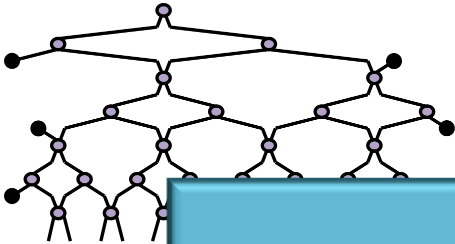
"New Development of Numerical Simulations in Low-Dimensional Quantum Systems: From Density Matrix Renormalization Group to Tensor Network Formulations"

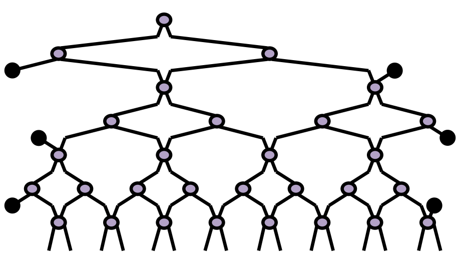
YITP, Kyoto University, October 19th 2010

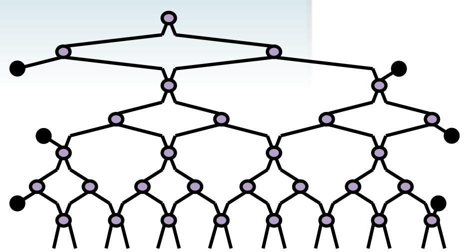


Beyond the *"entropic boundary law"* with entanglement renormalization

Glen Evenbly

THE UNIVERSITY OF QUEENSLAND
AUSTRALIA





Tensor Network Methods

(DMRG, PEPS, TERG, MERA)

Potentially offer **general formalism** to efficiently describe many-body wave functions

- ground states of large systems in D=1,2 (maybe 3) spatial dimensions
- strong or weak interactions, frustrated interactions etc
- different particle statistics (e.g. spins/bosons, fermions, or even anyons),

Only limited by the amount of entanglement in the state!

As Numerical Methods:

- Given Hamiltonian H what are properties x,y,z of the ground state?

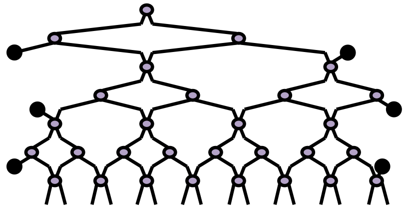
Conceptual Aspects:

- Framework for describing many-body systems - entanglement structure!

Outline

- Entanglement and tensor network methods
  - Scaling of entanglement entropy in ground states
  - Scaling of entanglement entropy in tensor network ansatz
    - physical geometry vs holographic geometry
  - Comparison of entropy scaling:
    - ground states vs tensor network ansatz

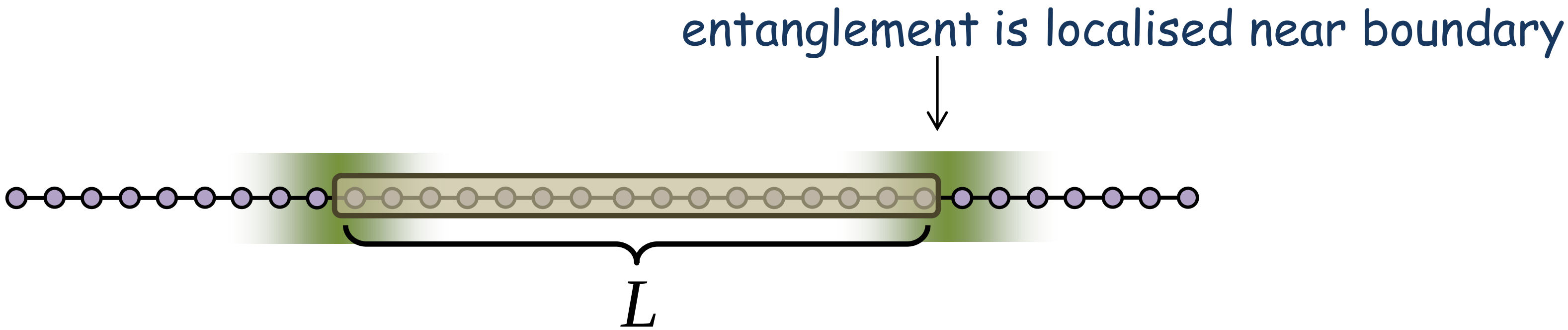
- Introduction of branching MERA



  - Decoupling a many-body theory
  - Holographic trees
  - Scaling of entropy in the branching MERA
  - Example: $S_L = L \log L$ entropy scaling in 2D fermions
  - Example: $S_L = (\log L)^2$ entropy scaling in 1D fermions

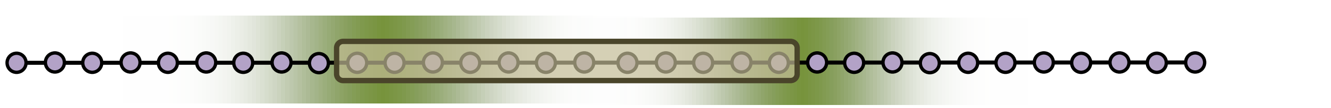
Entanglement entropy scaling in 1D systems

1D Gapped



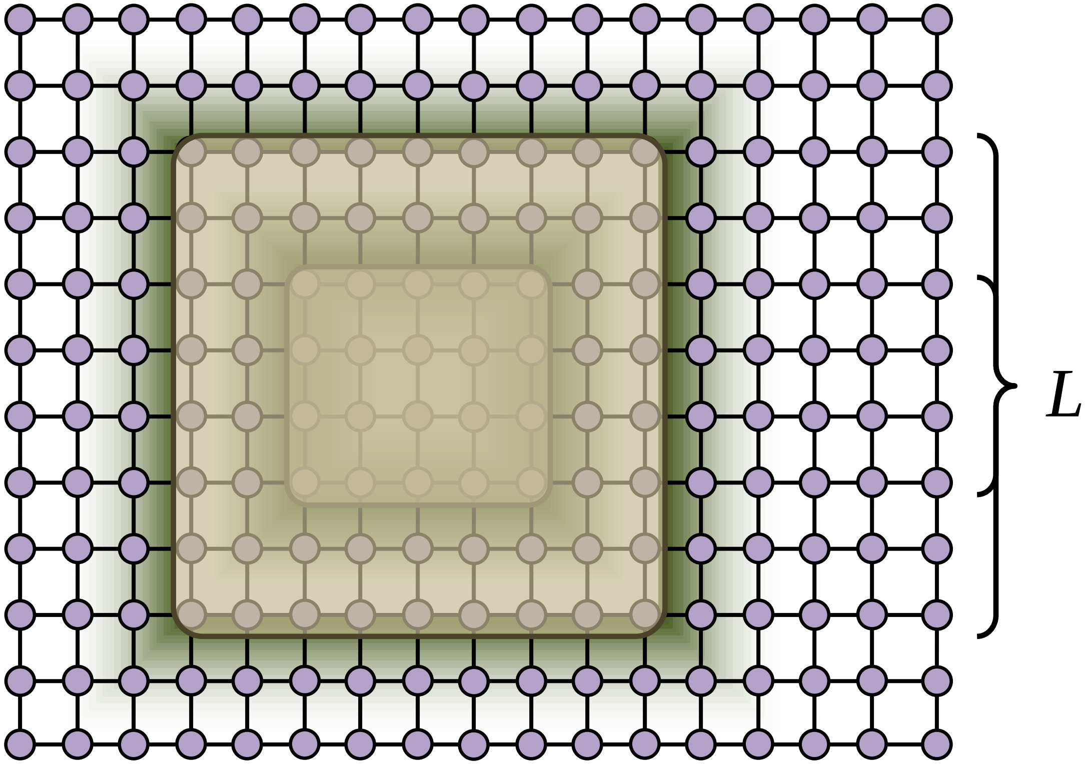
- Boundary law: $S_L = \text{const.}$ as opposed to bulk: $S_L = L$

1D Critical



- All sites of the block contribute to entanglement entropy!
- Logarithmic Correction: $S_L \propto \log(L)$

Entanglement entropy scaling in 2D systems



2D Gapped

- Boundary law: $S_L = L$

(boundary as opposed to bulk $S_L \approx L^2$)

2D Critical

- Boundary law: $S_L = L$

or

- Logarithmic violation: $S_L = L \log L$

Scaling of entanglement entropy for free fermions

1D

	Gap.	Crit.
S_L	const.	$\log(L)$

2D

	Gap.	Crit.I	Crit.II
S_L	L	L	$L\log(L)$

1D

Vidal, Latorre, Rico, Kitaev, PRL 2003
Srednicki, PRL 1993
Callan, Wilczek, Phys Lett B 1994.
Fiola, Preskill, Strominger, Trivedi, PRD 1994.
Holzhey, Larsen, Wilczek, Nucl.Phys.B 1994.
Jin, Korepin, J. Stat. Phys. 2004
Calabrese, Cardy, J. Stat. Mech. 2004

2D

Wolf, PRL 2006.
Gioev, Klich, PRL 2006.
Barthel, Chung, Schollwock, PRA 2006.
Li, Ding, Yu, Haas, PRB 2006.
Ding, Bray-Ali, Yu, Haas, PRL 2008.
Helling, Leschke, Spitzer 2009, arXiv:0906.4946.
Swingle 2009, arXiv:0908.1724.

- Can Tensor Network methods reproduce the appropriate entanglement entropy?

