab}n_a n_b = -1$$

  - the negative sign is introduced so that the direction of n corresponds to the direction to which t increases
  - n^a is the unit normal vector to Σ

The spatial metric of Σ : γ_{ab}

- The **spatial metric γ_{ab}** induced by g_{ab} onto Σ is defined by

$$\gamma_{ab} = g_{ab} + n_a n_b$$
$$\gamma^{ab} = g^{ac} g^{bd} \gamma_{cd} = g^{ab} + n^a n^b$$

- Using this 'induced' metric, a tensor on M is decomposed into two parts: components tangent and normal to Σ
- The tangent-projection operator is defined as

$$\perp^a_b = \delta^a_b + n^a n_b$$

  - The normal-projection operator is $\mathrm{N}^a_b = -n^b n_a = \delta^a_b - \perp^a_b$
- Then, projection of a tensor into Σ is defined by

$$\perp T^{a_1 \ldots a_r}{}_{b_1 \ldots b_s} \equiv \perp^{a_1}_{c_1} \ldots \perp^{a_r}_{c_r} \perp^{b_1}_{d_1} \ldots \perp^{b_s}_{d_s} T^{c_1 \ldots c_r}{}_{d_1 \ldots d_s}$$

  - It is easy to check $\perp g_{ab} = \perp^c_a \perp^d_b g_{cd} = \gamma_{ab}$

Covariant derivative associated with γ_{ab}

- **Covariant derivative** *acting on spatial tensors* is defined by

$$
\begin{aligned}
D_e T^{a_1 \dots a_r}{}_{b_1 \dots b_s} &\equiv \perp \nabla_e T^{a_1 \dots a_r}{}_{b_1 \dots b_s} \\
&= \perp_e^f \perp_{c_1}^{a_1} \dots \perp_{c_r}^{a_r} \perp_{d_1}^{b_1} \dots \perp_{d_s}^{b_s} \nabla_f T^{c_1 \dots c_r}{}_{d_1 \dots d_s}
\end{aligned}
$$

- The covariant derivative must satisfy the following conditions
  - It is a linear operator : (obviously holds from linearity of ∇)
  - Torsion free : $D_a D_b f = D_b D_a f$, (easy to check by direct calculation)
  - Compatible with the metric : $D_c g_{ab} = 0$, (easy to check also)
  - Leibnitz's rule holds :

$$
\begin{aligned}
D_a(v^c w_c) &= \perp_a^b \nabla_b (v^c w_c) = \perp_a^b v^d \delta_d^c \nabla_b w_c + \perp_a^b w_d \delta_c^d \nabla_b v^c \\
&= \perp_a^b v^d (\perp_d^c + \mathrm{N}_d^c) \nabla_b w_c + \perp_a^b w_d (\perp_c^d + \mathrm{N}_c^d) \nabla_b v^c \\
&= v^c D_a w_c + w_d D_a v^d + \perp_a^b (\mathrm{N}_d^c v^d \nabla_b w_c + \mathrm{N}_c^d w_d \nabla_b v^c) \\
&= v^c D_a w_c + w_d D_a v^d \qquad \text{for} \quad \mathrm{N}_d^c v^d = \mathrm{N}_c^d w_d = 0
\end{aligned}
$$

Intrinsic and extrinsic curvature for Σ

- The **Riemann tensor for the slice Σ** is defined by

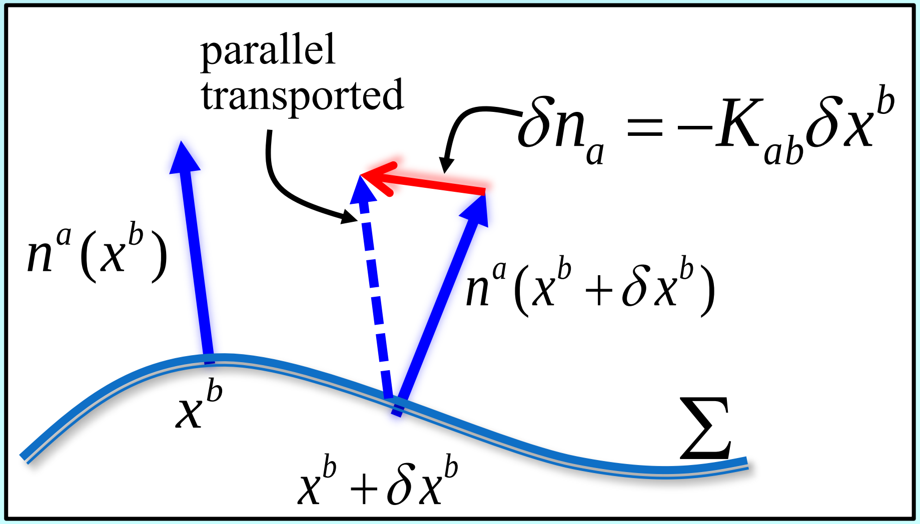
$$(D_a D_b - D_b D_a) v_c = v_d R^d{}_{abc}$$

- An other curvature tensor, **the extrinsic curvature for Σ** is defined by

$$K_{ab} \equiv -\perp \nabla_{(a} n_{b)} , \qquad n^a \text{ : unit normal to } \Sigma$$

  - extrinsic curvature provides information on how much the normal direction changes and hence, how Σ is curved
  - the antisymmetric part vanish due to Frobenius's theorem : "For unit normal n^a to a slice $n_{[a} \nabla_b n_{c]} = 0$ "

$$0 = \perp 3(n^a n_{[a} \nabla_b n_{c]}) = -\perp \nabla_{[b} n_{c]}$$



The other expressions of K_{ab}

- First, note that

$$\nabla_a n_b = \delta_a^c \delta_b^d \nabla_c n_d = (\perp_a^c + N_a^c)(\perp_b^d + N_b^d)\nabla_c n_d$$
$$= \perp \nabla_a n_b - n_a \perp a_b = -K_{ab} - n_a a_b$$

  - where a_b is the acceleration of n^b which is purely spatial

$$n^b a_b = n^a n^b \nabla_a n_b = n^a \nabla_a (n_b n^b) = 0$$

- Because the extrinsic curvature is symmetric, we have

$$K_{ab} = -\nabla_{(a} n_{b)} - n_{(a} a_{b)} = -\frac{1}{2} \mathcal{L}_n (g_{ab} + n_a n_b) = -\frac{1}{2} \mathcal{L}_n \gamma_{ab}$$

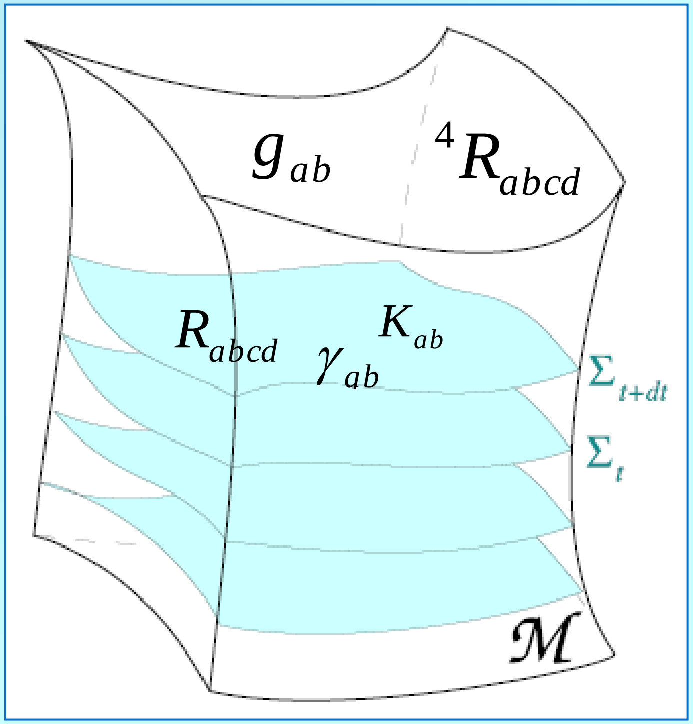
  - where $\mathcal{L}_n$ is the Lie derivative with respect to n
  - Also, we simply have $K_{ab} = -\perp \nabla_{(a} n_{b)} = -\frac{1}{2} \perp \mathcal{L}_n g_{ab}$

- Thus the extrinsic curvature is related to the "velocity" of the spatial metric γ_{ab}

3+1 decomposition of 4D Riemann tensor

- Geometry of a slice Σ is described by γ_{ab} and K_{ab}
  - γ_{ab} and K_{ab} represent the "instantaneous" gravitational fields in Σ
- In order that the foliation {Σ} to "fits" the spacetime manifold, γ_{ab} and K_{ab} must satisfy certain conditions known as Gauss, Codazzi, and Ricci relations
  - They are related to 3+1 decomposition of Einstein's equations
- These equations are obtained by taking the projections of the 4D Riemann tensor

$$\perp {}^4R_{abcd}\,,\quad \perp {}^4R_{abcd}n^d\,,$$
$$\perp {}^4R_{abcd}n^b n^d$$



Gauss relation: *spatial projection to Σ*

- Let us calculate the spatial Riemann tensor

$$D_a D_b w_c = \perp \nabla_a (\perp \nabla_b w_c)$$
$$= \perp \nabla_a \nabla_b w_c + \perp (\nabla_b w_d)(\nabla_a \perp_c^d) + \perp (\nabla_d w_c)(\nabla_a \perp_b^d)$$
$$= \perp \nabla_a \nabla_b w_c - \perp K_{ac} K_b^d w^d - \perp K_{ab} n^d \nabla_d w_c$$

  - where we used (also note that $\perp n$ vanishes if n is uncontracted)

$$\nabla_a \perp_b^d = n_a (\nabla_b n^d) + n^d (\nabla_a n_b) = -n_b (K_a^d + n_a a^d) - n^d (K_{ab} + n_a a_b)$$

- Then we obtain the Gauss relation

$$(D_a D_b - D_b D_a) w_c = R_{abc}{}^d w_d = \perp {}^4R_{abc}{}^d w_d - K_{ac} K_b^d w_d + K_{bc} K_a^d w_d$$
$$\perp {}^4R_{abcd} = R_{abcd} + K_{ac} K_{bd} - K_{ad} K_{bc}$$

  - The contracted Gauss relations are

$$\perp {}^4R_{ac} + \perp {}^4R_{abcd} n^b n^d = R_{ac} + K K_{ac} - K_{ab} K_c^b$$

$${}^4R + 2\,{}^4R_{ab} n^a n^b = R + K^2 - K_{ab} K^{ab}$$

Codazzi relation : *mixed projection to Σ and $\boldsymbol{n}$*

- Next, let us consider the "mixed" projection

$$\perp {}^4R_{abc}{}^d n_d = \perp(\nabla_a \nabla_b n_c - \nabla_b \nabla_a n_c)$$

  - where the right hand side is calculated as

$$\begin{aligned}\perp \nabla_a \nabla_b n_c &= \perp \nabla_a(-K_{bc} - n_b a_c) = -D_a K_{bc} - \perp a_c \nabla_a n_b \\ &= -D_a K_{bc} + a_c K_{ab}\end{aligned}$$

- Then we obtain the Codazzi relation

$$\perp {}^4R_{abc}{}^d n_d = D_b K_{ac} - D_a K_{bc}$$

  - The contracted Codazzi relation is

$$\perp {}^4R_{ab} n^b = D_a K - D_b K_a^b$$

Gauss and Codazzi relations

- Note that the Gauss and Codazzi relations depend only on the spatial metric γ_{ab}, the extrinsic curvature K_{ab}, and their spatial derivatives
- This implies that **the Gauss-Codazzi relations represent integrability conditions** that γ_{ab} and K_{ab} must satisfy for any slice to be embedded in the spacetime manifold
- The Gauss-Codazzi relations are directly associated with the constraint equations of Einstein's equation

Ricci relation (1)

- Let us start from the following equation

$$
\begin{aligned}
\perp {}^4R_{abcd} n^b n^d &= \perp n^b (\nabla_a \nabla_b n_c - \nabla_b \nabla_a n_c) \\
&= \perp n^b [-\nabla_a (K_{bc} + n_b a_c) + \nabla_b (K_{ac} + n_a a_c)] \\
&= \perp [K_{bc} \nabla_a n^b + \nabla_a a_c + n^b \nabla_b K_{ac} + a_c a_a + n_a n^b \nabla_b a_c] \\
&= \perp [K_{bc} (-K_a^b - n_a a^b) + \nabla_a a_c + n^b \nabla_b K_{ac} + a_c a_a + n_a n^b \nabla_b a_c] \\
&= -K_{bc} K_a^b + D_a a_c + \perp n^b \nabla_b K_{ac} + a_c a_a
\end{aligned}
$$

- The Lie derivative of K_{ac} is

$$
\perp \mathbf{L}_n K_{ac} = \perp (n^b \nabla_b K_{ac} + K_{ab} \nabla_c n^b + K_{cb} \nabla_a n^b) = \perp n^b \nabla_b K_{ac} - K_{ab} K_c^b - K_{cb} K_a^b
$$

- Then we obtain the Ricci relation

$$
\perp {}^4R_{abcd} n^b n^d = \perp \mathbf{L}_n K_{ac} + K_{ab} K_c^b + D_a a_c + a_c a_a
$$

Ricci relation (2)

- Note that the Lie derivative of $\boldsymbol{K_{ab}}$ is purely spatial, as

$$n^a \mathcal{L}_n K_{ab} = n^a n^c \nabla_c K_{ab} + n^a K_{ac} \nabla_b n^c + n^a K_{bc} \nabla_a n^c = -K_{ab} a^a + K_{bc} a^c = 0$$

- Thus the Ricci relation is

$$\perp {}^4R_{abcd} n^b n^d = \mathcal{L}_n K_{ac} + K_{ab} K^b_c + D_a a_c + a_c a_a$$

Ricci relation (3)

- The acceleration a^b is related to the lapse function α, as

$$a_b = n^c \nabla_c n_b = 2n^c \nabla_{[c} n_{b]} = -2n^c \nabla_{[c} \alpha \Omega_{b]} = -n^c (\Omega_b \nabla_c \alpha - \Omega_c \nabla_b \alpha)$$
$$= n^c n_b \nabla_c \ln \alpha + \delta_b^c \nabla_c \ln \alpha \ = \ D_b \ln \alpha$$

  - where we have used the fact that Ω is closed one-form

- Then the Ricci relation can be written as

$$\perp {}^4R_{abcd} n^b n^d = \mathcal{L}_n K_{ac} + K_{ab} K^b_c + D_a D_c \ln \alpha + D_a \ln \alpha D_a \ln \alpha$$
$$= \mathcal{L}_n K_{ac} + K_{ab} K^b_c + \frac{1}{\alpha} D_a D_c \alpha$$

- Furthermore, using the contracted Gauss relation

$$\perp {}^4R_{ac} + \perp {}^4R_{abcd} n^b n^d = R_{ac} + K K_{ac} - K_{ab} K^b_c$$

- we obtain

$$\perp {}^4R_{ac} = -\mathcal{L}_n K_{ac} - \frac{1}{\alpha} D_a D_c \alpha + R_{ac} + K K_{ac} - 2K_{ab} K^b_c$$

“Evolution vector” and αn^a

- What is the natural “evolution” vector ?
  - As stated before, the foliation is characterized by the closed one-form Ω
- **Dual vectors t^a to Ω will be the evolution vector : $\Omega_a t^a = 1$**
  - One simple candidate is $t^a = \alpha n^a$
  - Note that n^a is not the natural evolution vector because

$$\mathcal{L}_n \perp^a_b = n^c \nabla_c \perp^a_b - \perp^c_b \nabla_c n^a + \perp^a_c \nabla_b n^c = n^c \nabla_c (n^a n_b) + K^a_b - (K^a_b + n_b a^a))$$
$$= n^a a_b \neq 0$$

  - This means that the Lie derivative with respect to n^a of a tensor tangent to Σ is NOT a tensor tangent to Σ
  - On the other hand, $\mathcal{L}_{\alpha n} \perp^a_b = 0$ and any tensor field tangent to Σ is Lie transported by αn^a to a tensor field tangent to Σ

The shift vector

- We have a degree of freedom to add any spatial vector, called **"shift vector", β^a** to αn^a because $\Omega_a \beta^a = 0$
- Therefore the general **evolution vector is : $t^a = \alpha n^a + \beta^a$**
  - This freedom in the definition of the evolution time vector stems from the general covariance of Einstein's equations
- It is convenient to rewrite the Ricci relation in terms of the Lie derivative of the evolution time vector, as

$$\perp {}^4R_{ac} = -\frac{1}{\alpha}(\mathcal{L}_t - \mathcal{L}_\beta) K_{ac} - \frac{1}{\alpha} D_a D_c \alpha + R_{ac} + K K_{ac} - 2 K_{ab} K^b_c$$

  - where we have used

$$\mathcal{L}_t K_{ab} = \mathcal{L}_{\alpha n} K_{ab} + \mathcal{L}_\beta K_{ab} = \alpha n^c \nabla_c K_{ab} + K_{ac} \nabla_b (\alpha n^c) + K_{bc} \nabla_a (\alpha n^c) + \mathcal{L}_\beta K_{ab} = \alpha \mathcal{L}_n K_{ab} + \mathcal{L}_\beta K_{ab}$$

3+1 decomposition of Einstein's equations (1)

- *Decomposition of T_{ab}*

- Now we proceed 3+1 decomposition of Einstein's equations

$$G_{ab} = {}^4R_{ab} - \frac{1}{2}{}^4R g_{ab} = 8\pi T_{ab}$$

using the Gauss, Codazzi, and Ricci relations

- To do it, let us decompose the stress-energy tensor as

$$T_{ab} = E n_a n_b + 2P_{(a} n_{b)} + S_{ab}$$

  - where $E \equiv n_a n_b T_{ab}$, $P_a \equiv -\perp(n^b T_{ab})$, and $S_{ab} \equiv \perp T_{ab}$ are the energy density, momentum density/momentum flux, and stress tensor of the source field measured by the Eulerian observer
  - the trace is $T = S - E$

- We shall also use Einstein's equations in the form of

$$^4R_{ab} = 8\pi\left(T_{ab} - \frac{1}{2} g_{ab} T\right)$$

3+1 decomposition of Einstein's equations (2)

- *Hamiltonian constraint*

- We first project Einstein's equation into the direction perpendicular to Σ to obtain

$$ {}^4R_{ab}n^a n^b + \frac{1}{2}{}^4R = 8\pi E $$

  - For the left-hand-side, we use the contracted Gauss relation

$$ {}^4R + 2\,{}^4R_{ab}n^a n^b = R + K^2 - K_{ab}K^{ab} $$

- We finally obtain the **Hamiltonian constraint**

$$ R + K^2 - K_{ab}K^{ab} = 16\pi E $$

  - This is a single elliptic equation which must be satisfied everywhere on the slice

3+1 decomposition of Einstein's equations (3)

- *Momentum constraint*

- Similary, "mixed" projection of Einstein's equations gives

$$\perp {}^4R_{ab} n^b = -8\pi P_a$$

  - Using the contracted Codazzi relation

$$\perp {}^4R_{ab} n^b = D_a K - D_b K^b_a$$

- We reach the **momentum constraint**

$$D_b K^b_a - D_a K = 8\pi P_a$$

  - includes 3 elliptic equations

3+1 decomposition of Einstein's equations (4)

- *Evolution equations*

- The evolution part of Einstein's equations is given by the full projection onto Σ of Einstein's equations :

$$\perp {}^4R_{ab} = \perp 8\pi\left(S_{ab} + 2n_{(a}P_{b)} - \frac{1}{2}\gamma_{ab}(S-E) + \frac{1}{2}n_a n_b (S+E) \right) = 8\pi\left(S_{ab} - \frac{1}{2}\gamma_{ab}(S-E) \right)$$

  - Using a version of the Ricci relation

$$\perp {}^4R_{ac} = -\frac{1}{\alpha}(\boldsymbol{L}_t - \boldsymbol{L}_\beta)K_{ac} - \frac{1}{\alpha}D_a D_c \alpha + R_{ac} + KK_{ac} - 2K_{ab}K^b_c$$

- We obtain the **evolution equation for K_{ab}**

$$(\boldsymbol{L}_t - \boldsymbol{L}_\beta)K_{ab} = -D_a D_b \alpha + \alpha[R_{ab} + KK_{ab} - 2K_{ac}K^c_b] - 8\pi\alpha\left(S_{ab} - \frac{1}{2}\gamma_{ab}(S-E) \right)$$

- The **evolution equation for γ_{ab}** is given by an expression of K_{ab}

$$(\boldsymbol{L}_t - \boldsymbol{L}_\beta)\gamma_{ab} = -2\alpha K_{ab}$$

Summary of 3+1 decomposition

- Einstein's equations are 3+1 decomposed as follows

$$G_{ab}n^a n^b = 8\pi T_{ab} n^a n^b$$

Gauss rel. →

Hamiltonian constraint

$$R + K^2 - K_{ab}K^{ab} = 16\pi E$$

$$\perp G_{ab} n^b = 8\pi \perp T_{ab} n^b$$

Codazzi rel. →

Momentum constraint

$$D_b K^b_a - D_a K = 8\pi P_a$$

$$\perp G_{ab} = 8\pi \perp T_{ab}$$

Ricci rel. →

Evolution Eq. of K_{ab}

$$(\mathcal{L}_t - \mathcal{L}_\beta) K_{ab} = -D_a D_b \alpha + \alpha [R_{ab} + K K_{ab} - 2 K_{ac} K^c_b] - 4\pi\alpha\left(2S_{ab} - \gamma_{ab}(S - E)\right)$$

Definition of K_{ab}

$$K_{ab} \equiv - \perp \nabla_{(a} n_{b)}$$

→

Evolution Eq. of γ_{ab}

$$(\mathcal{L}_t - \mathcal{L}_\beta)\gamma_{ab} = -2\alpha K_{ab}$$

- 3+1 decomposition of the stress-energy tensor

$$T_{ab} = E n_a n_b + 2 P_{(a} n_{b)} + S_{ab}$$

$$E \equiv n_a n_b T_{ab}, \quad P_a \equiv - \perp (n^b T_{ab}), \quad S_{ab} \equiv \perp T_{ab}$$

Similarity with the Maxwell's equations

$$\nabla_b F^{ab} = 4\pi \mathrm{J}^{\,a} = 4\pi(\rho_e n^a + J^a)$$

$$\nabla_{[a} F_{bc]} = 0 \quad \Leftrightarrow \quad \nabla_a {}^*F^{ab} = 0$$

$$F^{ab} = n^a E^b - n^b E^a + \varepsilon^{abc} B_c$$

$$^*F^{ab} = n^a B^b - n^b B^a - \varepsilon^{abc} E_c$$

$$E^a n_a = 0, \quad B^a n_a, \quad J^a n_a = 0$$

$$n_a \nabla_b F^{ab} = n_a (4\pi \mathrm{J}^{\,a})$$

$$n_a \nabla_b {}^*F^{ab} = 0$$

Normal projection →

$$D_a E^a = 4\pi\rho_e$$

$$D_a B^a = 0$$

Gauss's law, No monopole
no time derivatives of E, B

Constraint equations

$$\perp \nabla_b F^{ab} = \perp (4\pi \mathrm{J}^{\,a})$$

$$\perp \nabla_b {}^*F^{ab} = 0$$

Spatial projection →

$$\left(\partial_t - \mathcal{L}_\beta\right) E^a = \varepsilon^{abc} D_b(\alpha B_c) - 4\pi\alpha J^a + \alpha K E^a$$

$$\left(\partial_t - \mathcal{L}_\beta\right) B^a = -\varepsilon^{abc} D_b(\alpha E_c) + \alpha K B^a$$

Faraday's law, Ampere's law

Evolution equations

Evolution of constraints

- It can be shown that the "evolution" equations for the Hamiltonian (C_H) and Momentum (C_M) constraints becomes

$$(\partial_t - \mathcal{L}_\beta)C_H = -D_k(\alpha C_M^k) - C_M^k D_k\alpha + \alpha K(2C_H - F) + \alpha K^{ij}F_{ij}$$
$$(\partial_t - \mathcal{L}_\beta)C_M^i = -D_j(\alpha F^{ij}) + 2\alpha K_j^i C_M^j + \alpha K C_M^i + \alpha D^k(F - C_H) + (F - 2H)D^i\alpha$$

  - Where F_{ij} is the spatial projection : the evolution equation

$$F_{ab} \equiv \perp\left[{}^4R_{ab} - 8\pi\left(T_{ab} - \frac{1}{2}Tg_{ab}\right)\right]$$

- The evolution equations for the constraints show that the constraints are "preserved" or "satisfied" , if
  - They are satisfied initially ($C_H = C_M = 0$)
  - The evolution equation is solved correctly ($F_{ab}=0$)

Coordinate-basis vectors

- Let us choose the coordinate basis vectors
- First, we choose the evolution timelike vector t^a as the time-basis vector : $t^a = (e_0)^a$
- The spatial basis vectors are chosen such that $\Omega_a (e_i)^a = 0$
  - The spatial basis vectors are Lie transported along t^a :

    $$\mathcal{L}_t (e_i)^a = t^b \nabla_b (e_i)^a - (e_i)^b \nabla_b t^a = [t, e_i]^a = 0$$
  - $(e_i)^a$ remains purely spatial because

    $$\mathcal{L}_t (\Omega_a (e_i)^a) = (\mathcal{L}_t \Omega_a)(e_i)^a - \Omega_a \mathcal{L}_t (e_i)^a = (\mathcal{L}_t \Omega_a)(e_i)^a$$
    $$= (t^b \nabla_b \Omega_a + \Omega_b \nabla_a t^b)(e_i)^a = 2 t^b \nabla_{[b} \Omega_{a]} (e_i)^a = 0$$
- $(e_\mu)^a$ constitute the commutable coordinate basis
  - Then $\mathcal{L}_t = \partial_t$
  - We define the dual basis vectors by $(\xi^\mu)_a : \ (\xi^\mu)_a (e_\mu)^a$

Components of geometrical quantities (1)

- Now we have set the coordinate basis we proceed to calculate the components of geometrical quantities
- Because the evolution time vector is the time-coordinate basis we have $t^a = t^\mu (e_\mu)^a = (e_0)^a \;\Rightarrow\; t^\mu = [1\,0\,0\,0]$
- From the property of the spatial basis, we have

$$n_i = 0, \;\; \because\; 0 = \Omega_a (e_i)^a = \Omega_\mu \delta_i^\mu = \alpha n_i$$

  - Then, 0th contravariant components of spatial tensors vanish

$$\beta^\mu = [0 \;\; \beta^i], \;\; \gamma^{\mu\nu} = \begin{bmatrix} 0 & 0 \\ 0 & \gamma^{ij} \end{bmatrix}, \;\; K^{\mu\nu} = \begin{bmatrix} 0 & 0 \\ 0 & K^{ij} \end{bmatrix}$$

- From the definition of the time vector and normalization condition of n^a, we obtain

$$t^a = \alpha n^a + \beta^a \;\Rightarrow\; n^\mu = [\alpha^{-1} \;\; -\alpha^{-1}\beta^i]$$

$$n^a n_a = -1 \;\Rightarrow\; n_\mu = [-\alpha \;\; 0 \;\; 0 \;\; 0]$$

Components of geometrical quantities (2)