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- Entanglement and tensor network methods
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    - Scaling of entanglement entropy in tensor network ansatz
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    - Comparison of entropy scaling:
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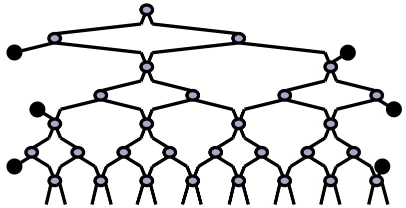

    - Decoupling a many-body theory
    - Holographic trees
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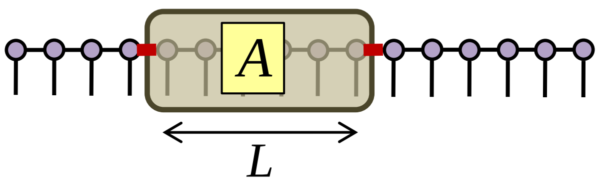
Tensor Networks in Physical Geometry

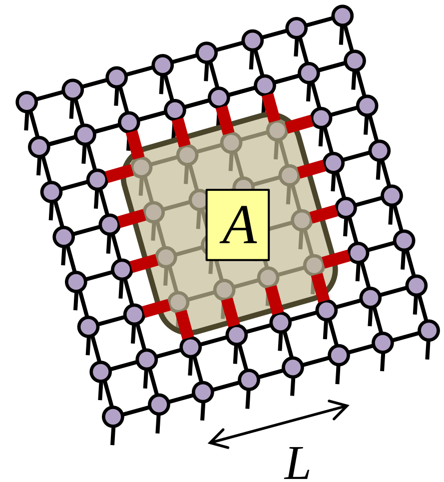
- Entanglement entropy scaling?

1D: MPS



$$S_L = \text{const.}$$

2D: PEPS

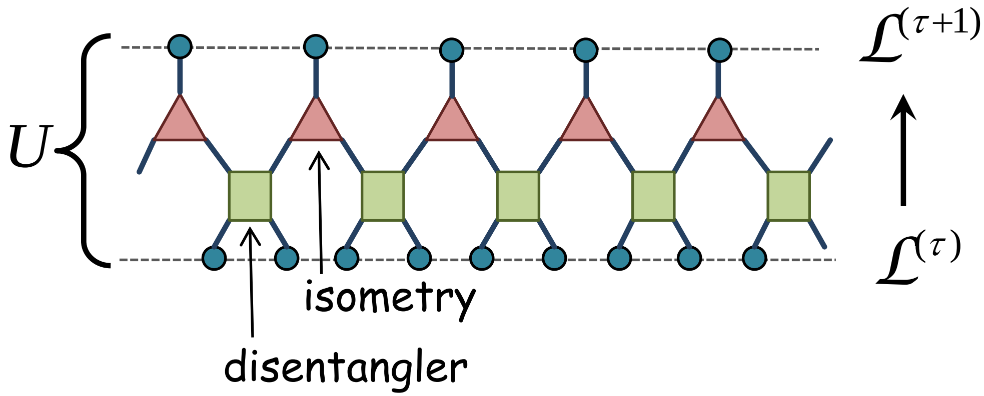


$$S_L = L$$

- Tensor networks based upon **physical geometry** produce **boundary law** for scaling of entropy: $S_L \approx L^{D-1}$

- Entropy scales as boundary in **physical geometry**: $$S(A) \sim |\partial A|$$

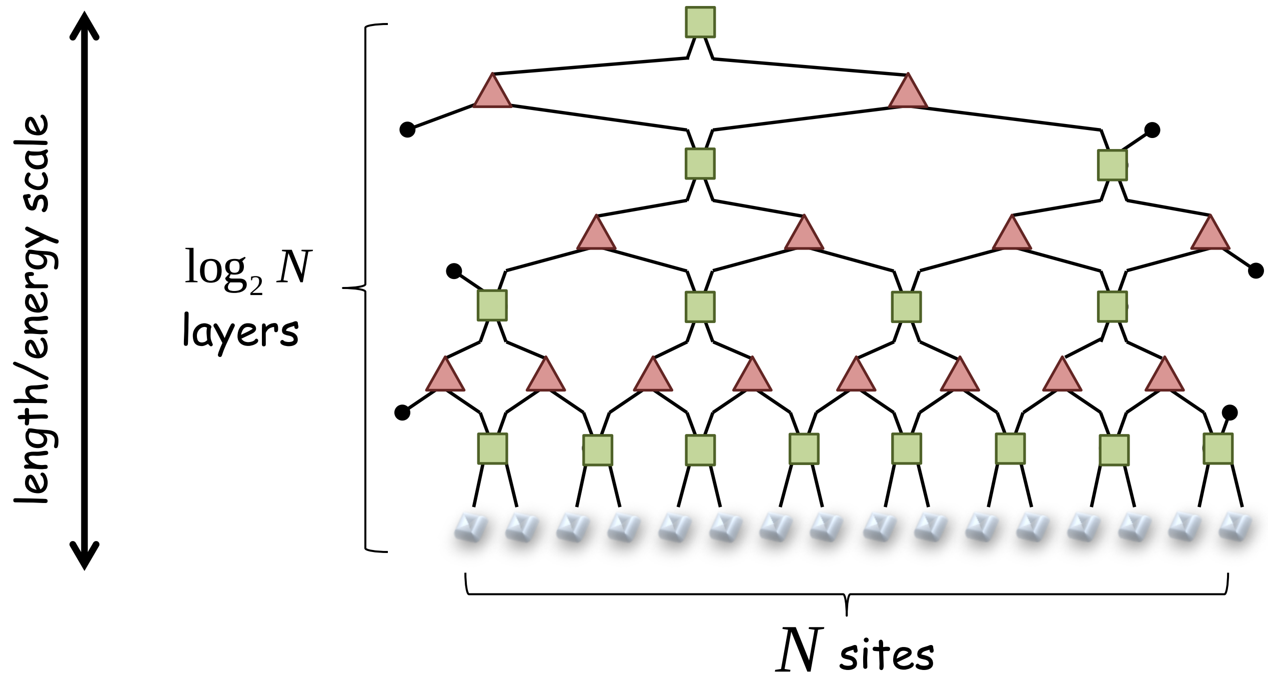
Entanglement Renormalization and the MERA



Coarse-graining transformation:

Entanglement Renormalization

MERA (multi-scale entanglement renormalization ansatz)



MERA ↔ Holography

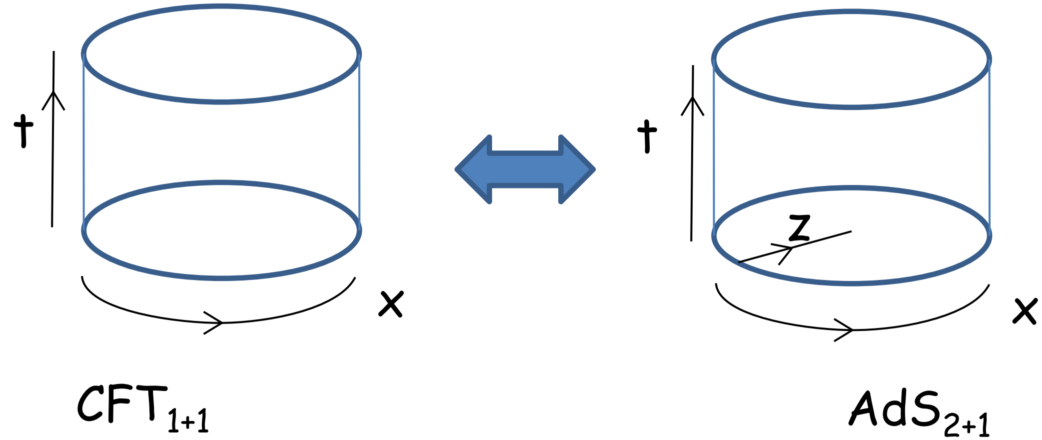
Brian Swingle

arXiv:0905.1317

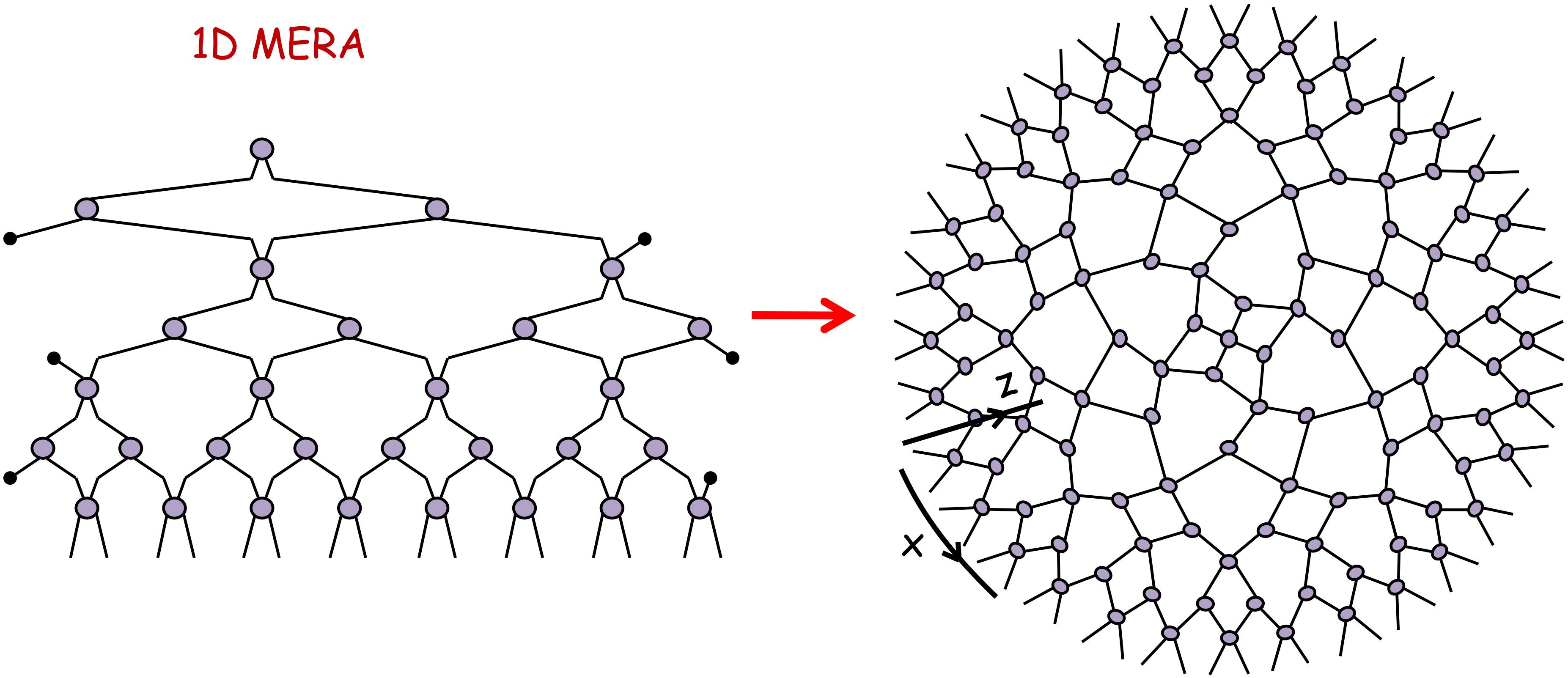
Holographic geometry

• Reproduce the pattern of entanglement in the ground state

AdS/CFT Correspondance

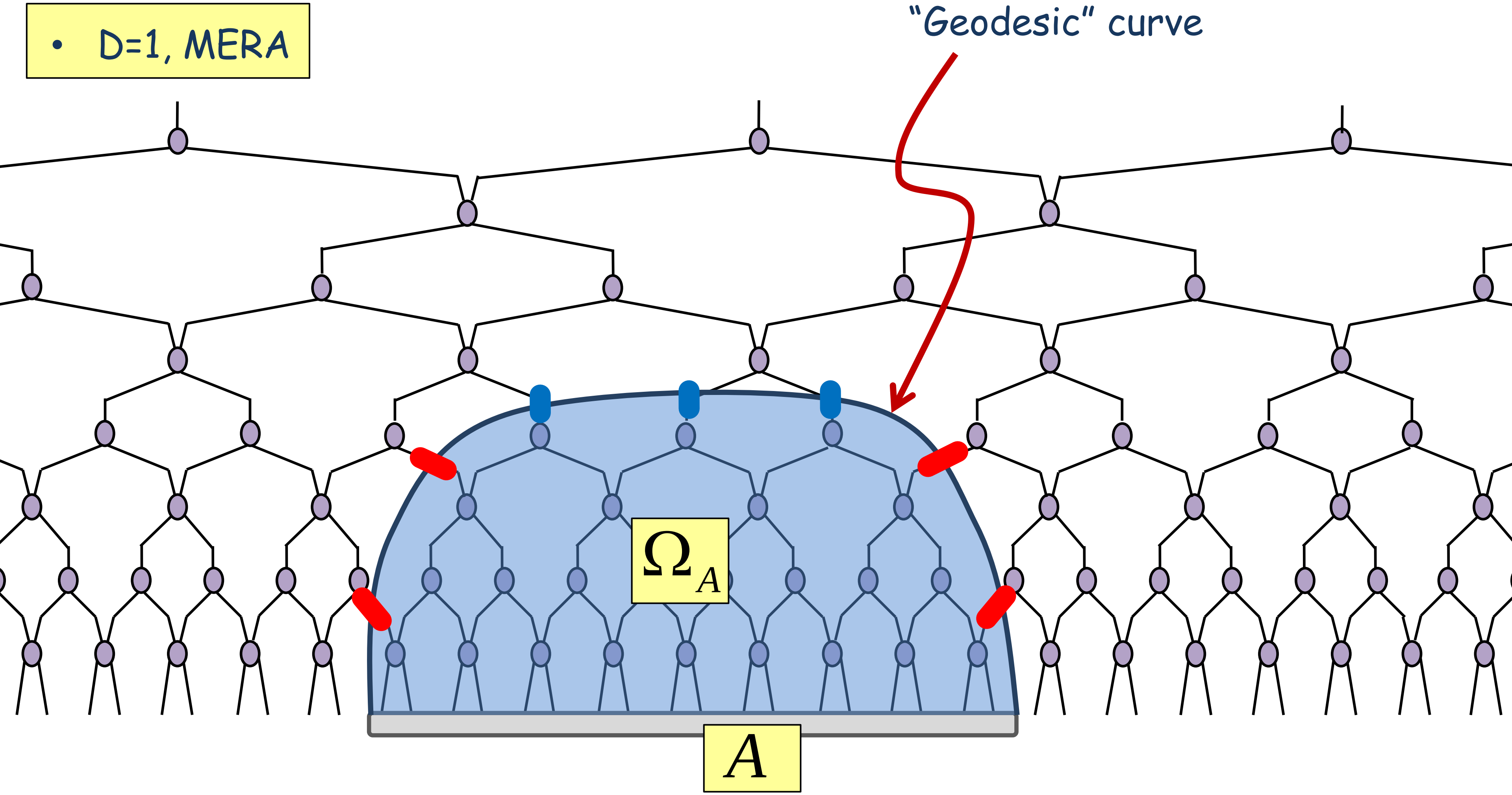


1D MERA



Computation of entanglement entropy

- D=1, MERA

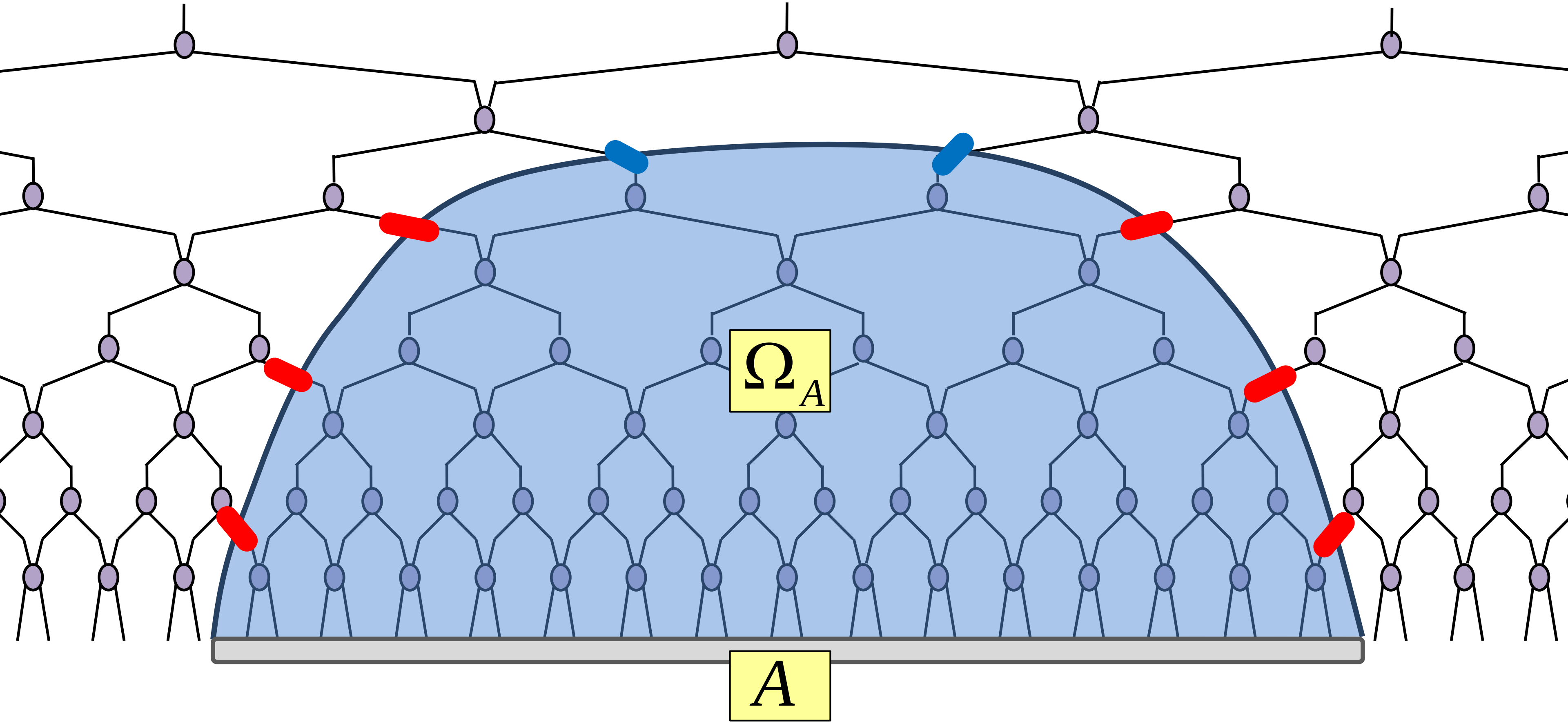


- Entanglement entropy as boundary in holographic geometry: $S(A) \sim |\partial\Omega_A|$

Holographic Entanglement Entropy, e.g. Nishioka, Ryu, Takayanagi, J. Phys. A 2009