- From the definition of spatial metric, we have

$$g^{ab} = \gamma^{ab} - n^a n^b \;\Rightarrow\; g^{\mu\nu} = \begin{bmatrix} -\alpha^{-2} & \alpha^{-2}\beta^i \\ \alpha^{-2}\beta^i & \gamma^{ij} - \alpha^{-2}\beta^i\beta^j \end{bmatrix}$$

$$g_{ab} = \gamma_{ab} - n_a n_b \;\Rightarrow\; g_{ij} = \gamma_{ij}$$

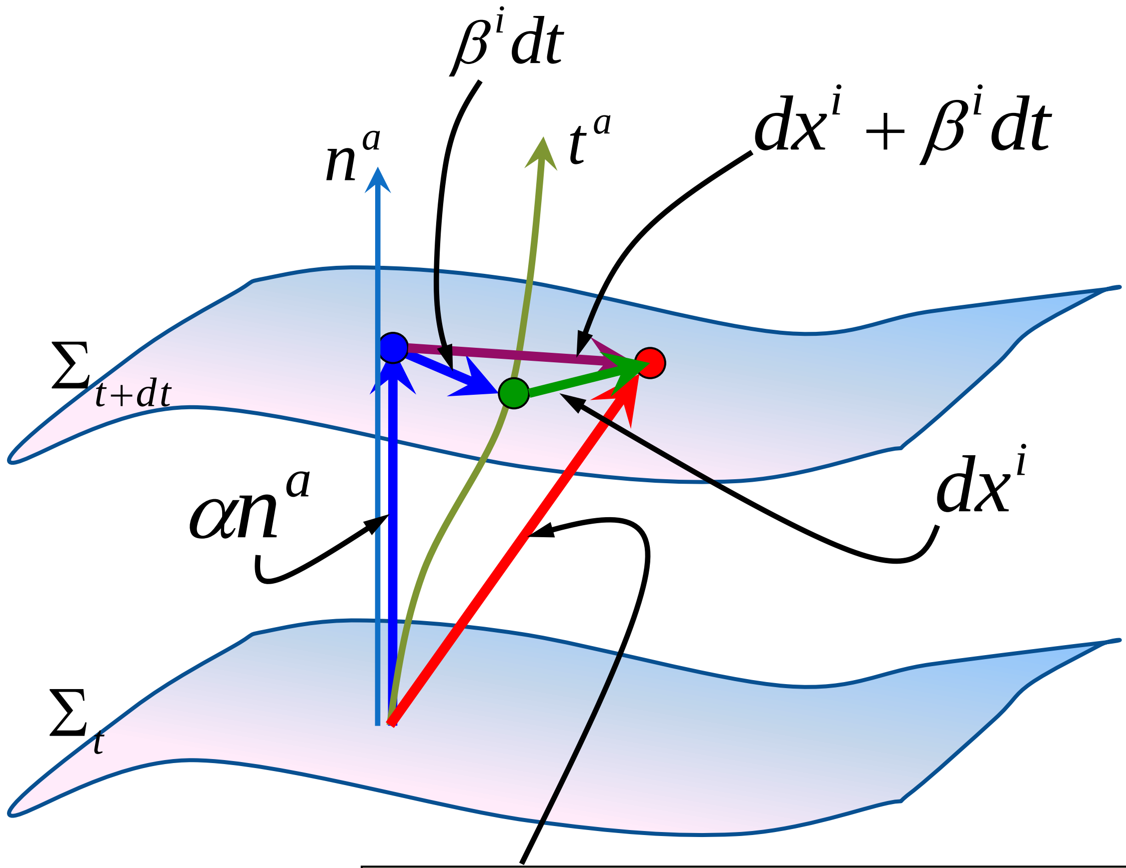
- We here note that from the spatial component of the following equation, we have

$$\gamma^{\mu\sigma} g_{\sigma\nu} = (g^{\mu\sigma} + n^\mu n^\sigma) g_{\sigma\nu} = \delta^\mu_\nu + n^\mu \delta^0_\nu \;\Rightarrow\; \gamma^{ik}\gamma_{kj} = \delta^i_j$$

- This means that the indices of spatial tensors can be lowered and raised by the spatial metric
- Then, from the inverse of g^{ab}, we obtain

$$g_{\mu\nu} = \begin{bmatrix} -\alpha^2 + \beta_k\beta^k & \beta_i \\ \beta_i & \gamma_{ij} \end{bmatrix}, \quad ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

An intuitive interpretation



- The lapse function measures proper time between two adjacent slices
- The shift vector gives relation of the spatial origin between slices

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

Conformal decomposition

- The importance of the conformal decomposition in the time evolution problem was first noted by *York* (*PRL 26, 1656 (1971); PRL 28, 1082 (1972)*)
  - He showed that the two degrees of freedom of the gravitational field are carried by the conformal equivalence classes of 3-metric, which are related each other by the conformal transformation :

$$\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij}$$

  - In the initial data problems, the conformal decomposition is a powerful tool to solve the constraint equations, as studied by *York* and *O'Murchadha* (*J. Math. Phys. 14, 456 (1973); PRD 10, 428 (1974)*) (see for reviews , e.g., *Cook, G.B., Living Rev. Rel. 3, 5 (2000); Pfeiffer, H. P. gr-qc/0412002*)
- In the following, we shall derive conformal decomposition of Einstein's equations

"Conformal" decomposition of Ricci tensor (1)

- The covariant derivative associated with the conformal metric is characterized by

$$\tilde{D}_c \tilde{\gamma}_{ab} = 0$$

- The two covariant derivatives are related by (e.g. Wald)

$$D_k T^i_j = \tilde{D} T^i_j + C^i_{kl} T^l_j - C^l_{kj} T^i_l$$

  - where C^i_{jk} is a tensor defined by difference of Christoffel symbols

$$C^k{}_{ij} \equiv \Gamma^k{}_{ij} - \tilde{\Gamma}^k{}_{ij} = \frac{1}{2} \gamma^{kl} (\tilde{D}_i \gamma_{lj} + \tilde{D}_j \gamma_{il} - \tilde{D}_l \gamma_{ij})$$
$$= 2(\delta^k_i D_j \ln \psi + \delta^k_j D_i \ln \psi - \tilde{\gamma}_{ij} \tilde{D}^k \ln \psi)$$

- By a straightforward calculation, we can show (e.g Wald)

$$R_{ij} v^j = (D_j D_i - D_i D_j) v^j = (\tilde{D}_j \tilde{D}_i - \tilde{D}_i \tilde{D}_j) v^j + (\tilde{D}_k C^k_{ij} - \tilde{D}_i C^k_{kj} + C^l_{lk} C^k_{ij} - C^k_{il} C^l_{kj}) v^j$$
$$= \tilde{R}_{ij} v^j + (\tilde{D}_k C^k_{ij} - \tilde{D}_i C^k_{kj} + C^l_{lk} C^k_{ij} - C^k_{il} C^l_{kj}) v^j$$

"Conformal" decomposition of Ricci tensor (2)

- Thus the Ricci tensor is decomposed into two parts, one which is the Ricci tensor associated with the conformal metric and one which contains the conformal factor ψ
  - More explicitly one can show (see e.g. Wald (1984))

$$
\begin{aligned}
R_{ij} = \tilde{R}_{ij} - 2\tilde{D}_i\tilde{D}_j \ln\psi - 2\tilde{\gamma}_{ij}\tilde{D}_k\tilde{D}^k \ln\psi \\
+ 4(\tilde{D}_i \ln\psi)(\tilde{D}_j \ln\psi) - 4\tilde{\gamma}_{ij}(\tilde{D}_k \ln\psi)(\tilde{D}^k \ln\psi) \\
\equiv \tilde{R}_{ij} + R^{\phi}_{ij}
\end{aligned}
$$

- Then, the Ricci scalar is decomposed as

$$
\begin{aligned}
R = \psi^{-4}[\tilde{R} - 8(\tilde{D}_k\tilde{D}^k \ln\psi + (\tilde{D}_k \ln\psi)(\tilde{D}^k \ln\psi))] \\
= \psi^{-4}\tilde{R} - 8\psi^{-5}\tilde{D}_k\tilde{D}^k\psi
\end{aligned}
$$

Conformal decomposition of extrinsic curvature

- The first step is to decompose K_{ij} into trace (K) and traceless (A_{ij}) parts as

$$K_{ij} = A_{ij} + \frac{1}{3}\gamma_{ij}K, \quad K^{ij} = A^{ij} + \frac{1}{3}\gamma^{ij}K$$

- Then, we perform the conformal decomposition of the traceless part as

$$A_{ij} = \psi^{4}\tilde{A}_{ij}, \qquad A^{ij} = \gamma^{ik}\gamma^{jl}A_{kl} = \psi^{-4}\tilde{A}_{ij}$$

- Under these conformal decompositions of the spatial metric and the extrinsic curvature, let us consider the conformal decomposition of Einstein's equation

"Conformal" decomposition of the evolution equations (0) – *an additional constraint*

- In the following, with BSSN reformulation in mind, we set the determinant of the conformal metric to be unity:

$$\tilde{\gamma} = \det \tilde{\gamma}_{ij} = 1$$

  - with this setting, the conformal factor becomes

$$\ln \psi = \frac{1}{12} \ln \gamma$$

  - In the BSSN formulation, the conformal factor is defined by $\phi = \ln \psi$ so that $\phi = \ln\gamma/12$
  - In the case that we do not impose the above condition to the background conformal metric, the equations derived in the following are modified slightly (for this, see *Gourgoulhon, E., gr-qc/0703035*)

"Conformal" decomposition of the evolution equations (1) : *the conformal factor*

- Let us start from the evolution equation of the spatial metric γ_{ij} :

$$(\mathcal{L}_t - \mathcal{L}_\beta)\gamma_{ab} = \mathcal{L}_{\alpha n}\gamma_{ab} = -2\alpha K_{ab}$$

- Taking the trace of this equation, we have

$$\gamma^{ab}\mathcal{L}_{\alpha n}\gamma_{ab} = -2\alpha K$$

  - Now we use an identity for any matrix A : $\det[\exp A] = \exp[\mathrm{tr}\, A]$
  - By setting $\gamma_{ij} = \exp A$ and taking the Lie derivative, we obtain

$$\mathcal{L}_{\alpha n}\gamma = \exp[\mathrm{tr}(\ln \gamma_{ij})]\mathcal{L}_{\alpha n}(\mathrm{tr}(\ln \gamma_{ij})) = \gamma\gamma^{ij}\mathcal{L}_{\alpha n}\gamma_{ij}$$

- Now we can derive the evolution equation for the conformal factor :

$$\gamma^{ij}\mathcal{L}_{\alpha n}\gamma_{ij} = \mathcal{L}_{\alpha n}\ln\gamma = 12\mathcal{L}_{\alpha n}\ln\psi = -2\alpha K$$

$$(\mathcal{L}_t - \mathcal{L}_\beta)\ln\psi = -\frac{1}{6}\alpha K$$

"Conformal" decomposition of the evolution equations (2) : *the conformal metric*

- Again, we start from the evolution equation for γ_{ij} :

$$(\mathcal{L}_t - \mathcal{L}_\beta)\gamma_{ab} = \mathcal{L}_{\alpha n}\gamma_{ab} = -2\alpha K_{ab}$$

- Substituting the decomposition of γ_{ij} and K_{ij}, we obtain

$$\psi^4(\mathcal{L}_t - \mathcal{L}_\beta)\tilde{\gamma}_{ij} + 4\psi^3\tilde{\gamma}_{ij}(\mathcal{L}_t - \mathcal{L}_\beta)\psi = -2\alpha\left(\psi^4\tilde{A}_{ij} + \frac{1}{3}\psi^4\tilde{\gamma}_{ij}K\right)$$

$$(\mathcal{L}_t - \mathcal{L}_\beta)\tilde{\gamma}_{ij} + 4\tilde{\gamma}_{ij}(\mathcal{L}_t - \mathcal{L}_\beta)\ln\psi = -2\alpha\left(\tilde{A}_{ij} + \frac{1}{3}\tilde{\gamma}_{ij}K\right)$$

- Now, we shall use the evolution equation for the conformal factor, and finally, we get

$$(\mathcal{L}_t - \mathcal{L}_\beta)\tilde{\gamma}_{ij} = -2\alpha\tilde{A}_{ij}$$

"Conformal" decomposition of the evolution equations (3) : *the inverse conformal metric*

- For the later purpose, let us derive the evolution equation for the inverse of the conformal metric
- It is easily obtained from the evolution equation for the conformal metric, as

$$\tilde{\gamma}^{ik}\tilde{\gamma}^{jl}(\mathcal{L}_t - \mathcal{L}_\beta)\tilde{\gamma}_{kl} = -2\alpha\tilde{A}^{ij}$$

$$\tilde{\gamma}^{ik}[(\mathcal{L}_t - \mathcal{L}_\beta)\delta_k^j - \tilde{\gamma}_{kl}(\mathcal{L}_t - \mathcal{L}_\beta)\tilde{\gamma}^{jl}] = -2\alpha\tilde{A}^{ij}$$

$$(\mathcal{L}_t - \mathcal{L}_\beta)\tilde{\gamma}^{ij} = 2\alpha\tilde{A}^{ij}$$

"Conformal" decomposition of the evolution equations (4a) : *the trace of the extrinsic curvature*

- We start from the evolution equation for K_{ij} :

$$\mathcal{L}_{\alpha n} K_{ij} = -D_i D_j \alpha + \alpha [R_{ij} + K K_{ij} - 2 K_{ik} K_j^k] + 4\pi\alpha(\gamma_{ij}(S-E) - 2S_{ij})$$

- We first simply take the trace of this equation

$$\mathcal{L}_{\alpha n} K - K_{ij} \mathcal{L}_{\alpha n} \gamma^{ij} = -D_i D^i \alpha + \alpha [R + K^2 - 2 K_{ij} K^{ij}] + 4\pi\alpha(3(S-E) - 2S)$$

- Here, let make use of the evolution equation for the inverse of the spatial metric, $(\mathcal{L}_t - \mathcal{L}_\beta)\gamma^{ab} = 2\alpha K^{ab}$

  then we obtain

$$\mathcal{L}_{\alpha n} K = -D_i D^i \alpha + \alpha [R + K^2] + 4\pi\alpha(S - 3E)$$

- Finally, **using the Hamiltonian constraint**, we obtain

$$(\mathcal{L}_t - \mathcal{L}_\beta) K = -D_i D^i \alpha + \alpha [K_{ij} K^{ij} + 4\pi(E+S)]$$

$$R + K^2 - K_{ab} K^{ab} = 16\pi E$$

"Conformal" decomposition of the evolution equations (4b) : *the trace of the extrinsic curvature*

- For convenience, let us express the right-hand-side in terms of the conformal quantities, as well as give a suggestion how to evaluate the derivative term :

$$D_k D^k \alpha = \frac{1}{\sqrt{\gamma}} \partial_k \left(\sqrt{\gamma} D^k \alpha \right) = \psi^{-6} \partial_k \left(\psi^6 \gamma^{jk} D_j \alpha \right) = \psi^{-6} \partial_k \left(\psi^2 \tilde{\gamma}^{jk} \partial_j \alpha \right)$$

$$K_{ij} K^{ij} = A_{ij} A^{ij} + \frac{1}{3} K^2 = \tilde{A}_{ij} \tilde{A}^{ij} + \frac{1}{3} K^2$$

"Conformal" decomposition of the evolution equations (5a) : *the traceless part of the extrinsic curvature*

- We start from the Lie derivative of K_{ij} :

$$\mathcal{L}_{\alpha n} K_{ij} = \mathcal{L}_{\alpha n} A_{ij} + \frac{1}{3}\gamma_{ij}\mathcal{L}_{\alpha n} K + \frac{1}{3} K \mathcal{L}_{\alpha n}\gamma_{ij}$$

- Substituting the following equations into this yields

$$\mathcal{L}_{\alpha n} K_{ij} = -D_i D_j \alpha + \alpha[R_{ij} + K K_{ij} - 2K_{ik}K^k_j] + 4\pi\alpha(\gamma_{ij}(S-E) - 2S_{ij})$$

$$\mathcal{L}_{\alpha n} K = -D_i D^i \alpha + \alpha[R + K^2] + 4\pi\alpha(S - 3E)$$

$$\mathcal{L}_{\alpha n}\gamma_{ij} = -2\alpha K_{ij}$$

$$\mathcal{L}_{\alpha n} A_{ij} = -(D_i D_j \alpha)^{\mathrm{TF}} + \alpha(R_{ij}{}^{\mathrm{TF}} - 8\pi S_{ij}{}^{TF}) + \alpha\left[\frac{5}{3}K K_{ij} - 2K_{ik}K^k_j - \frac{1}{3}K^2\gamma_{ij}\right]$$