Computation of entanglement entropy

- D=1, MERA



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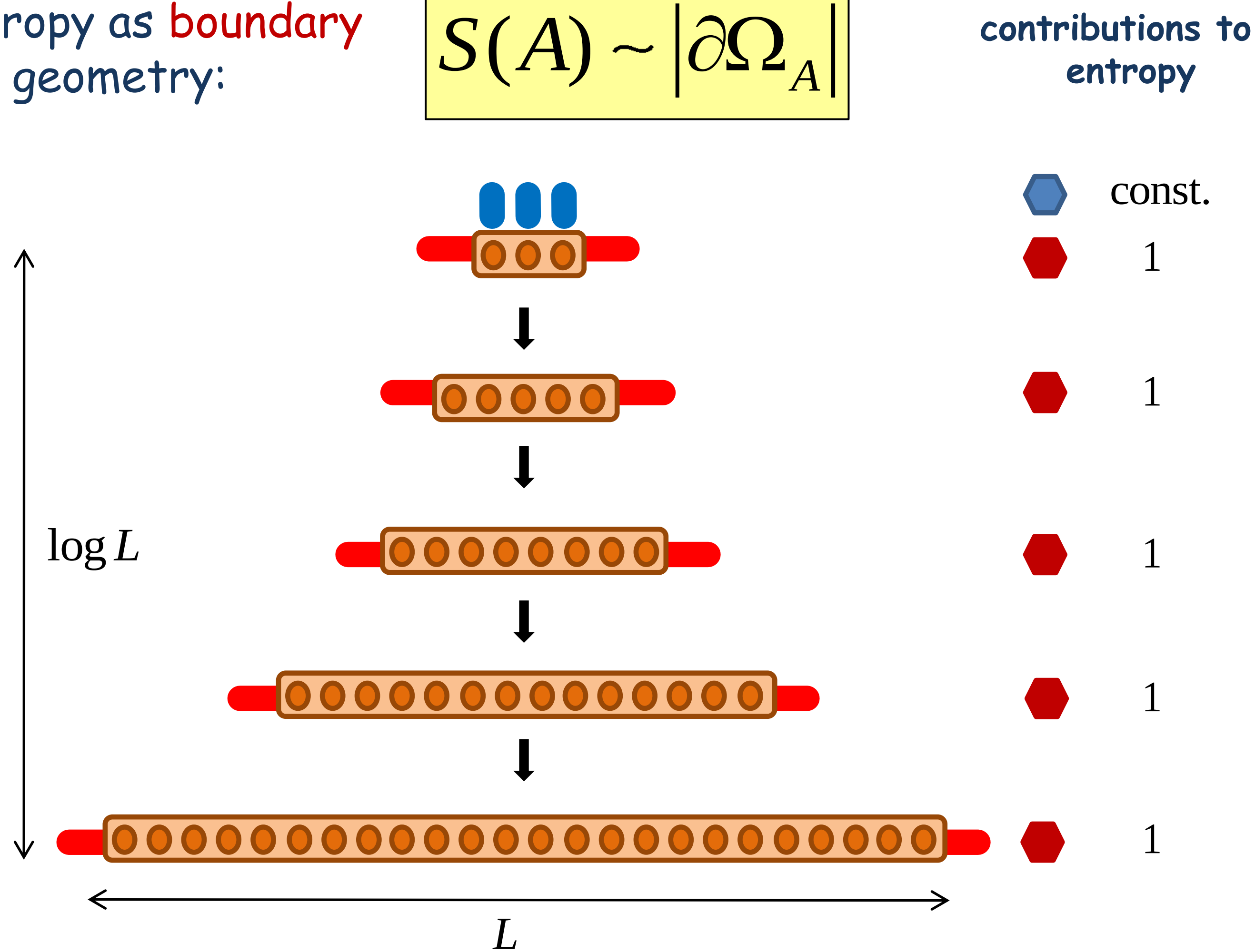
Holographic Entanglement Entropy, e.g. Nishioka, Ryu, Takayanagi, J. Phys. A 2009

MERA for D=1 spatial dimensions

- Entanglement entropy as boundary in holographic geometry:

$$S(A) \sim |\partial \Omega_A|$$

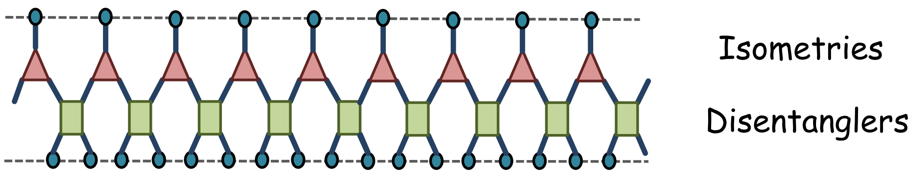


- Tensor network (MERA) based on holographic geometry can produce violations of boundary law, logarithmic violation:

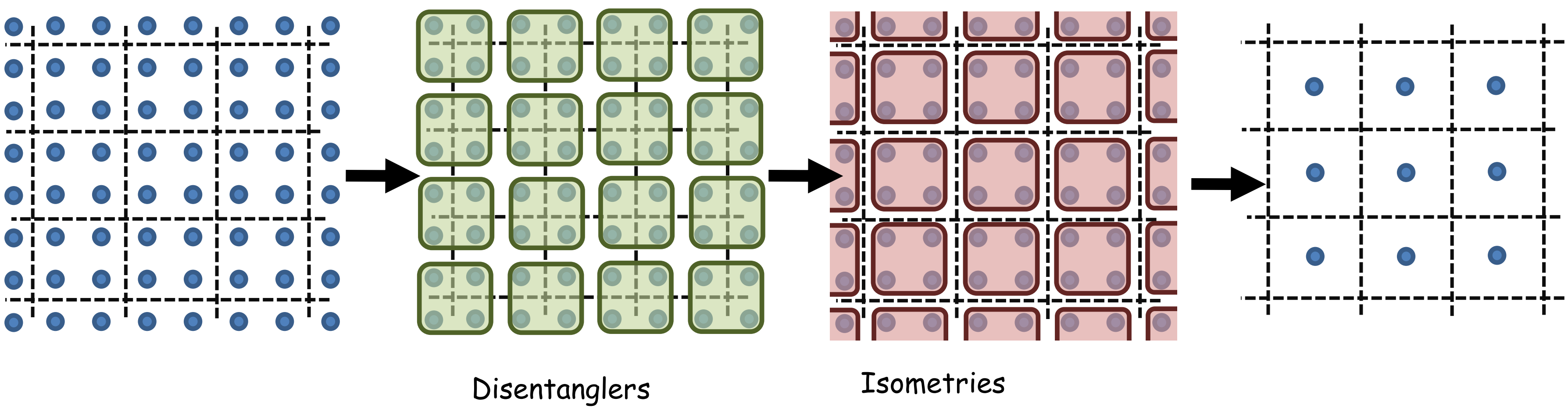
$$S(L) \approx \log L$$

MERA for D=2 spatial dimensions

D=1 spatial dimensions



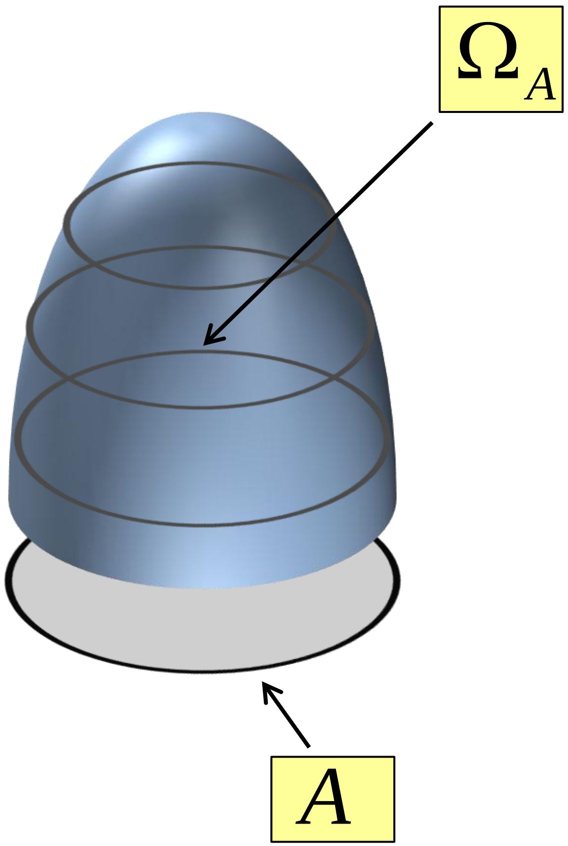
D=2 spatial dimensions

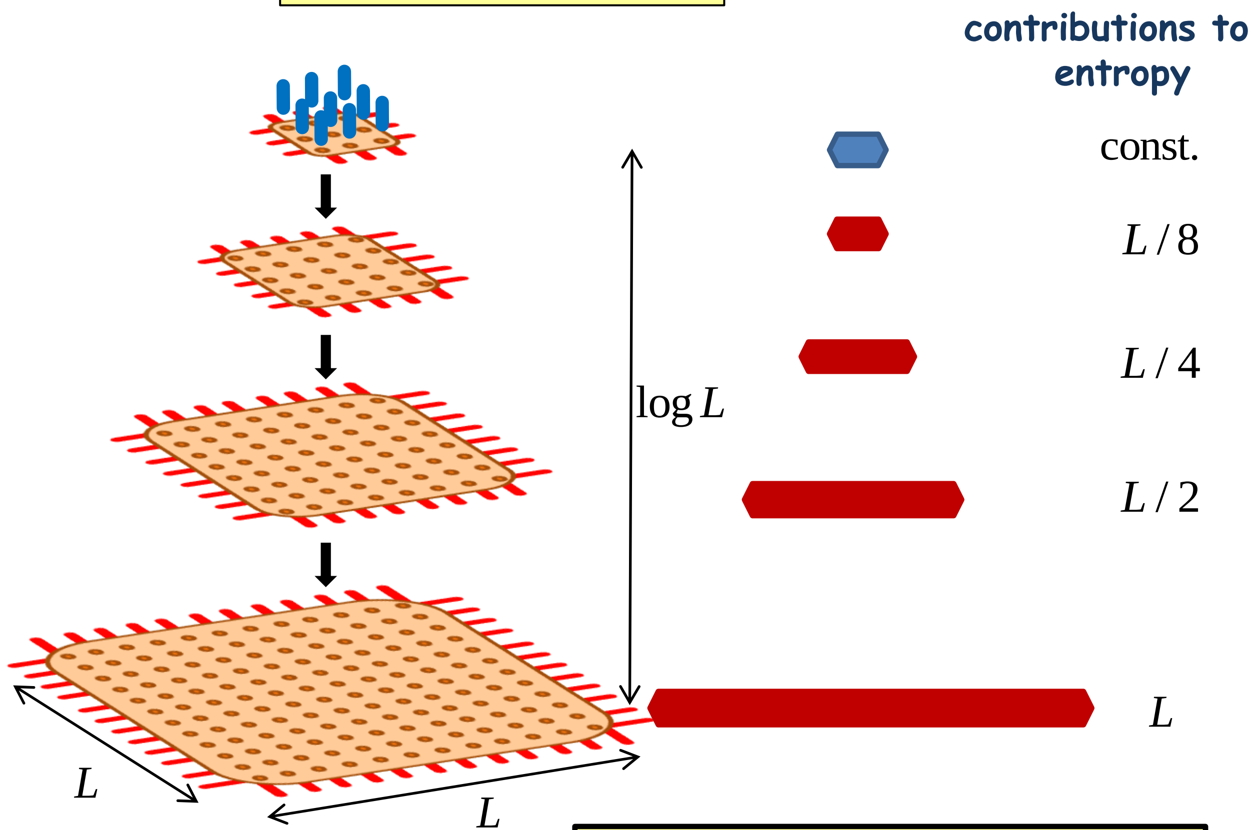


MERA for D=2 spatial dimensions

- Entanglement entropy as boundary in holographic geometry:

$$S(A) \sim |\partial\Omega_A|$$





$$S_L \approx L\underbrace{(1+\frac{1}{2}+\frac{1}{4}+\cdots)}_{\log L} \approx L$$

boundary law for entropy scaling!