  - where TF denotes the trace free part : $T_{ij}{}^{\mathrm{TF}} = T_{ij} - (1/3)\gamma_{ij}(\mathrm{tr}\,T)$
  - The terms that involve K in the right-hand-side can be written as

$$\frac{5}{3}K K_{ij} - 2K_{ik}K^k_j - \frac{1}{3}K^2\gamma_{ij} = \frac{1}{3}K A_{ij} - 2A_{ik}A^k_j = \psi^4\left[\frac{1}{3}K\tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}^k_j\right]$$

"Conformal" decomposition of the evolution equations (5b) : *the traceless part of the extrinsic curvature*

- We further proceed to decompose the left-hand-side :

$$\mathcal{L}_{\alpha n} A_{ij} = \psi^4 \left[\mathcal{L}_{\alpha n} \tilde{A}_{ij} + 4 \tilde{A}_{ij} \mathcal{L}_{\alpha n} \ln \psi \right] = \psi^4 \left[\mathcal{L}_{\alpha n} \tilde{A}_{ij} - \frac{2}{3} \alpha K \tilde{A}_{ij} \right]$$

- Combining all of the result, we finally reach

$$(\mathcal{L}_t - \mathcal{L}_\beta) \tilde{A}_{ij} = \psi^{-4} \left[-(D_i D_j \alpha)^{\mathrm{TF}} + \alpha (R_{ij}{}^{\mathrm{TF}} - 8\pi S_{ij}{}^{TF}) \right] + \alpha \left[K \tilde{A}_{ij} - 2 \tilde{A}_{ik} \tilde{A}^k_j \right]$$

  - We here note that the second-order covariant derivative of the lapse function may be calculated as

$$\begin{aligned} D_i D_j \alpha &= D_i \partial_j \alpha = \tilde{D}_i \partial_j \alpha - C^k_{ij} \partial_k \alpha \\ &= \left[\partial_i \partial_j \alpha - \tilde{\Gamma}^k_{ij} \partial_k \alpha \right] - 2 \left[2 \partial_{(i} \ln \psi \, \partial_{j)} \alpha - \tilde{\gamma}_{ij} \tilde{\gamma}^{kl} \partial_k \ln \psi \, \partial_l \alpha \right] \end{aligned}$$

  - NOTE: there is the same 2nd order derivative in $\boldsymbol{R}_{ij}{}^{\phi}$

"Conformal" decomposition of the constraint equations

- Let us turn now to consider the conformal decomposition of the constraint equations
- **Hamiltonian constraint**

$$R + K^2 - K_{ab}K^{ab} = 16\pi E$$

$$K_{ij}K^{ij} = \tilde{A}_{ij}\tilde{A}^{ij} + K^2/3$$
$$R = \psi^{-4}\tilde{R} - 8\psi^{-5}\tilde{D}_k\tilde{D}^k\psi$$

$$\tilde{D}_i\tilde{D}^i\psi - \frac{1}{8}\tilde{R}\psi + \left(\frac{1}{8}\tilde{A}_{ij}\tilde{A}^{ij} - \frac{1}{12}K^2 + 2\pi E\right)\psi^5 = 0$$

- **Momentum constraint**

$$D_b K^{ab} - D^a K = 8\pi P^a$$

$$D_j K^{ij} = D_j A^{ij} + D^i K/3$$
$$D_j A^{ij} = \tilde{D}_j A^{ij} + C^i_{jk}A^{kj} + C^j_{jk}A^{ik}$$
$$= \tilde{D}A^{ij} + 10 A^{ij}\tilde{D}_j \ln\psi$$
$$= \psi^{-4}\left[\tilde{D}_j\tilde{A}^{ij} + 6\tilde{A}^{ij}\tilde{D}_j \ln\psi\right]$$

$$\tilde{D}_j\tilde{A}^{ij} + 6\tilde{A}^{ij}\tilde{D}_j \ln\psi - \frac{2}{3}\tilde{D}^i K = 8\pi\psi^4 P^i$$

Summary of conformal decomposition

- With the conformal decomposition defined by

$$\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} \qquad K_{ij} = \psi^4 A_{ij} + \frac{1}{3}\gamma_{ij} K \qquad \tilde{\gamma} = \det \tilde{\gamma}_{ij} = 1$$

- The 3+1 decomposition (ADM formulation) of Einstein's equations becomes

Constraint equations

$$\tilde{D}_i \tilde{D}^i \psi - \frac{1}{8}\tilde{R}\psi + \left(\frac{1}{8}\tilde{A}_{ij}\tilde{A}^{ij} - \frac{1}{12}K^2 + 2\pi E\right)\psi^5 = 0$$

$$\tilde{D}_j \tilde{A}^{ij} + 6\tilde{A}^{ij}\tilde{D}_j \ln\psi - \frac{2}{3}\tilde{D}^i K = 8\pi\psi^4 P^i$$

Evolution equations

$$(\boldsymbol{L}_t - \boldsymbol{L}_\beta)\ln\psi = -\frac{1}{6}\alpha K$$

$$(\boldsymbol{L}_t - \boldsymbol{L}_\beta)\tilde{\gamma}_{ij} = -2\alpha\tilde{A}_{ij}$$

$$(\boldsymbol{L}_t - \boldsymbol{L}_\beta)K = -D_i D^i \alpha + \alpha[K_{ij}K^{ij} + 4\pi(E+S)]$$

$$(\boldsymbol{L}_t - \boldsymbol{L}_\beta)\tilde{A}_{ij} = \psi^{-4}\left[-(D_i D_j \alpha)^{\mathrm{TF}} + \alpha(R_{ij}{}^{\mathrm{TF}} - 8\pi S_{ij}{}^{TF})\right] + \alpha\left[K\tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}^k_j\right]$$

Lie derivatives of tensor density

- A tensor density of weight w is a object which is a tensor times $\gamma^{w/2}$: $\mathsf{T}_{ij} = \gamma^{w/2} T_{ij}$
- One should be careful because the Lie derivative of a tensor density is different from that of a tensor, as

$$\mathcal{L}_\beta \mathsf{T}^{a_1\ldots a_s}_{b_1\ldots b_r} = \left[\beta^c \partial_c \mathsf{T}^{a_1\ldots a_s}_{b_1\ldots b_r} - \sum_{i=1}^{s} \mathsf{T}^{a_1..c..a_s}_{b_1\ldots b_r} \partial_c \beta^{a_i} - \sum_{i=1}^{r} \mathsf{T}^{a_1\ldots a_s}_{b_1..c..b_r} \partial_{b_i} \beta^c \right] + w \mathsf{T}^{a_1\ldots a_s}_{b_1\ldots b_r} \partial_k \beta^k$$
$$= \left[\mathcal{L}_\beta \mathsf{T}^{a_1\ldots a_s}_{b_1\ldots b_r} \right]_{w=0} + w \mathsf{T}^{a_1\ldots a_s}_{b_1\ldots b_r} \partial_k \beta^k$$

$$\mathcal{L}_\beta \mathsf{T}^i_j = \mathcal{L}_\beta (\gamma^{w/2} T^i_j) = \gamma^{w/2} \left[\beta^k \partial_k T^i_j - T^k_j \partial_k \beta^i + T^i_k \partial_j \beta^k \right] + T^i_j (w/2) \gamma^{w/2-1} \mathcal{L}_\beta \gamma$$
$$= \left[\beta^k \partial_k (\gamma^{w/2} T^i_j) - T^i_j \beta^k \partial_k \gamma^{w/2} - \mathsf{T}^k_j \partial_k \beta^i + \mathsf{T}^i_k \partial_j \beta^k \right] + T^i_j (w/2) \gamma^{w/2-1} \left[\gamma \gamma^{ij} \mathcal{L}_\beta \gamma_{ij} \right]$$
$$= \left[\beta^k \partial_k \mathsf{T}^i_j - T^i_j w \gamma^{(w-1)/2} \beta^k \partial_k \gamma^{1/2} - \mathsf{T}^k_j \partial_k \beta^i + \mathsf{T}^i_k \partial_j \beta^k \right] + T^i_j w \gamma^{w/2} D_k \beta^k$$
$$= \left[\beta^k \partial_k \mathsf{T}^i_j - T^i_j w \gamma^{(w-1)/2} \beta^k \partial_k \gamma^{1/2} - \mathsf{T}^k_j \partial_k \beta^i + \mathsf{T}^i_k \partial_j \beta^k \right] + T^i_j w \gamma^{w/2} \left[\gamma^{-1/2} \partial_k (\gamma^{1/2} \beta^k) \right]$$
$$= \left[\beta^k \partial_k \mathsf{T}^i_j - \mathsf{T}^k_j \partial_k \beta^i + \mathsf{T}^i_k \partial_j \beta^k \right] + w T^i_j \gamma^{w/2} \partial_k \beta^k$$

Lie derivatives in conformal decomposition

- The weight factor of the conformal factor $\psi = \gamma^{1/12}$ is 1/6
- Thus the weight factor of the conformal metric and the conformal extrinsic curvature is −2/3, so that
- Note that the Lie derivative along t^a is equivalent to the partial derivative along the time direction
- Thus

$$(\mathcal{L}_t - \mathcal{L}_\beta)\ln\psi = \left(\partial_t - \beta^k\partial_k\right)\ln\psi - \frac{1}{6}\partial_k\beta^k$$

$$(\mathcal{L}_t - \mathcal{L}_\beta)\tilde{\gamma}_{ij} = \left(\partial_t - \beta^k\partial_k\right)\tilde{\gamma}_{ij} - \tilde{\gamma}_{ik}\partial_j\beta^k - \tilde{\gamma}_{jk}\partial_i\beta^k + \frac{2}{3}\tilde{\gamma}_{ij}\partial_k\beta^k$$

$$(\mathcal{L}_t - \mathcal{L}_\beta)\tilde{A}_{ij} = \left(\partial_t - \beta^k\partial_k\right)\tilde{A}_{ij} - \tilde{A}_{ik}\partial_j\beta^k - \tilde{A}_{jk}\partial_i\beta^k + \frac{2}{3}\tilde{A}_{ij}\partial_k\beta^k$$

Evolution of constraints

- It can be shown that the "evolution" equations for the Hamiltonian (C_H) and Momentum (C_M) constraints becomes

$$\left(\partial_t - \mathcal{L}_\beta\right)C_H = -D_k(\alpha C_M^k) - C_M^k D_k\alpha + \alpha K(2C_H - F) + \alpha K^{ij}F_{ij}$$

$$\left(\partial_t - \mathcal{L}_\beta\right)C_M^i = -D_j(\alpha F^{ij}) + 2\alpha K_j^i C_M^j + \alpha K C_M^i + \alpha D^k(F - C_H) + (F - 2H)D^i\alpha$$

  - Where F_{ij} is the spatial projection of the evolution equation

$$F_{ab} \equiv \perp\left[{}^4R_{ab} - 8\pi\left(T_{ab} - \frac{1}{2}Tg_{ab}\right)\right]$$

- The evolution equations for the constraints show that the constraints are "preserved" or "satisfied" , if
  - They are satisfied initially ($C_H = C_M = 0$)
  - The evolution equation is solved correctly ($F_{ab}=0$)

Numerical-relativity simulations based on the 3+1 decomposition is unstable !!

- It is known that simulations based on the 3+1 decomposition (ADM formulation), unfortunately ***crash in a rather short time***
- This crucial limitation may be captured in terms of notions of **hyperbolicity** (e.g. see textbook by *Alcubierre (2008)*)
  - Consider the following first-order system $\partial_t \boldsymbol{U} + \boldsymbol{A}^i \cdot \partial_i \boldsymbol{U} = 0$
  - The system is called
    - **Strongly Hyperbolic** ,if a matrix representation of **A** has real eigenvalues and complete set of eigenvectors
    - **Weakly Hyperbolic** , if **A** has real eigenvalues but not a complete set of eigenvectors
  - The key property of strongly and weakly hyperbolic systems :
    - Strongly hyperbolic system is well-posed, and hence, the solution for the finite-time evolution is **bounded**
    - Weakly hyperbolic system is ill-posed and the solution can be **unbounded**

Numerical-relativity simulations based on the 3+1 decomposition is unstable !!

- It is known that the ADM formulation is only weakly hyperbolic (*Alcubierre (2008)*)
  - Consequently, the ADM formulation is ill-posed and the numerical solution can be unbounded, leading to termination of the simulation
- We need formulations for the Einstein's equation which is (at least) strongly hyperbolic
- Let us consider Maxwell's equations in flat spacetime to capture what we should do to obtain a more stable system

$$\partial_i E^i = 4\pi\rho_e$$
$$\partial_i B^i = 0$$
$$\partial_t E_i = \varepsilon_{ijk}\partial^j B^k - 4\pi j_i$$
$$\partial_t B_i = -\varepsilon_{ijk}\partial^j E^k$$

$$A^\mu = (\Phi, A^k)$$
$$B_i = \varepsilon_{ijk}\partial^j A^k$$

→

$$\partial_i E^i = 4\pi\rho_e$$
$$\partial_t E_i = D_i D^j A_j - D^j D_j A_i - 4\pi j_i$$
$$\partial_t A_i = -E_i - D_i\Phi$$

Note the similarity of these equations to those in the ADM formulation

Consideration in Maxwell's equations

- First of all, let us note the similarity of the Maxwell's equations with the ADM equations (for simplicity in vacuum)

$$\partial_i E^i = 0 \quad \text{(constraint eq.)}$$
$$\partial_t E_i = \boxed{D_i D^j A_j} - \boxed{D^j D_j A_i}$$
$$\partial_t A_i = -E_i - D_i \Phi$$

(constraint eqs.)

$$2\mathcal{L}_{\alpha n} K_{ij} = -2\alpha R_{ij} + \cdots\cdots$$
$$= \alpha\left(\underbrace{\gamma^{kl}\partial_l\partial_i\gamma_{kj} + \gamma^{kl}\partial_j\partial_k\gamma_{il}}_{\text{"mixed-derivative" part}} - \overbrace{\gamma^{kl}\partial_k\partial_l\gamma_{ij}}^{\text{wave-like part}}\right) + \cdots\cdots$$
$$\mathcal{L}_{\alpha n}\gamma_{ij} = -2\alpha K_{ij}$$

- Second, the Maxwell's equations are 'almost' wave equation

$$-\partial_t^2 A_i + D^k D_k A_i - D_i D^j A_j = D_i \partial_t \Phi$$

  - Recall that in the Coulomb gauge $D_j A^j = 0$, the longitudinal part (associated with divergence part) of the electric field E does not obey a wave equation but is described by a Poisson equation (see a standard textbook, e.g., *Jakson*)

Reformulating Maxwell's equations (1)

- *Introducing auxiliary variables*

- A simple but viable approach is to introduce independent auxiliary variables to the system
- Let us introduce a new independent variable defined by

$$F = D^k A_k$$

- The evolution equation for this is

$$\partial_t F = \partial_t D^k A_k = -D^i E_i - D_k D^k \Phi$$

- Then, the Maxwell's equations for the vector potential become a wave equation in the form :

$$-\partial_t^2 A_i + D^k D_k A_i = D_i \partial_t \Phi + D_i F$$

Reformulating Maxwell's equations (2)

- Imposing a better gauge

- A second approach is to impose a good gauge condition
- In the Lorenz gauge, the Maxwell's equations in the flat spacetime are wave equations

$$\partial_\mu \partial^\mu A_\nu = 0$$

- Alternatively, by introducing a source function, one may "generalize" the Coulomb gauge condition so that Poisson-like equations do not appear

$$D^k A_k = H(x^\mu)$$

  - Recall again, that in the Coulomb gauge $D_j A^j = 0$, the longitudinal part (associated with divergence part) of the electric field E is described by a Poisson-type equation

Reformulating Maxwell's equations (3)

- Using the constraint equations

- A third approach is to use the constraint equations
- To see this, let us back to the example considered in "introducing auxiliary variables"

$$D_i E^i = 4\pi\rho_e$$
$$\partial_t E_i = D_i F - D^k D_k A_i - 4\pi j_i$$
$$\partial_t A_i = -E_i - D_i \Phi$$
$$\partial_t F = -D^i E_i - D_k D^k \Phi$$

$$D_i E^i = 4\pi\rho_e$$
$$\partial_t E_i = D_i F - D^k D_k A_i - 4\pi j_i$$
$$\partial_t A_i = -E_i - D_i \Phi$$
$$\partial_t F = -4\pi\rho_e - D_k D^k \Phi$$

  - The constraint equation can be used to rewrite the evolution equation for the auxiliary variable
  - Seen as the first-order system, the hyperbolic properties of the two system is different: **the hyperbolicity could be changed !**
- It is important and sometimes even crucial to use the constraint equations to change the hyperbolic properties of the system

Reformulating Einstein's equations

- The lessons learned from the Maxwell's equations are
  - Introducing new, independent variables
    - **BSSN** (*Shibata & Nakamura PRD* ***52****, 5428 (1995); Baumgarte & Shapiro PRD* ***59****, 024007 (1999)*)
      - see also *Nakamura et al. Prog. Theor. Phys. Suppl.* ***90****, 1 (1987)*
    - Kidder-Scheel-Teukolsky (*Kidder et al. PRD* ***64****, 064017 (2001)*)
    - Bona-Masso (*Bona et al. PRD* ***56****, 3405 (1997)*)
    - Nagy-Ortiz-Reula (*Nagy et al. PRD* ***70****, 044012 (2004)*)
  - Choosing a better gauge
    - Generalized harmonic gauge (*Pretorius, CQG* ***22****, 425 (2005)*)
    - Z4 formalism (*Bona et al. PRD* ***67****, 104005 (2003)*)
  - Using the constraint equations to improve the hyperbolicity
    - adjusted ADM/BSSN (*Shinkai & Yoneda, gr-qc/0209111*)
- **BSSN outperforms** (*Alcubierre (2008)*) !
  - Exact reason is not clear

BSSN formalism (1)

- Let first analyze the conformal Ricci tensor
  - By noting that $2\tilde{\Gamma}^k_{ik} = \partial_i \ln \tilde{\gamma} = 0$ the conformal Ricci tensor is

$$\tilde{R}_{ij} = \partial_k \tilde{\Gamma}^k_{ij} - \partial_j \tilde{\Gamma}^k_{ik} + \tilde{\Gamma}^k_{ij}\tilde{\Gamma}^l_{kl} - \tilde{\Gamma}^k_{il}\tilde{\Gamma}^l_{kj}$$

$$= -\frac{1}{2}\tilde{\gamma}^{kl}\left(\partial_k\partial_l\tilde{\gamma}_{ij} - \partial_i\partial_l\tilde{\gamma}_{jk} - \partial_j\partial_l\tilde{\gamma}_{ik}\right) + (\text{terms with } \partial\gamma\partial\gamma)$$

W_{ij}

  - If we divide the conformal metric formally as $\tilde{\gamma}^{ij} = \delta^{ij} + f^{ij}$, we have

$$W_{ij} = \partial_k\partial^k\tilde{\gamma}_{ij} - \left(\partial_i\partial^k\gamma_{kj} + \partial_j\partial^k\gamma_{ki}\right) + (\text{terms with } f\partial f)$$

- Thus we can eliminate the "mixed derivative" terms by introducing new auxiliary variable (*Shibata & Nakamura (1995)*)

$$F_i \equiv \delta^{jk}\partial_k\tilde{\gamma}_{ij} = \partial^j\tilde{\gamma}_{ij}$$

$$W_{ij} = \partial_k\partial^k\tilde{\gamma}_{ij} - \left(\partial_i F_j + \partial_j F_i\right) + (\text{terms with } f\partial f)$$

BSSN formalism (2)

- Baumgarte and Shapiro introduced the slightly different auxiliary variables

$$\Gamma^i = -\partial_j \tilde{\gamma}^{ij}$$

- In this case, the mixed-second-derivative terms are encompassed as

$$\tilde{R}_{ij} = -\frac{1}{2}\left(\tilde{\gamma}^{kl}\partial_k\partial_l\tilde{\gamma}_{ij} - \tilde{\gamma}_{ik}\partial_j\Gamma^k - \tilde{\gamma}_{jk}\partial_i\Gamma^k\right) + (\text{terms with } \partial\gamma\partial\gamma)$$

  - In linear regime, SN and BS are equivalent

BSSN formalism (3)

- Finally let us consider the evolution equation for the auxiliary variables (giving only a rough sketch of derivation)
  - Let us start from the momentum constraint equation

$$\tilde{D}_k(\tilde{\gamma}^{jk}\tilde{A}_{ij}) + 6\tilde{A}_{ij}\tilde{\gamma}^{jk}\tilde{D}_k \ln\psi - \frac{2}{3}\tilde{D}_i K = 8\pi\psi^4 P_i$$

$$\tilde{D}_j\tilde{A}^{ij} + 6\tilde{A}^{ij}\tilde{D}_j \ln\psi - \frac{2}{3}\tilde{D}^i K = 8\pi\psi^4 P^i$$

  - Substituting the evolution equation for the conformal extrinsic curvature

$$(\boldsymbol{L}_t - \boldsymbol{L}_\beta)\tilde{\gamma}_{ij} = -2\alpha\tilde{A}_{ij}$$

$$(\boldsymbol{L}_t - \boldsymbol{L}_\beta)\tilde{\gamma}^{ij} = 2\alpha\tilde{A}^{ij}$$

  - We obtain the evolution equations for F_i and Γ^i, respectively
    - It can be seen from the above sketch of derivation, the evolution equation for Γ^i is slightly simpler

BSSN formalism (4)

- The explicit forms of the evolution equations are

$$\left(\partial_t - \beta^k \partial_k\right) F_i = -16\pi\alpha P_i + 2\alpha\left[f^{jk}\partial_k \tilde{A}_{ij} + \tilde{A}_{ij}\partial_k \tilde{\gamma}^{jk} - \frac{1}{2}\tilde{A}^{jk}\partial_i \tilde{\gamma}_{jk} + 6\tilde{A}_i^{\ j}\partial_j \ln\psi - \frac{2}{3}\partial_i K \right]$$
$$+ \delta^{jk}\left[-2\tilde{A}_{ij}\partial_k \alpha + (\partial_k \beta^l)(\partial_l \tilde{\gamma}_{ij}) + \partial_k\left(\tilde{\gamma}_{il}\partial_j \beta^l + \tilde{\gamma}_{jl}\partial_i \beta^l - \frac{2}{3}\tilde{\gamma}_{ij}\partial_l \beta^l \right)\right]$$

$$\left(\partial_t - \beta^k \partial_k\right) \Gamma^i = -16\pi\alpha P^i + 2\alpha\left[\tilde{\Gamma}^i_{jk}\tilde{A}^{jk} + 6\tilde{A}^{ij}\partial_j \ln\psi - \frac{2}{3}\tilde{\gamma}^{ij}\partial_j K \right] - 2\tilde{A}^{ij}\partial_j \alpha$$
$$+ \beta^j \partial_j \Gamma^i - \Gamma^j \partial_j \beta^i + \frac{2}{3}\Gamma^i \partial_j \beta^j + \frac{1}{3}\tilde{\gamma}^{ij}\partial_j \partial_k \beta^k + \tilde{\gamma}^{jk}\partial_j \partial_k \beta^i$$

BSSN formalism : summary (1)

$$\tilde{D}_i \tilde{D}^i \psi - \frac{1}{8} \tilde{R} \psi + \left(\frac{1}{8} \tilde{A}_{ij} \tilde{A}^{ij} - \frac{1}{12} K^2 + 2\pi E \right) \psi^5 = 0$$

$$\tilde{D}_j \tilde{A}^{ij} + 6 \tilde{A}^{ij} \tilde{D}_j \ln \psi - \frac{2}{3} \tilde{D}^i K = 8\pi \psi^4 P^i$$

$$\left(\partial_t - \beta^k \partial_k \right) \ln \psi = -\frac{1}{6} \alpha K + \frac{1}{6} \partial_k \beta^k$$

$$\left(\partial_t - \beta^k \partial_k \right) \tilde{\gamma}_{ij} = -2\alpha \tilde{A}_{ij} + \tilde{\gamma}_{ik} \partial_j \beta^k + \tilde{\gamma}_{jk} \partial_i \beta^k - \frac{2}{3} \tilde{\gamma}_{ij} \partial_k \beta^k$$

$$\left(\partial_t - \beta^k \partial_k \right) K = -D_i D^i \alpha + \alpha [K_{ij} K^{ij} + 4\pi (E + S)]$$

Hamiltonian constraint is used

$$\left(\partial_t - \beta^k \partial_k \right) \tilde{A}_{ij} = \psi^{-4} \left[-(D_i D_j \alpha)^{\mathrm{TF}} + \alpha (R_{ij}{}^{\mathrm{TF}} - 8\pi S_{ij}{}^{TF}) \right] + \alpha \left[K \tilde{A}_{ij} - 2 \tilde{A}_{ik} \tilde{A}_j^k \right]$$
$$+ \tilde{A}_{ik} \partial_j \beta^k + \tilde{A}_{jk} \partial_i \beta^k - \frac{2}{3} \tilde{A}_{ij} \partial_k \beta^k$$

BSSN formalism : summary (2)

$$\left(\partial_t - \beta^k \partial_k\right) F_i = -16\pi\alpha P_i + 2\alpha\left[f^{jk}\partial_k \tilde{A}_{ij} + \tilde{A}_{ij}\partial_k \tilde{\gamma}^{jk} - \frac{1}{2}\tilde{A}^{jk}\partial_i \tilde{\gamma}_{jk} + 6\tilde{A}_i^j \partial_j \ln\psi - \frac{2}{3}\partial_i K \right]$$

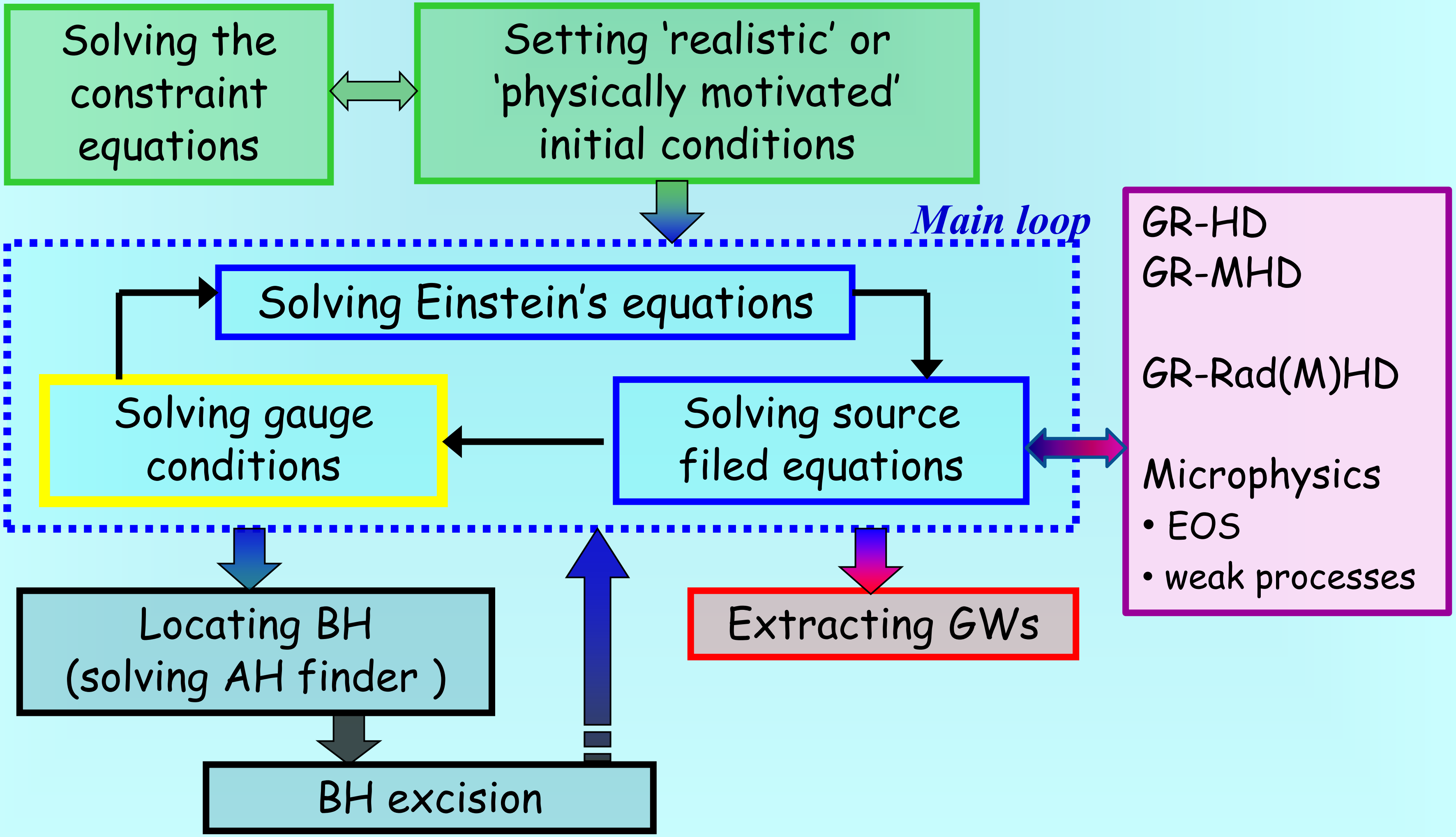
$$+\delta^{jk}\left[-2\tilde{A}_{ij}\partial_k \alpha + (\partial_k \beta^l)(\partial_l \tilde{\gamma}_{ij}) + \partial_k\left(\tilde{\gamma}_{il}\partial_j \beta^l + \tilde{\gamma}_{jl}\partial_i \beta^l - \frac{2}{3}\tilde{\gamma}_{ij}\partial_l \beta^l \right)\right]$$