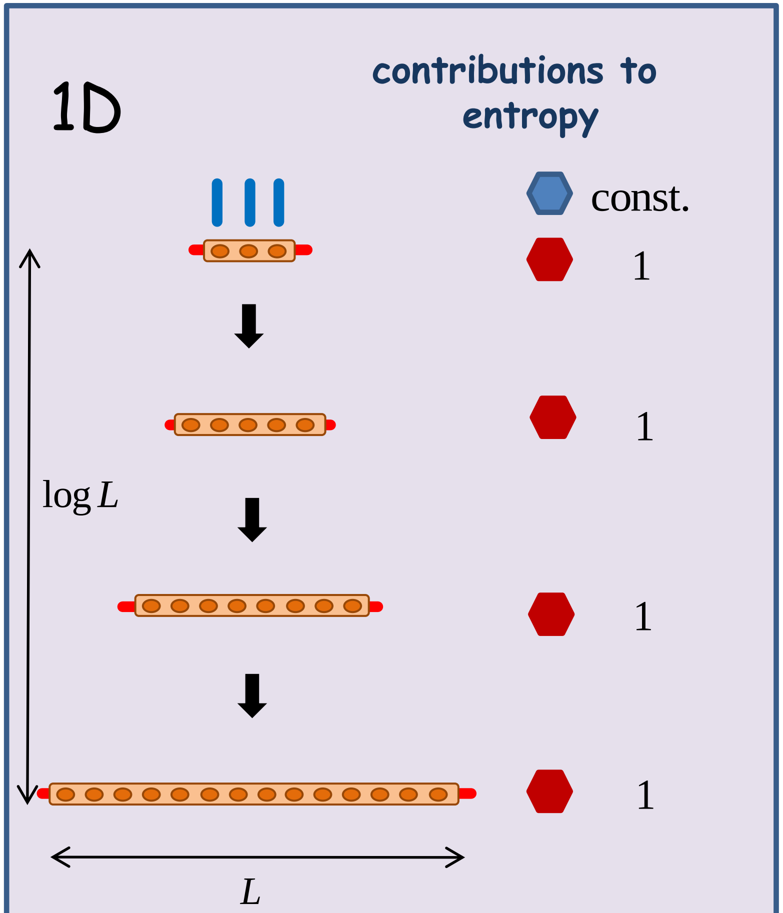
Entanglement entropy in the MERA

Vidal, quant-ph/0610099

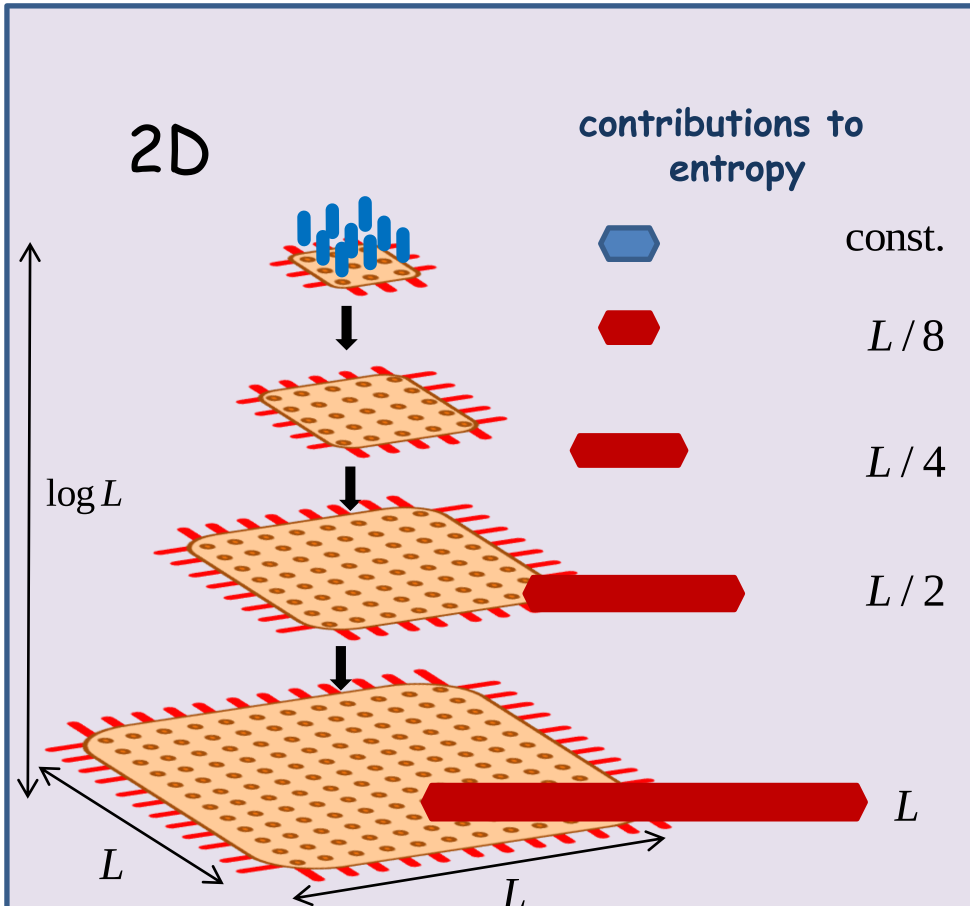
left out of PRL 101, 110501 (2008) !!!

- Entanglement entropy as boundary in holographic geometry:

$$S(A) \sim |\partial\Omega_A|$$



$$S_L \approx \underbrace{(1+1+\cdots+1)}_{\log L} \approx \log L$$



$$S_L \approx L\underbrace{\left(1+\frac{1}{2}+\frac{1}{4}+\cdots\right)}_{\log L} \approx L$$

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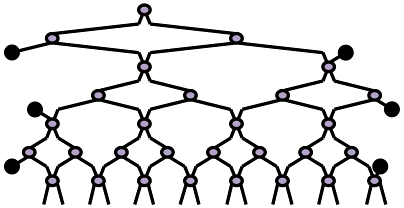

    - Decoupling a many-body theory
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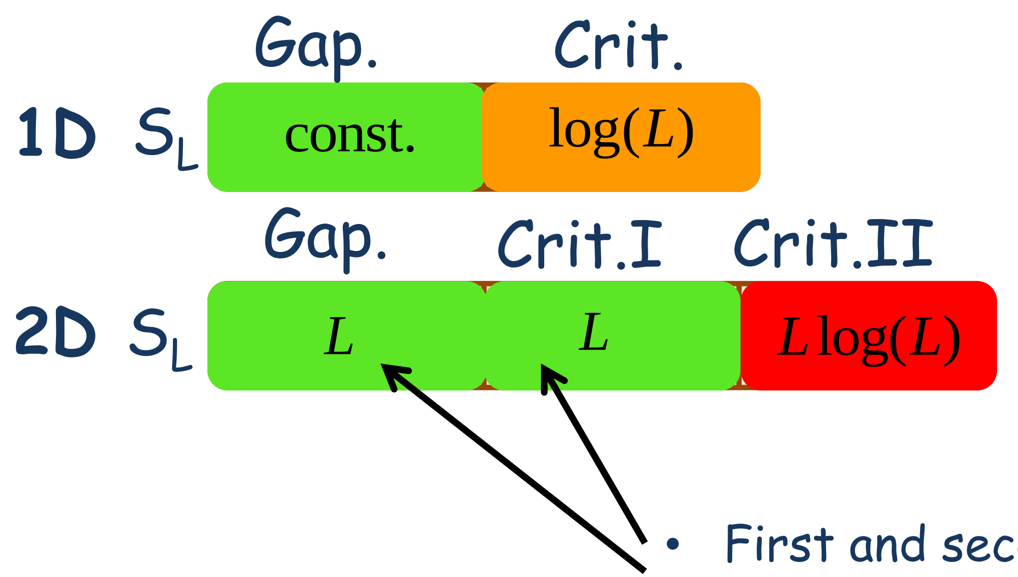
Entanglement entropy and Tensor Networks

- Can Tensor Network methods reproduce the proper entanglement entropy?

Tensor networks in physical geometry:

1D	MPS:	$S_L = \text{const.}$
2D	PEPS:	$S_L = L$



- First and second order phase transitions
- Frustrated Magnets
- Interacting Fermions

All on large (or infinite) 2D lattices!

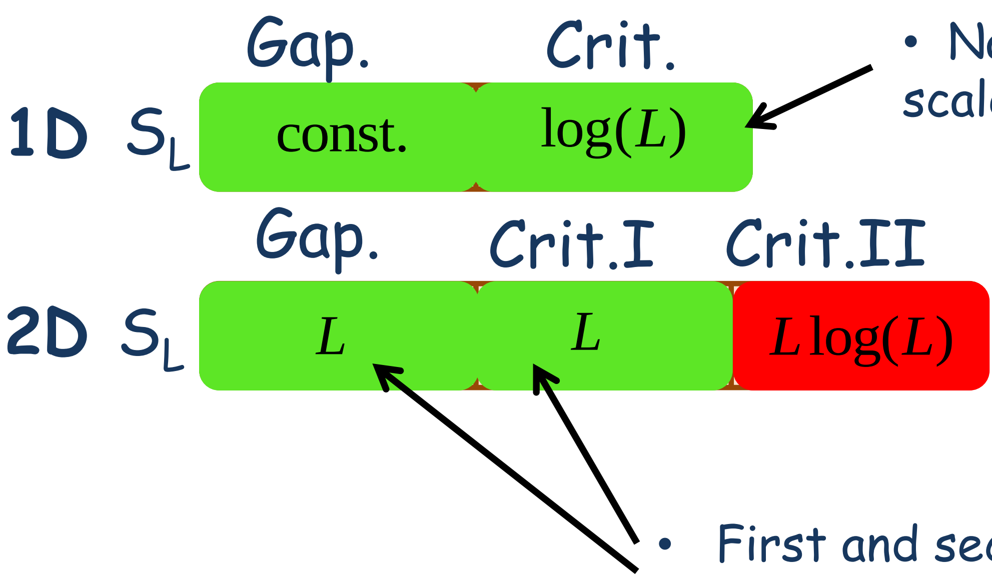
Entanglement entropy and Tensor Networks

- Can Tensor Network methods reproduce the proper entanglement entropy?