Momentum constraint is used

$$\left(\partial_t - \beta^k \partial_k\right)\Gamma^i = -16\pi\alpha P^i + 2\alpha\left[\tilde{\Gamma}^i_{jk}\tilde{A}^{jk} + 6\tilde{A}^{ij}\partial_j \ln\psi - \frac{2}{3}\tilde{\gamma}^{ij}\partial_j K \right] - 2\tilde{A}^{ij}\partial_j \alpha$$

$$+\beta^j \partial_j \Gamma^i - \Gamma^j \partial_j \beta^i + \frac{2}{3}\Gamma^i \partial_j \beta^j + \frac{1}{3}\tilde{\gamma}^{ij}\partial_j \partial_k \beta^k + \tilde{\gamma}^{jk}\partial_j \partial_k \beta^i$$

Overview of numerical relativity



Gauge conditions

- Associated directly with the general covariance in general relativity, there are degrees of freedom in choosing coordinates (gauge freedom)
  - **Slicing condition** is a prescription of choosing the lapse function
  - **Shift condition** is that of choosing the shift vector
- Einstein's equations say nothing about how the gauge conditions should be imposed
- As we have seen in the reformulation of the ADM system, choosing "good" gauge conditions are very important to achieve stable and robust numerical simulations
  - An improper slicing conditions in a stellar-collapse problem will lead to appearance of (coordinate and physical) singularities
  - Also, the shift vector is important in resolving the frame dragging effect in simulations of e.g. compact binary merger

Preliminary

– decomposition of covariant derivative of n^a –

- The covariant derivative of a **timelike unit vector z^a** can be decomposed as

$$\nabla_a z_b = \omega_{ab} + \sigma_{ab} + \frac{1}{3} h_{ab}\theta - z_a \zeta_b$$

  - Where, the deformation of the congruence of the timelike vector is characterized by these tensors

$$\begin{aligned} h_{ab} &\equiv g_{ab} + z_a z_b, && \text{(induced metric)} \\ \omega_{ab} &\equiv \perp \nabla_{[a} z_{b]}, && \text{(twist)} \\ \sigma_{ab} &\equiv \perp \nabla_{(a} z_{b)}^{\text{TF}}, && \text{(shear)} \\ \theta &\equiv \nabla_c z^c, && \text{(expansion)} \\ \zeta^a &\equiv z^c \nabla_c z^a, && \text{(acceleration)} \end{aligned}$$

- For the unit normal vector to Σ, n^a we have
  - **The expansion is $-K$**
  - **The shear is $-A_{ab}$**
  - **The twist vanishes**

$$\nabla_a n_b = -A_{ab} - \frac{1}{3}\gamma_{ab} K - n_a a_b$$

Geodesic slicing $\alpha=1$, $\beta^i=0$

- In the geodesic slicing, the evolution equation of the trace of the extrinsic curvature is $\partial_t K = K_{ij}K^{ij} + 4\pi(E+3S)$
  - For normal matter (which satisfies the strong energy condition), the right-hand-side is positive
- Thus the expansion of time coordinate ($-K$) decreases monotonically in time
- In terms of the volume element $\gamma^{1/2}$, this means that the volume element goes to zero, as

$$\partial_t \ln \gamma^{1/2} = \frac{1}{2}\gamma^{ij}\partial_t \gamma_{ij} = -\alpha K + D_k \beta^k \;\Rightarrow\; -K$$

- This behavior results in a <u>coordinate singularity</u>
  - As can be seen in this example, how to impose a <u>slicing condition is closely related to the trace of the extrinsic curvature</u>

Maximal slicing

- Because the decrease in time of the volume element results in a coordinate singularity, let us maximize the volume element
- We take the volume of a 3D-domain S : $V[S]=\int_S \sqrt{\gamma} d^3x$
  and consider a variation along the time vector $t^a = \alpha n^a + \beta^a$
  - At the boundary of S, we set $\alpha=1, \beta^i=0$

$$\mathcal{L}_t V[S] = \int_S d^3x \left[-\alpha K \sqrt{\gamma} + \partial_i (\sqrt{\gamma} \beta^i) \right] = -\int_S \alpha K \sqrt{\gamma} d^3x$$

- Thus if $K=0$ on a slice, the volume is extremal (maximal)
- We shall demand that this maximal slicing condition holds for all slices and set $0 = (\mathcal{L}_t - \mathcal{L}_\beta) K = -D_i D^i \alpha + \alpha [K_{ij} K^{ij} + 4\pi (E+S)]$
  - The maximal slicing has a **strong singularity avoidance** property (E.g. *Estabrook & Wahlquist PRD 7, 2814 (1973); Smarr & York, PRD 17, 1945/2529 (1978)*)
  - However this is a elliptic equation and is computationally expensive

(K-driver)/(approximate maximal) condition

- As a generalization of the maximal slicing condition, let us consider the following condition with a positive constant c

$$\partial_t K = -cK$$

  - This (elliptic) condition drives $\boldsymbol{K}$ back to zero even when $\boldsymbol{K}$ deviates from zero due to some error or insufficient convergence
- Balakrishna et al. (*CQG 13, L135 (1996)*) and Shibata (*Prog. Theor. Phys. 101, 251 (1999)*) converted this equation into a parabolic one by adding a "time" derivative of the lapse :

$$\partial_\lambda \alpha = D_i D^i \alpha - \alpha[K_{ij}K^{ij} + 4\pi(E+S)] - \beta^i D_i K + cK$$

  - If a certain degree of the "convergence" is achieved and the lapse relaxes to a "stationary state", it suggests $\partial_t K = -cK$
- This condition is called K-driver or approximate maximal slicing condition

$$\partial_t \alpha = -\varepsilon\left(\partial_t K + cK\right), \qquad \lambda = \varepsilon t$$

Harmonic slicing

- The harmonic gauge condition $\nabla_c \nabla^c x^a = 0$ have played an important role in theoretical developments (*Choquet-Bruhat's textbook*)
  - Existence and uniqueness of the solution of the Cauchy problem of Einstein's equations (somewhat similar to Lorenz gauge in EM)
- The harmonic slicing condition is defined by

$$\nabla_c \nabla^c t = 0 \quad \Leftrightarrow \quad \partial_\mu \left(\sqrt{-g}\, g^{\mu 0} \right) = 0$$

  - Note that $\sqrt{-g} = \alpha \sqrt{\gamma}$
- The harmonic slicing condition can be written as

$$\left(\partial_t - \partial_k \beta^k \right) \alpha = -\alpha^2 K$$

  - This is **an evolution equation**
- It is known that the harmonic slicing condition has some singularity avoidance property, although weaker than that of the maximal slicing (e.g. *Cook & Scheel PRD 56, 4775 (1997), Alcubierre's textbook*)

Generalized harmonic slicing

- Bona et al. (*PRL 75, 600 (1995)*) generalized the harmonic slicing condition to

$$\left(\partial_t - \partial_k \beta^k\right)\alpha = -\alpha^2 f(\alpha) K$$

  - This family of slicing includes the geodesic slicing ($f=0$), the harmonic slicing ($f=1$), and formally the maximal slicing ($f=\infty$)
- The choice $f(\alpha)=2/\alpha$, which is called **1+log slicing** , has stronger singularity avoidance properties than the harmonic slicing (*Anninos et al. PRD 52, 2059 (1995)*)
  - The 1+log slicing has been widely used and has proven to be a successful and robust slicing condition

Minimal distortion (shift) condition

- Smarr and York (*PRD 17, 1945/2529 (1978)*) proposed a well motivated shift condition called the **minimal distortion condition**
  - As seen in the preliminary, the "distortion" part of the congruence is contained in the shear tensor
- They define a distortion functional by $I \equiv \int \Sigma_{ab}\Sigma^{ab}\sqrt{\gamma}dx^3$ and take a variation in terms of the shift
  - here the distortion tensor is defined by

$$\Sigma_{ab} \equiv \frac{1}{2}\perp \mathcal{L}_t \gamma_{ab}{}^{\mathrm{TF}} \sim -K_{ab}{}^{\mathrm{TF}} = \perp \nabla_{(a} n_{b)}{}^{\mathrm{TF}}$$

- The resulting shift condition is $D_a \Sigma^{ab} = 0$
  - Beautiful and physical but vector elliptic equations (computationally expensive)

$$D_c D^c \beta^a + D_a D_c \beta^c + R_{ab}\beta^b = D^b\left[2\alpha A_{ab}\right] = 2A^{ab}D_b\alpha + \alpha\left(\frac{4}{3}\gamma^{ab}D_b K + 16\pi P_a\right)$$

Γ-Freezing and approximate minimal condition

- With some calculations, one can show that the minimal distortion condition is written as
  $$\tilde{D}^j(\psi^6\partial_t\tilde{\gamma}_{ij}) = 0$$
  - The conformal factor is coupled !
- Modifications of the minimal distortion condition are proposed by Nakamura et al. (*Prog. Theor. Phys. Suppl. 128, 183 (1997)*) and Shibata (*Prog. Theor. Phys. 101, 1199 (1999)*)
  - E.g., Nakamura et al. proposed instead to solve the decoupled pseudo-minimal distortion condition :
  $$\tilde{D}^j(\partial_t\tilde{\gamma}_{ij}) = 0$$
- Alcubierre and Brugmann (*PRD 63, 104006 (2001)*) proposed an approximate minimal distortion condition called Gamma-Freezing :
  $$\tilde{D}_j(\partial_t\tilde{\gamma}^{ij}) = \partial_t\Gamma^i = 0$$
- Anyway, these conditions are elliptic-type !

Γ-Driver condition

- Alcubierre and Brugmann (*PRD 63, 104006 (2001)*) converted the Γ-freezing elliptic condition into **a parabolic** one by adding a time derivative of the shift (somewhat similar to the K-driver)

$$\partial_t \beta^i = k \partial_t \Gamma^i$$

- Alcubierre et al. (*PRD 67, 084023 (2003)*) and others (*Lindblom & Scheel PRD 67, 124005 (2003); Bona et al. PRD 72, 104009 (2005)*) extended the Γ-freezing condition to **hyperbolic** conditions
  - There are several alternative conditions

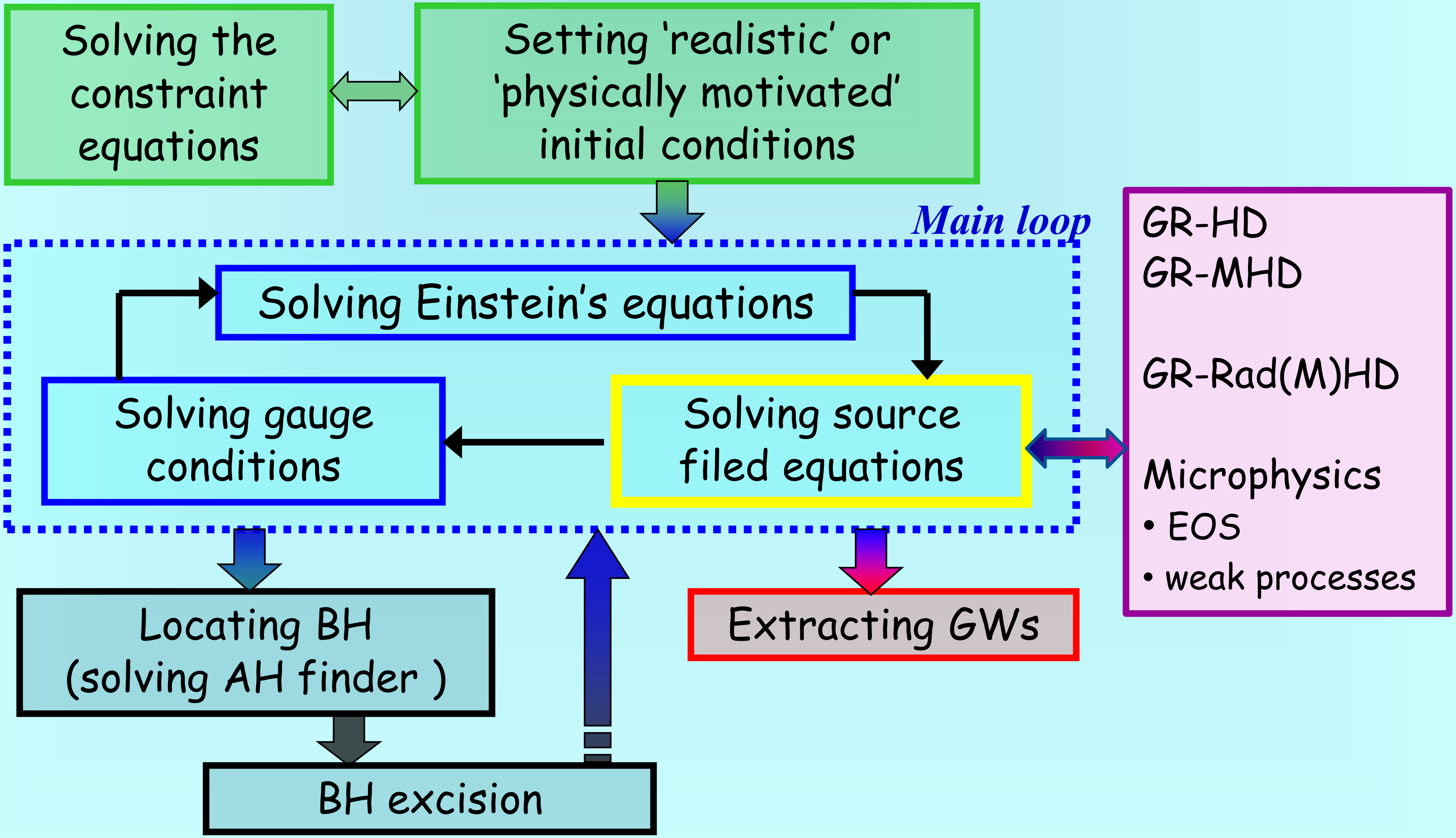
$$\partial_t \beta^i = k B^i$$
$$\partial_t B^i = \partial_t \Gamma^i - \eta B^i$$

- Shibata (*ApJ 595, 992 (2003)*) proposed a **hyperbolic** shift condition

$$\partial_t \beta^i = \tilde{\gamma}^{ij} \left(F_i + \Delta t_{\rm step} \partial_t F_i \right), \qquad \Delta t_{\rm step} : \text{time-step used in simulation}$$

- To date, the above two families of shift conditions are known to be robust

Overview of numerical relativity



3+1 decomposition of $\nabla_a T^{ab} = 0$
- *Energy Conservation Equation (1)*

- First, substitute the **3+1 decomposition of T_{ab}** to obtain

$$0 = \nabla_b T_a^b = \nabla_b (E n^b n_a + P^b n_a + P_a n^b + S_a^b)$$
$$= n_a n^b \nabla_b E + E a_a - K E n_a - P^b K_{ba} + n_a \nabla_b P^b + n^b \nabla_b P_a - K P_a + \nabla_b S_a^b$$

- Then, let us project it onto normal direction to Σ. Noting that P^a, K^{ab}, and a^b is purely spatial, we obtain

$$-n^b \nabla_b E + KE - \nabla_a P^a + n^a n^b \nabla_b P_a + n^a \nabla_b S_a^b = 0$$

- Because $n^a S_{ab}=0$, we have

$$n^a \nabla_b S_a^b = -S_a^b \nabla_b n^a = S_a^b (K_b^a + n_b a^a) = S^{ab} K_{ab}$$

- Similarly, $$n^a n^b \nabla_b P_a = -P_a n^b \nabla_b n^a = -P_b a^b$$

- The divergence term of P^a is

$$D_a P^a = \perp_a^b \nabla_b P^a = (\delta_a^b + n_a n^b) \nabla_b P^a = \nabla_a P^a - P_a a^a$$

3+1 decomposition of $\nabla_a T^{ab} = 0$
- *Energy Conservation Equation (2)*

- Combining altogether, we reach the **energy conservation equation**

$$n^b \nabla_b E + D_b P^b + 2P^b a_b - KE - K_{ab} S^{ab} = 0$$
$$(\partial_t - \beta^k D_k) E + \alpha [D_b P^b - KE - K_{ab} S^{ab}] + 2P^b D_b \alpha = 0$$
$$\partial_t E + D_k (\alpha P^k - E\beta^k) + E(D_k \beta^k - \alpha K) - \alpha K_{ab} S^{ab} + P^b D_b \alpha = 0$$

  - where we have used

$$n^b \nabla_b E = \mathcal{L}_n E = \alpha^{-1} (\mathcal{L}_t - \mathcal{L}_\beta) E = \alpha^{-1} (\partial_t - \beta^k D_k) E$$

$$a_b = D_b \ln \alpha$$

  - The last equation will be used to derive the conservative forms of the energy equation

3+1 decomposition of $\nabla_a T^{ab} = 0$

- *Momentum Conservation Equation (1)*

- To this turn, let us project the equation onto Σ to obtain

$$E a_a - P^b K_{ba} + \perp_a^c n^b \nabla_b P_c - K P_a + \perp_a^c \nabla_b S_c^b = 0$$

- The spacetime-divergence term of $S^b{}_c$ can be replaced by the spatial-divergence by

$$D_b S_c^b = \perp_b^d \perp_c^e \nabla_d S_e^b = \perp_c^e (\delta_b^d + n_b n^d) \nabla_d S_e^b = \perp_c^e \nabla_b S_e^b - S_c^d a_d$$

- The projection term with the covariant derivative of P_c is

$$\perp_a^c n^b \nabla_b P_c = \alpha^{-1} \perp_a^c (\alpha n^b) \nabla_b P_c = \alpha^{-1} \perp_a^c (\mathcal{L}_{\alpha n} P_c - P_d \nabla_c (\alpha n^d))$$

  - Note that (αn)-Lie derivative of any spatial tensor is spatial, and

$$\nabla_b (\alpha n^a) = n^a \nabla_b \alpha + \alpha \nabla_b n^a = n^a \nabla_b \alpha - \alpha (K_b^a + n_b a^a)$$

  - so that

$$\perp_a^c n^b \nabla_b P_c = \alpha^{-1} \mathcal{L}_{\alpha n} P_a + K_{ab} P^b$$

3+1 decomposition of $\nabla_a T^{ab} = 0$

- *Momentum Conservation Equation (2)*

- Combining altogether, we obtain the momentum conservation equation :

$$(\mathcal{L}_t - \mathcal{L}_\beta)P_a + \alpha[D_b S_a^b + S_a^b a_b - KP_a + Ea_a] = 0$$

$$(\partial_t - \mathcal{L}_\beta)P_a + \alpha[D_b S_a^b - KP_a] + S_a^b D_b \alpha + ED_a \alpha = 0$$

$$(\partial_t - \beta^c D_c)P_a + \alpha D_b S_a^b + (D_c \beta^c - \alpha K)P_a + S_a^b D_b \alpha + ED_a \alpha = 0$$

$$\partial_t P_a + D_c(\alpha S_a^c - \beta^c P_c) + (D_c \beta^c - \alpha K)P_a - P_c D_a \beta^c + ED_a \alpha = 0$$

  - Where we have expressed the Lie derivative by spatial covariant derivative
  - The last equation will be used in conservative reformulation
  - NOTE: In *York (1979)*, because he used P^a instead of P_a, a extra term appear in the equation.

3+1 decomposition of $\nabla_a T^{ab} = 0$

- *Conservative Formulation (1)*

- Now we will show the energy and momentum conservation equations can be recast to conservative form

$$\partial_t E + D_c(\alpha P^c - E\beta^c) + (D_c\beta^c - \alpha K)E - \alpha K_{ab}S^{ab} + P^b D_b\alpha = 0$$
$$\partial_t P_a + D_c(\alpha S^c_a - \beta^c P_a) + (D_c\beta^c - \alpha K)P_a - P_c D_a\beta^c + ED_a\alpha = 0$$

- First, by taking the trace of evolution eq. of γ_{ab}, we get

$$\gamma^{ab}(\partial_t\gamma_{ab} - D_a\beta_b - D_b\beta_a) = -2\alpha K \;\Rightarrow\; D_a\beta^a - \alpha K = \frac{1}{2}\gamma^{ij}\partial_t\gamma_{ij} = \frac{1}{\sqrt{\gamma}}\partial_t\sqrt{\gamma}$$

- Second, note that for any rank-(1,1) spatial tensor,

$$D_k T_i^k = \partial_k T_i^k + \Gamma^k_{jk}T_i^j - \Gamma^j_{ik}T_j^k = \partial_k T_i^k + (\partial_j \ln\sqrt{\gamma})T_i^j - \Gamma^j_{ik}T_j^k$$
$$= \frac{1}{\sqrt{\gamma}}\partial_k\left(\sqrt{\gamma}T_i^k\right) - \Gamma^j_{ik}T_j^k$$

3+1 decomposition of $\nabla_a T^{ab} = 0$

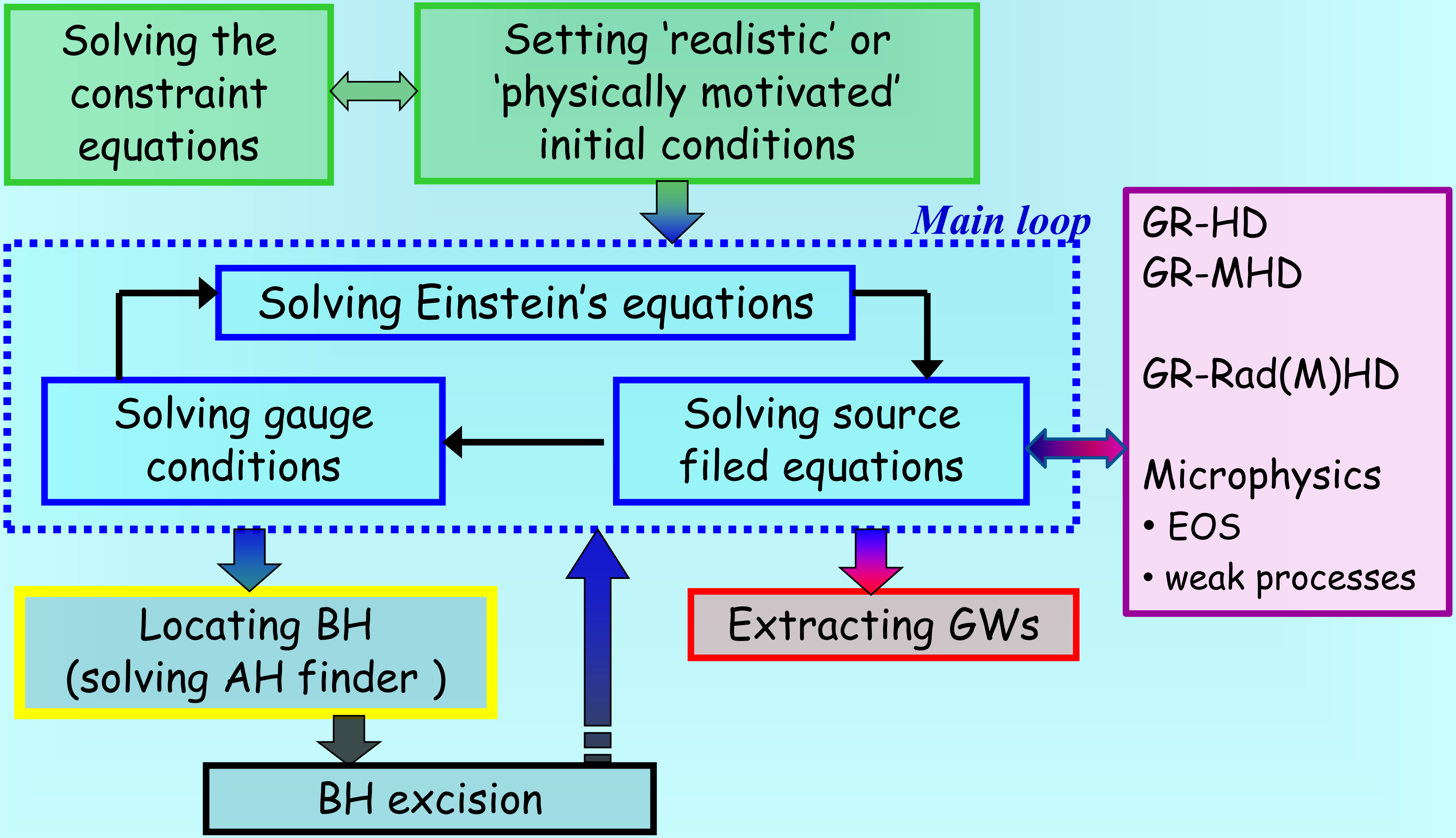
- Conservative Formulation (2)

- Using the equations derived the above, we can finally reach the **conservative forms of the energy and momentum equations**

$$\partial_t\left(\sqrt{\gamma}E\right)+\partial_k\left(\sqrt{\gamma}(\alpha P^k - E\beta^k)\right)=\sqrt{\gamma}(\alpha K_{ij}S^{ij} - P^k D_k\alpha)$$
$$\partial_t\left(\sqrt{\gamma}P_i\right)+\partial_k\left(\sqrt{\gamma}(\alpha S_i^k - \beta^k P_i)\right)=\sqrt{\gamma}\left[P_k D_i\beta^k - ED_i\alpha + \Gamma^k_{ij}(\alpha S_k^j - \beta^j P_k)\right]$$

- For the perfect fluid, for instance, these equations may be solved by high resolution shock capturing schemes

Overview of numerical relativity



Locating the apparent horizon (1)

- **Apparent horizon (e.g. Wald (1984))**: the apparent horizon is the boundary of the (total) trapped region
  - Trapped region: the trapped region is collections of points where the expansion of the null geodesics is negative or zero
- Thus, to locate the apparent horizon, we must calculate the expansion of the null geodesics and determine the points where the expansion vanishes

- Recall that the expansion is related to the trace of the extrinsic curvature : $K \Leftrightarrow$ expansion
- So that let us first define the extrinsic curvature of a null surface N generated by **an outgoing null** vector on a slice Σ:

Locating the apparent horizon (2)

- Let S to be an intersection of the slice Σ and the null surface N
  - We denote the unit normal of S in Σ, as s^a
- Then the outgoing (k^a) and ingoing (l^a) null vectors on S are

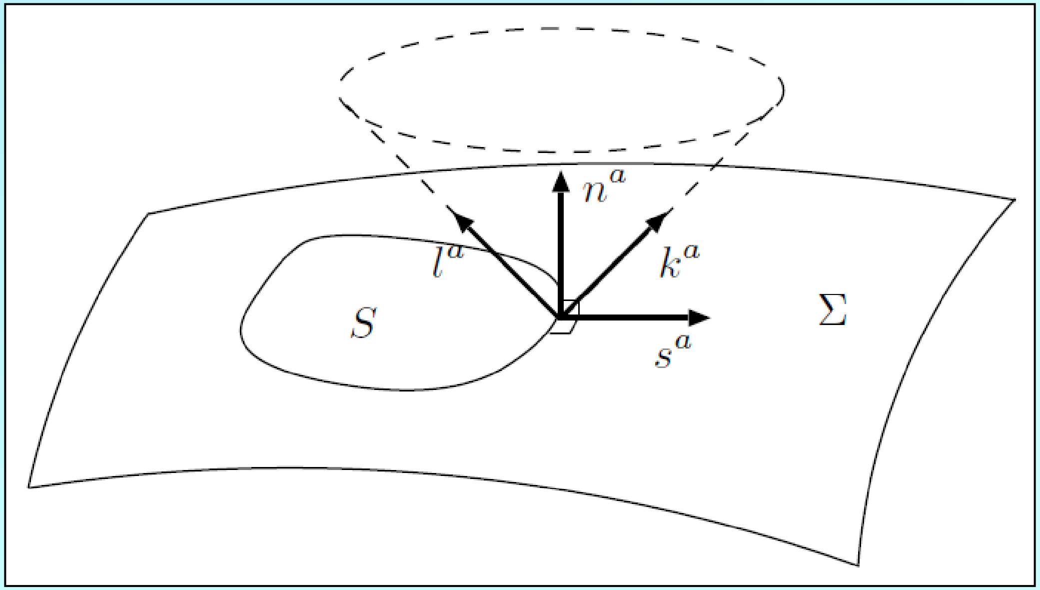
$$k^a \equiv \frac{1}{\sqrt{2}}\left(n^a + s^a\right), \quad l^a \equiv \frac{1}{\sqrt{2}}\left(n^a - s^a\right)$$

- Using k^a and l^a , the metric on S induced by g_{ab} is given by

$$\begin{aligned}\chi_{ab} &= g_{ab} + k_a l_b + k_b l_a \\ &= g_{ab} + n_a n_b - s_a s_b\end{aligned}$$