Tensor networks in holographic geometry:

1D MERA: $S_L = \log L$

2D MERA: $S_L = L$

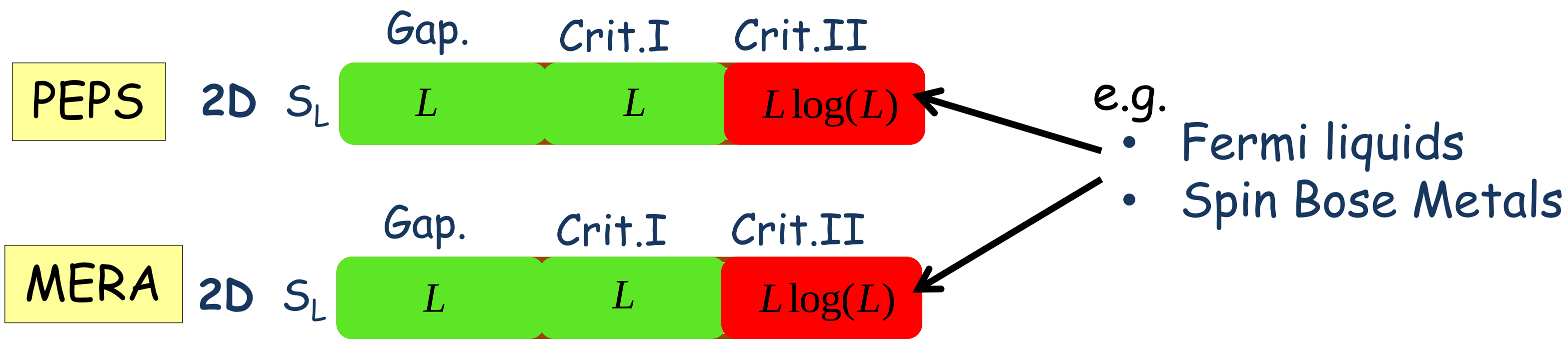


- Natural description of scale-invariant 1D systems

- First and second order phase transitions
- Frustrated Magnets
- Interacting Fermions

All on large (or infinite) 2D lattices!

Entanglement entropy and Tensor Networks



- Certain types of critical 2D phases cannot be properly addressed (large N simulations) with current tensor-network techniques

- The **entanglement structure** of these systems is not properly understood

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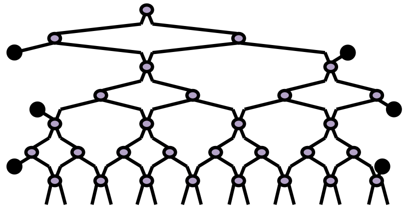

Entanglement entropy and Tensor Networks

- Entanglement entropy in MERA as boundary in holographic geometry: $S(A) \sim |\partial\Omega_A|$

- By considering exotic holographic geometries we obtain a more general class of MERA that reproduces more entanglement entropy

➡ **Branching MERA** — Evenbly, Vidal, in preparation


Guifre Vidal

	Gap.	Crit.I	Crit.II
2D S_L	L	L	$L\log(L)$

- Potentially a good ansatz for 2D systems with a 1D Fermi surface

- Can reproduce other violations to boundary law for entropy scaling
  - e.g. 1D quantum system with entropy: $S_L \propto (\log L)^2$

- Provides a framework for understanding entanglement structure

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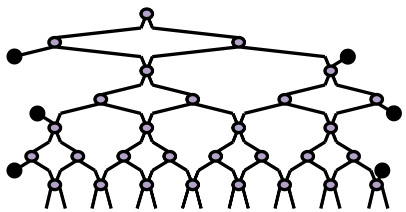

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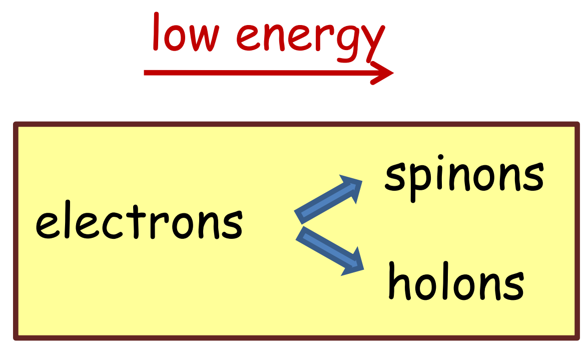
Low energy decoupling and holographic branching

MOTIVATION:

at low energies, sometimes sets of degrees of freedom decouple

Examples:

- 1D system: spin-charge separation



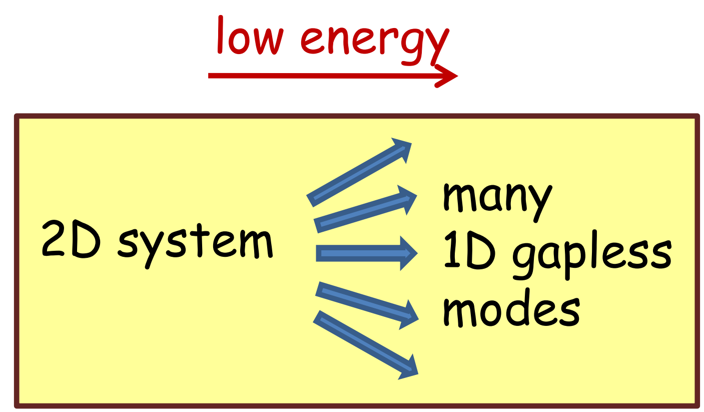
- 2D systems with 1D Fermi surface (or 1D Bose surface)

  - free fermions
  - Fermi liquids

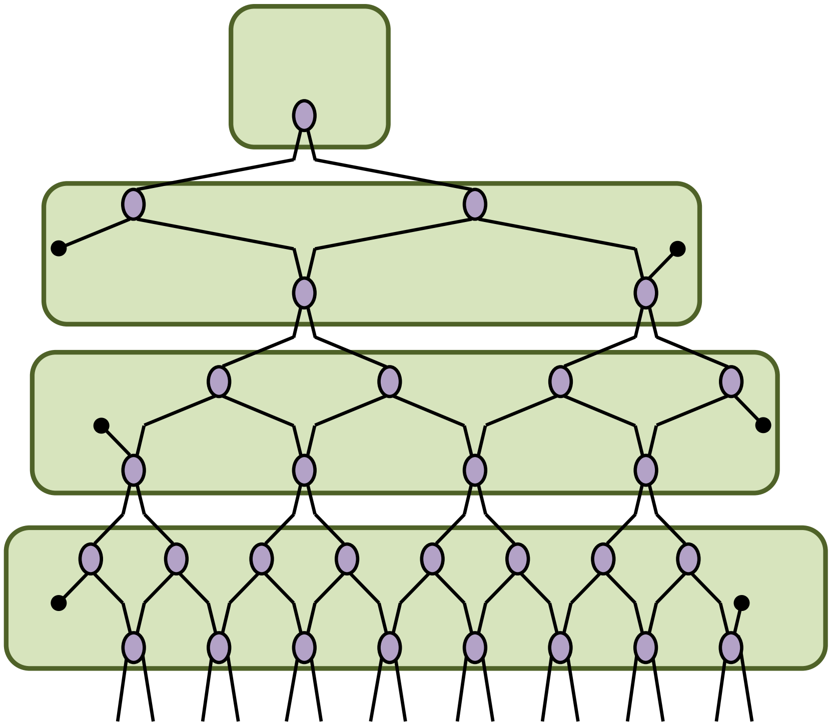
    e.g. Swingle, arXiv:0908.1724 and arXiv:1002.4635.
    Senthil, Phys. Rev. B 2008.
    Faulkner, Liu, McGreevy, Vegh, arXiv:0907.2694.

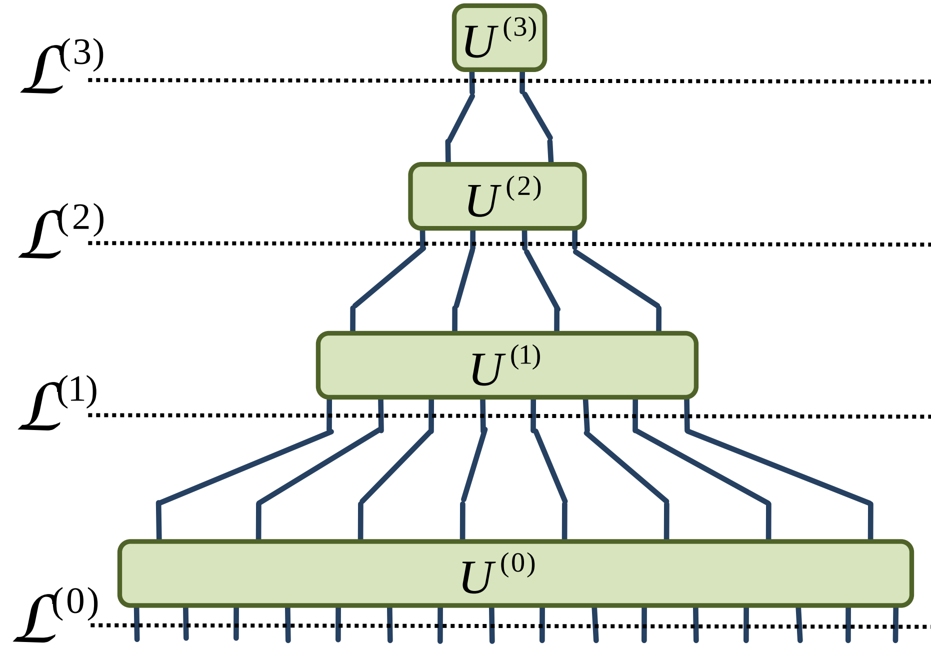
  - spin Bose metal

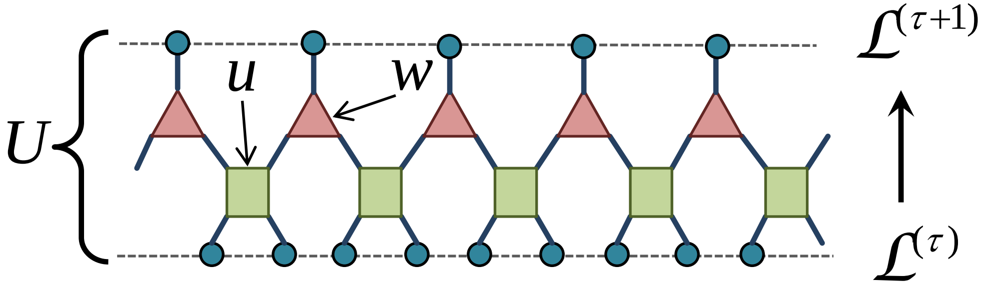
    e.g. Block, Sheng, Motrunich, Fisher, arXiv:1009.1179.
    Sheng, Motrunich, Fisher, arXiv:0902.4210.