- Thus we can define the projection operator to S :

$$P^a_b = \delta^a_b + n^a n_b - s^a s_b$$



Locating the apparent horizon (3)

- Using the projection operator, the extrinsic curvature for ***N*** is defined by

$$\kappa_{ab} = -P_a^c P_b^d \nabla_{(c} k_{d)}$$

- Because k^a is the outgoing null vector on ***S***, the 2D-surface ***S*** is the apparent horizon if **tr[*κ*] = *κ^a_a* = *κ* = *0***
- This condition can be written in terms of s^a as

$$D_k s^k - K + K_{ij} s^i s^j = 0$$

- This is a single equation for the three unknown "functions" s^k !
- However, the condition that ***S*** is closed 2-sphere and that s^a is a unit normal vector bring two additional relation to s^k
  - For detail, see (e.g. *Bowen, J. M. & York, J. W., PRD* ***21****, 2047 (1980); Gundlach, C. PRD* ***57****, 863 (1998)*)

Energy and Momentums

Canonical formulation (1)

- The Lagrangian density of gravitational field in General Relativity is (e.g. *Wald (1984)*)

$$L_G \equiv \sqrt{-g}\,{}^4R$$

- Because the 4D Ricci scalar is written as

$$\begin{aligned} {}^4R &= 2(G_{ab}n^a n^b - {}^4R_{ab}n^a n^b) \\ &= \alpha\sqrt{\gamma}(R + K_{ab}K^{ab} - K^2) + (\text{Divergence terms}) \end{aligned}$$

- Noting that the extrinsic curvature is

$$K_{ab} = \frac{1}{2\alpha}(\dot{\gamma}_{ab} - D_a\beta^b - D_b\beta^a)$$

- The conjugate momentum π^{ab} is defined by

$$\pi^{ab} \equiv \frac{\partial L_G}{\partial \dot{\gamma}_{ab}} = \sqrt{\gamma}(K^{ab} - K\gamma^{ab})$$

Canonical Formulation (2)

- Now we obtain the Hamiltonian density as

$$\mathcal{H}_G \equiv \pi^{ab}\dot{\gamma}_{ab} - \mathcal{L}_G$$
$$= \sqrt{\gamma}\left[\frac{\alpha}{\gamma}\left(-\gamma R + \pi^{ab}\pi_{ab} - \frac{1}{2}\pi^2\right) - 2\beta_b D_a\left(\gamma^{-1/2}\pi^{ab}\right)\right] + (\text{ Divergence terms })$$

- The Hamiltonian is defined by $H_G \equiv \int \mathcal{H}_G dx^3$

  - The constraint equations are derived by taking the variations with respect to the lapse and the shift, respectively, as

$$C_H \equiv -R + \gamma^{-1}\pi_{ab}\pi^{ab} - \frac{1}{2}\gamma^{-1}\pi^2 = 0 \quad : \text{Hamiltonian constraint}$$
$$C_M^b \equiv D_a\left(\gamma^{-1/2}\pi^{ab}\right) = 0 \quad : \text{Momentum constraint}$$

where we have dropped the surface term

Canonical Formulation (3)

- The evolution equations are derived by taking the variations with respect to the canonical variables (e.g. *Wald (1984)*) :

$$\dot{\gamma}_{ab} \equiv \frac{\delta H_G}{\delta \pi^{ab}} = 2\alpha\gamma^{-1/2}\left[\pi_{ab} - \frac{1}{2}\gamma_{ab}\pi\right] + 2D_{(a}\beta_{b)} \quad \equiv B_{ab}$$

$$\dot{\pi}^{ab} \equiv \frac{\delta H_G}{\delta \gamma_{\alpha\beta}} = -\alpha\gamma^{1/2}\left[R - \frac{1}{2}R\gamma^{ab}\right] + \frac{1}{2}\alpha\gamma^{-1/2}\gamma^{ab}\left[\pi^{cd}\pi_{cd} - \frac{1}{2}\pi^2\right] - 2\alpha\gamma^{-1/2}\left[\pi^{ac}\pi_c^b - \frac{1}{2}\pi\pi^{ab}\right]$$
$$+\gamma^{1/2}\left(D^a D^b \alpha - \gamma^{ab} D_c D^c \alpha\right) + \gamma^{1/2} D_c\left(\gamma^{-1/2}\beta^c \pi^{ab}\right) - 2\pi^{c(a} D_c \beta^{b)} \quad \equiv A^{ab}$$

  - again, we here dropped the divergence terms

Energy for Asymptotically Flat spacetime (1)

- Let us consider the **energy** of gravitational field in the asymptotically flat spacetime
  - *Although there is no unique definition of 'local' gravitational energy in General Relativity, we can consider the total energy in the asymptotically flat spacetime*
  - *Asymptotically flat spacetime represent ideally isolated spacetime, and hence, there will be the conserved energy*
- A simple consideration based on the Hamiltonian density,

$$H_G = \sqrt{\gamma}\left[\alpha C_H - 2\beta_b C_M^b\right] + (\text{ Divergence terms })$$

may lead to a conclusion that the energy of any spacetime is zero when the constraint equations are satisfied !

- This "contradiction" stems from the wrong treatment of the divergence (surface) terms (which we have dropped)

Energy for Asymptotically Flat spacetime (2)

- The boundary conditions to be imposed are not fixed ones $\delta Q|_{\mathrm{boundary}} = 0$ where Q denotes relevant geometrical variables, but the "asymptotic flatness" :

$$\alpha = 1 + O(r^{-1}), \quad \beta^i = O(r^{-1}), \quad \gamma_{ij} - \delta_{ij} = O(r^{-1}), \quad \pi^{ij} = O(r^{-2})$$

- Keeping the divergence terms, the variation of the Hamiltonian now becomes (*Regge & Teitelboim, Ann. Phys* ***88****. 286 (1974)*)

$$\delta H_G = -\oint M^{ijkl}\left[\alpha D_k(\delta\gamma_{ij}) - (D_k\alpha)\delta\gamma_{ij}\right]d\sigma_l \\ -\oint\left[2\beta_k\delta\pi^{kl} + (2\beta^k\pi^{jl} - \beta^l\pi^{jk})\delta\gamma_{jk}\right]d\sigma_l$$

  - where we have assumed that the constraint equations and the evolution equations are satisfied, $d\sigma_l$ is the volume element of the boundary sphere and M^{ijkl} is defined as

$$M^{ijkl} \equiv \frac{1}{2}\sqrt{\gamma}\left[\gamma^{ik}\gamma^{jl} + \gamma^{il}\gamma^{jk} - 2\gamma^{ij}\gamma^{kl}\right]$$

Energy for Asymptotically Flat spacetime (3)

- Under the boundary conditions of the asymptotic flatness, the non-zero contribution of the surface terms is,

$$-\oint M^{ijkl} D_k(\delta\gamma_{ij})\, d\sigma_l = -\delta \oint \sqrt{\gamma}\gamma^{ij}\gamma^{kl}\left(\partial_j \gamma_{ik} - \partial_k \gamma_{ij}\right) d\sigma_l$$

- Thus, we define the Hamiltonian of the asymptotically flat spacetime as

$$H_G^{\text{asympt.flat}} \equiv H_G + 16\pi E_G[\gamma_{ij}]$$

$$E_G[\gamma_{ij}] \equiv \frac{1}{16\pi} \oint \sqrt{\gamma}\gamma^{ij}\gamma^{kl}\left(\partial_j \gamma_{ik} - \partial_k \gamma_{ij}\right) d\sigma_l$$

  - Then, **the energy of the gravitational fields is not zero but $E[\gamma_{ij}]$**
  - The overall factor is determined by the requirement that the energy of an asymptotically flat spacetime is M

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(\delta_{ij} + \frac{2M x_i x_j}{r^3}\right)dx^i dx^j$$

Momentums for Asymptotically Flat spacetime (1)

- The action for the region V ($t_1<t<t_2$) is

$$S_G = \int_V L_G dx^4 = \int_{t_1}^{t_2} dt \int d^3x \left[\pi^{ij} \dot{\gamma}_{ij} - H_G \right]$$

- Taking the variation, we obtain (*Regge & Teitelboim, Ann. Phys* ***88***. *286 (1974)*)

$$\delta S_G = \int_{t_1}^{t_2} dt \frac{d}{dt} \int d^3x \left[\pi^{ij} \delta\gamma_{ij} + \text{terms vanishing by EOM} \right]$$

- When there is a Killing vector ξ^a, the action is invariant under the Lie transport by ξ^a
- Making use of $\delta\gamma_{ij} = -\mathcal{L}_\xi \gamma_{ij} = -(D_i\xi_j + D_j\xi_i)$, we obtain

$$\delta_\xi S_G = \int_{t_1}^{t_2} dt \frac{d}{dt} \int d^3x \left[-D_i\left(2\pi^{ij}\xi_j\right) + 2\xi_i D_j \pi^{ij} \right] = 0$$

  - Note that the second term in the integrand vanished thanks to the momentum constraint

Momentums for Asymptotically Flat spacetime (2)

- Finally the variation of the action is reduced to

$$\delta S_G = \left[\oint d\sigma_l \left(-2\pi^{kl}\xi_k\right)\right]_{t_1}^{t_2} = 0$$

  - Because the Killing vector approaches at the boundary (spacelike infinity) to a constant translation vector field τ_a , we have

$$\tau_k\left[P_G{}^k(t_2) - P_G{}^k(t_1)\right] = 0, \quad P_G{}^k[\gamma_{ij}] \equiv -\frac{1}{8\pi}\oint d\sigma_l \pi^{kl}$$

  - This equation means that ***$P_G{}^k$ represent the total linear momentum***
  - Similarly, the generator of the rotational Lie transport approaches $\varepsilon_{ijk}\varphi^j x^k$ (φ is a constant vector field, ε is the totally anti-symetric tensor) , we may define **the total angular momentum** by

$$\varphi^k\left[L_k^G(t_2) - L_k^G(t_1)\right] = 0, \quad L_k^G[\gamma_{ij}] \equiv \frac{1}{8\pi}\oint d\sigma_l \varepsilon_{ijk}\pi^{jl}x^k$$

Energy and Momentums : summary

- To summarize, we define the energy, the linear momentum, and the angular momentum in the asymptotically flat spacetime by

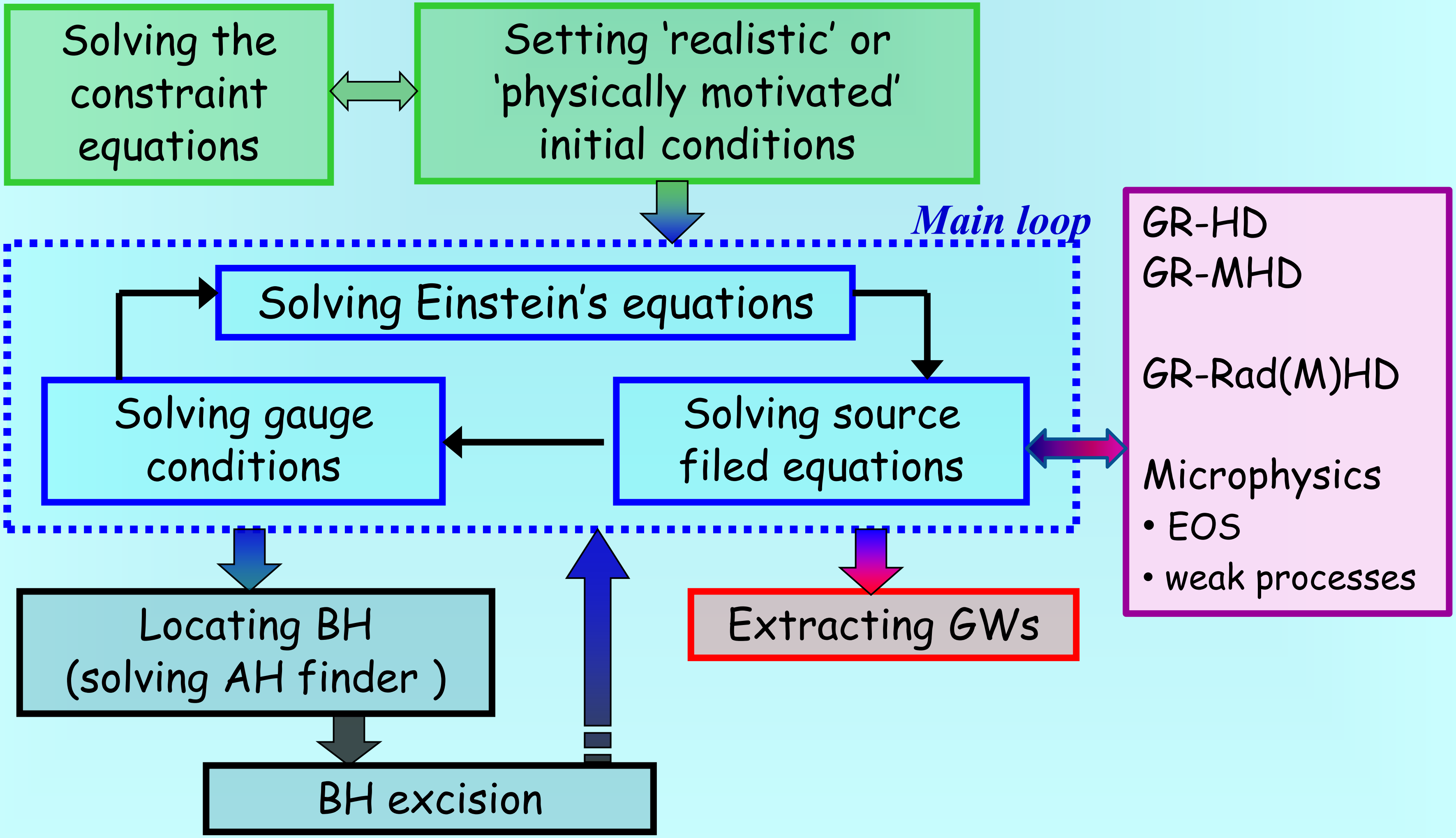
$$E_G[\gamma_{ij}] \equiv \frac{1}{16\pi}\oint \sqrt{\gamma}\gamma^{ij}\gamma^{kl}\left(\partial_j\gamma_{ik} - \partial_k\gamma_{ij}\right)d\sigma_l$$

$$P_G{}^k[\gamma_{ij}] \equiv -\frac{1}{8\pi}\oint d\sigma_l \pi^{kl}$$

$$L_k^G[\gamma_{ij}] \equiv \frac{1}{8\pi}\oint d\sigma_l \varepsilon_{ijk}\pi^{jl}x^k$$

  - A number of examples of the actual calculation will be found in a textbook (*Baumgarte & Shapiro (2010)*)

Overview of numerical relativity



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