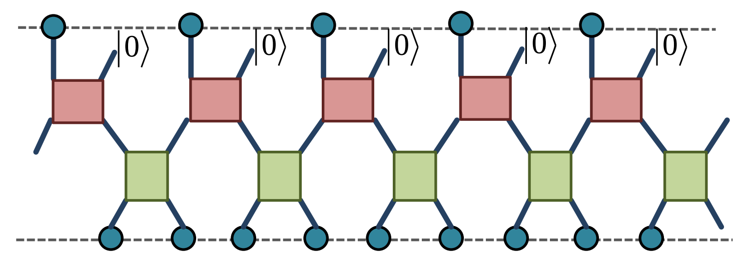


- simplified diagrammatic representation for the MERA

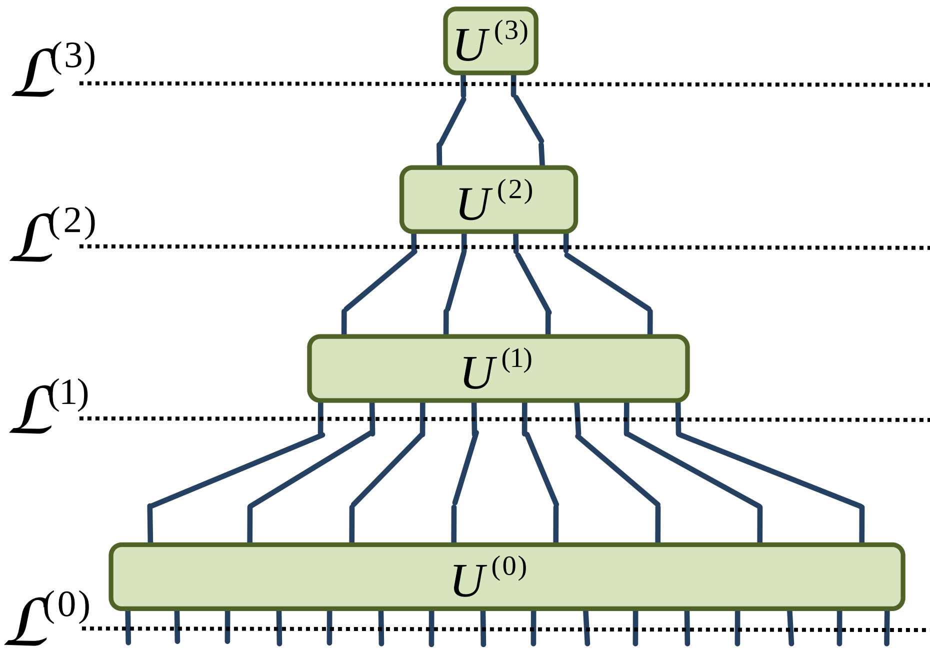




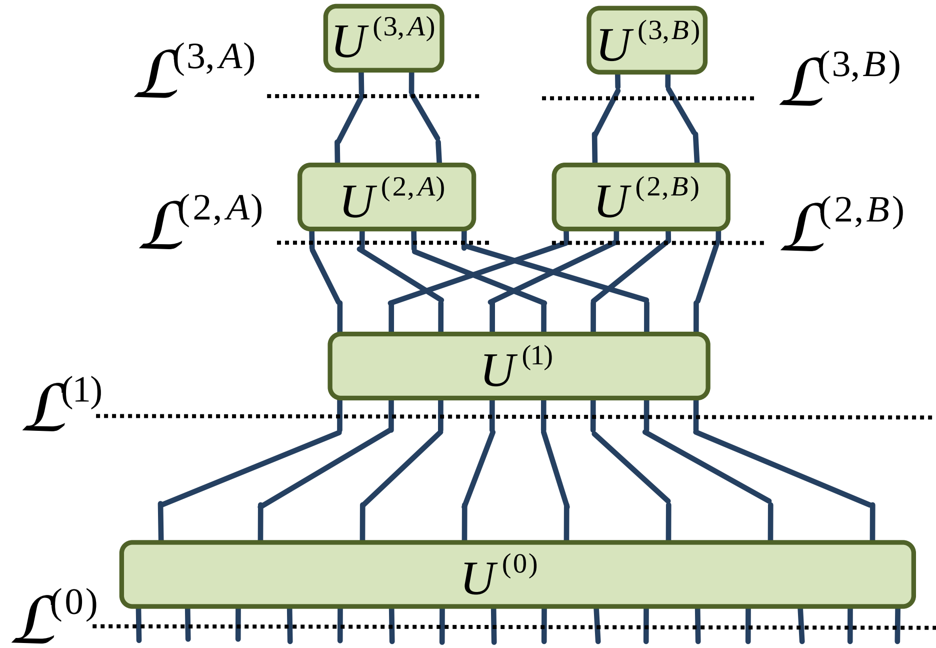


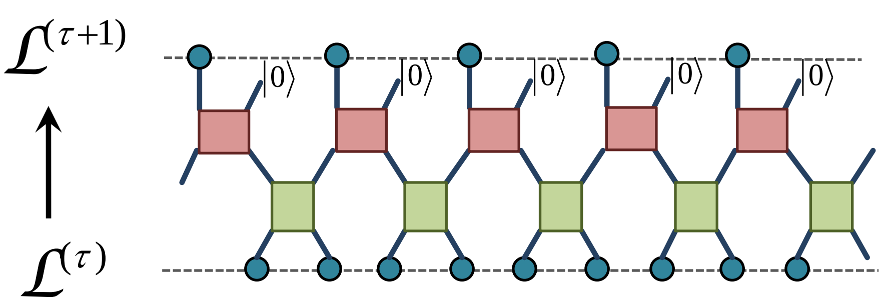


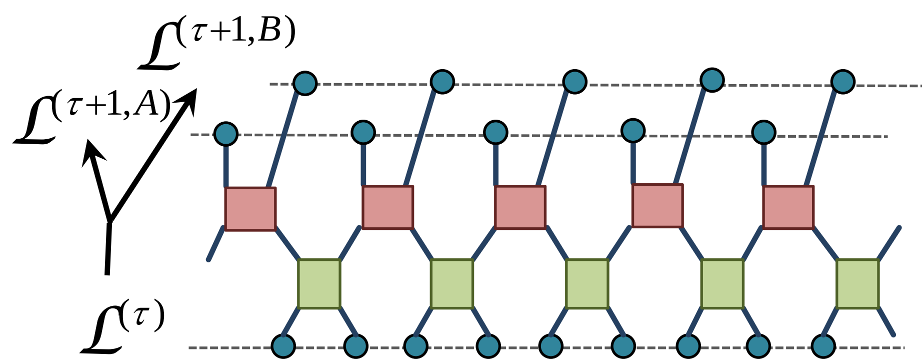
- Entanglement renormalization in the presence of decoupling



branching MERA



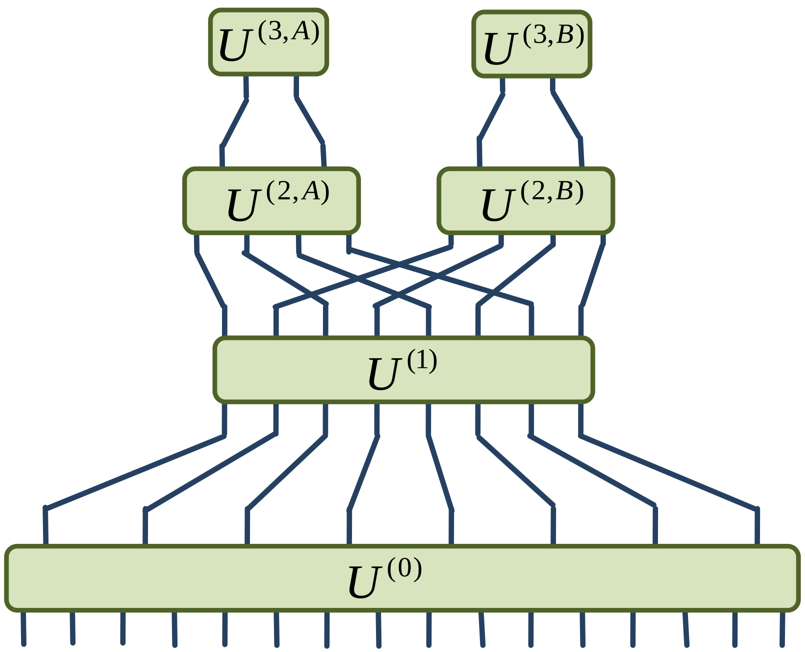


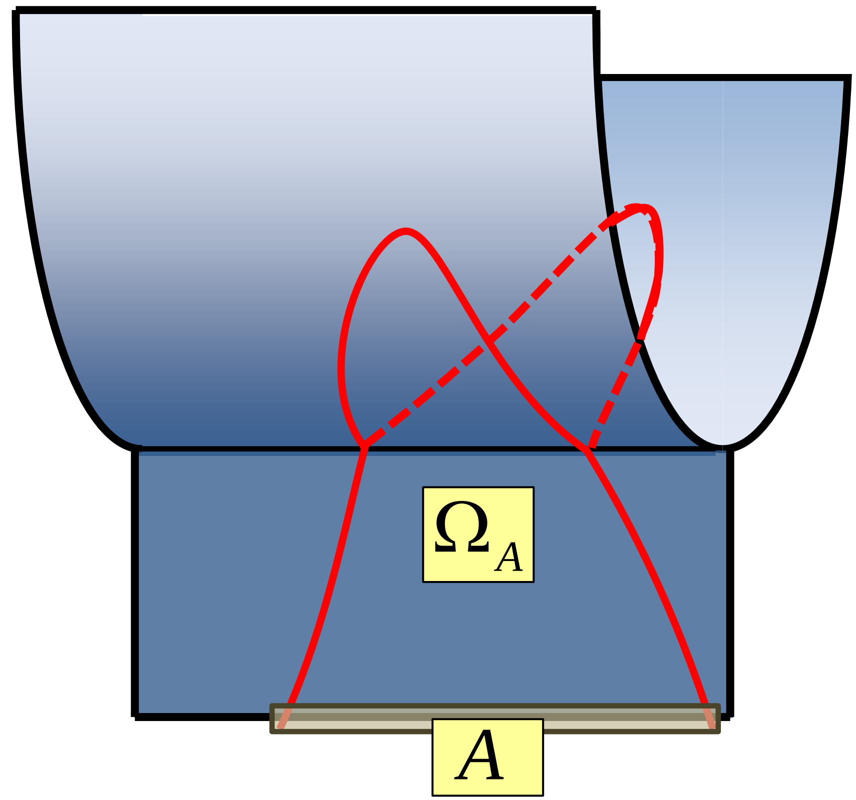


Entanglement entropy?

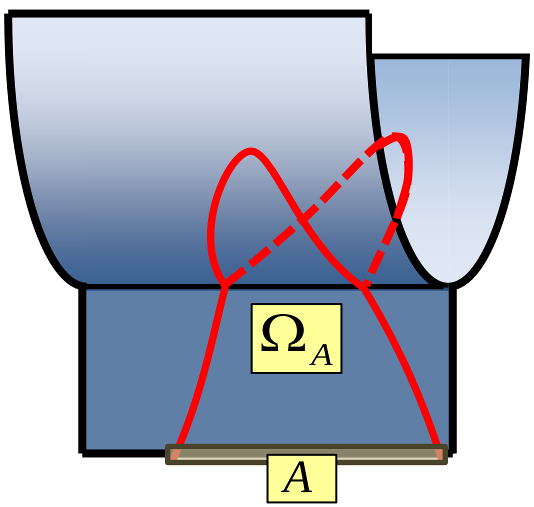
- given by boundary in exotic holographic geometry: $S(A) \sim |\partial\Omega_A|$

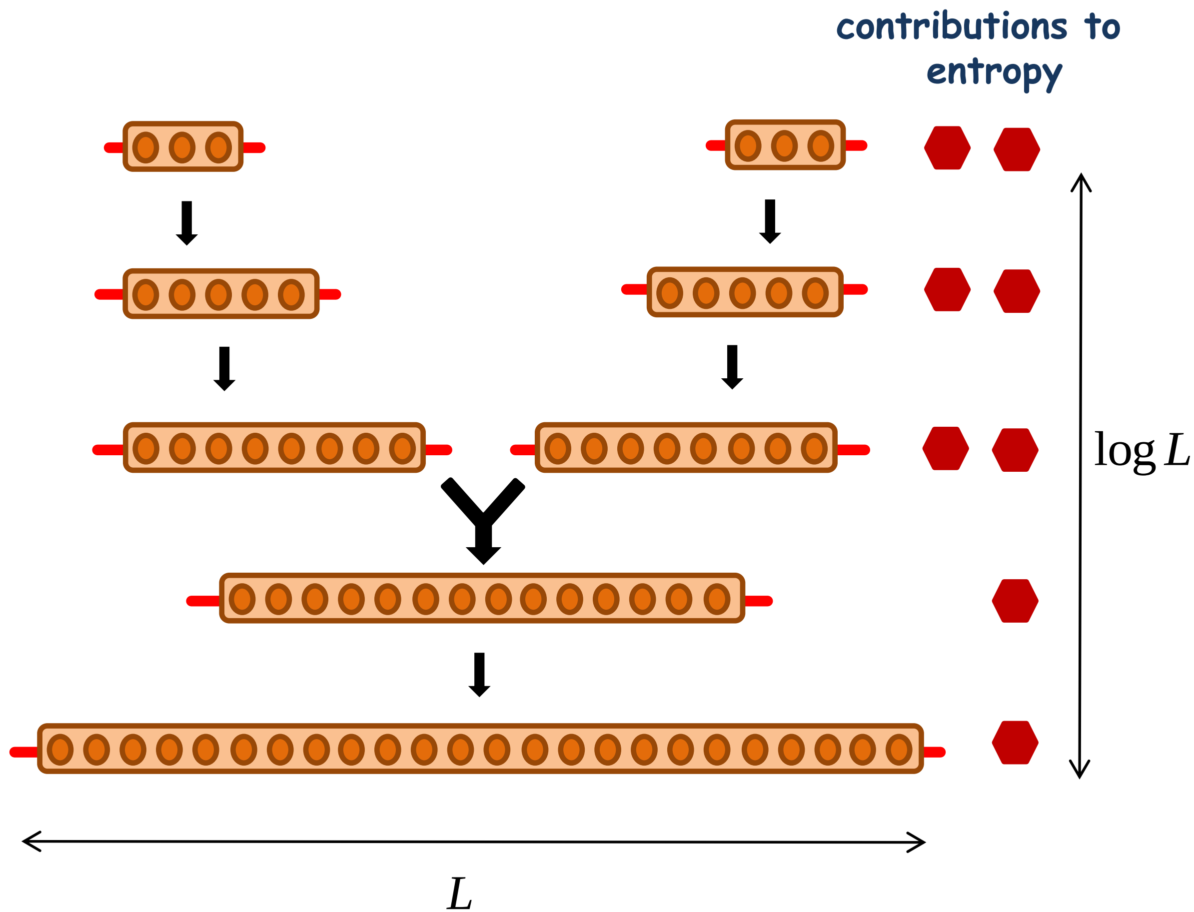
branching MERA





Entanglement entropy?





- Holographic branching can increase size of Ω_A

➡ branching MERA can reproduce more entanglement entropy!

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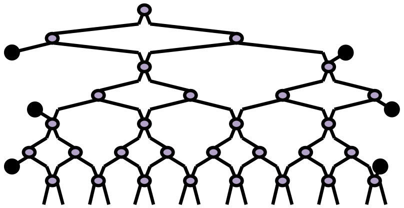
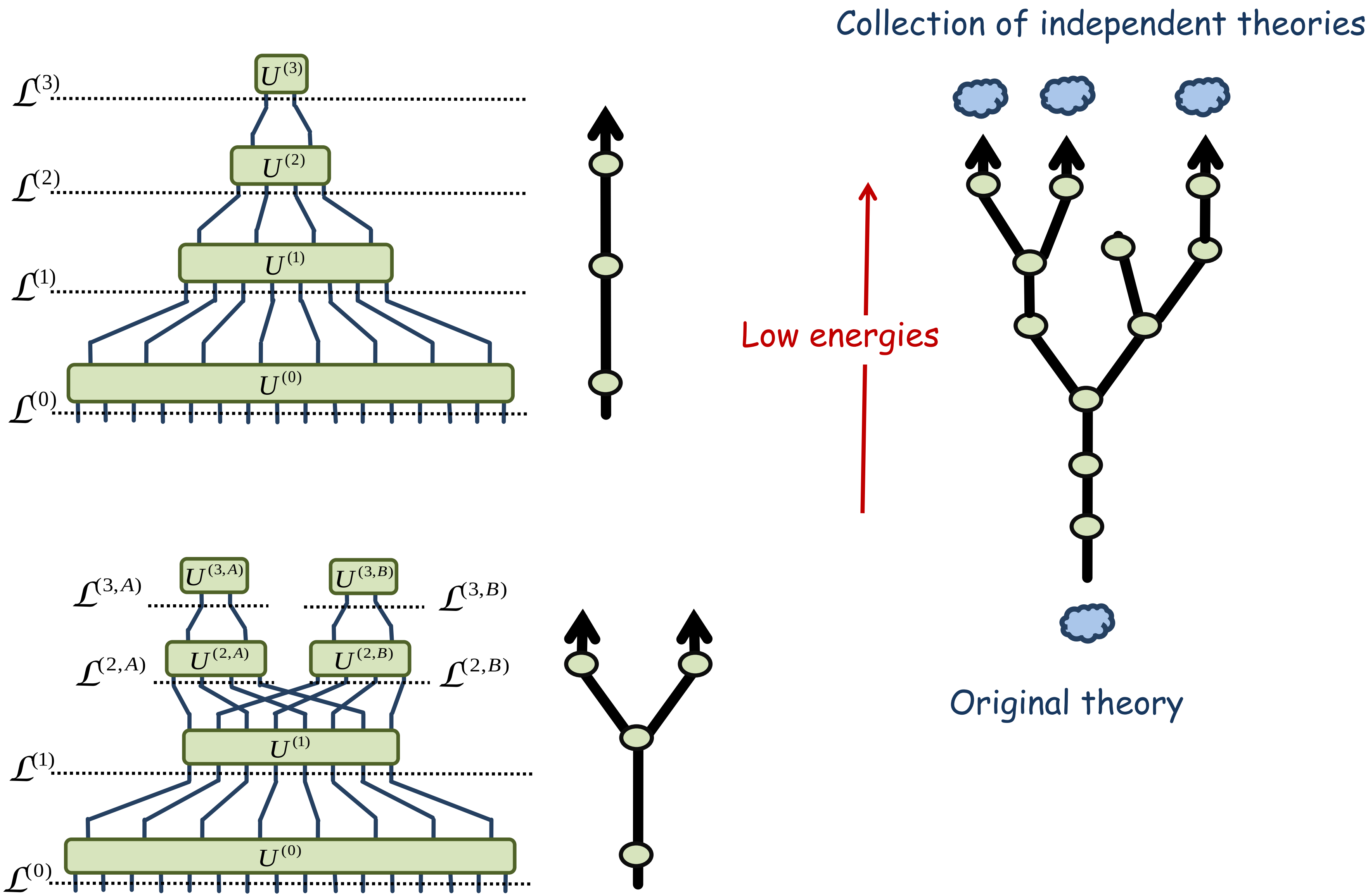
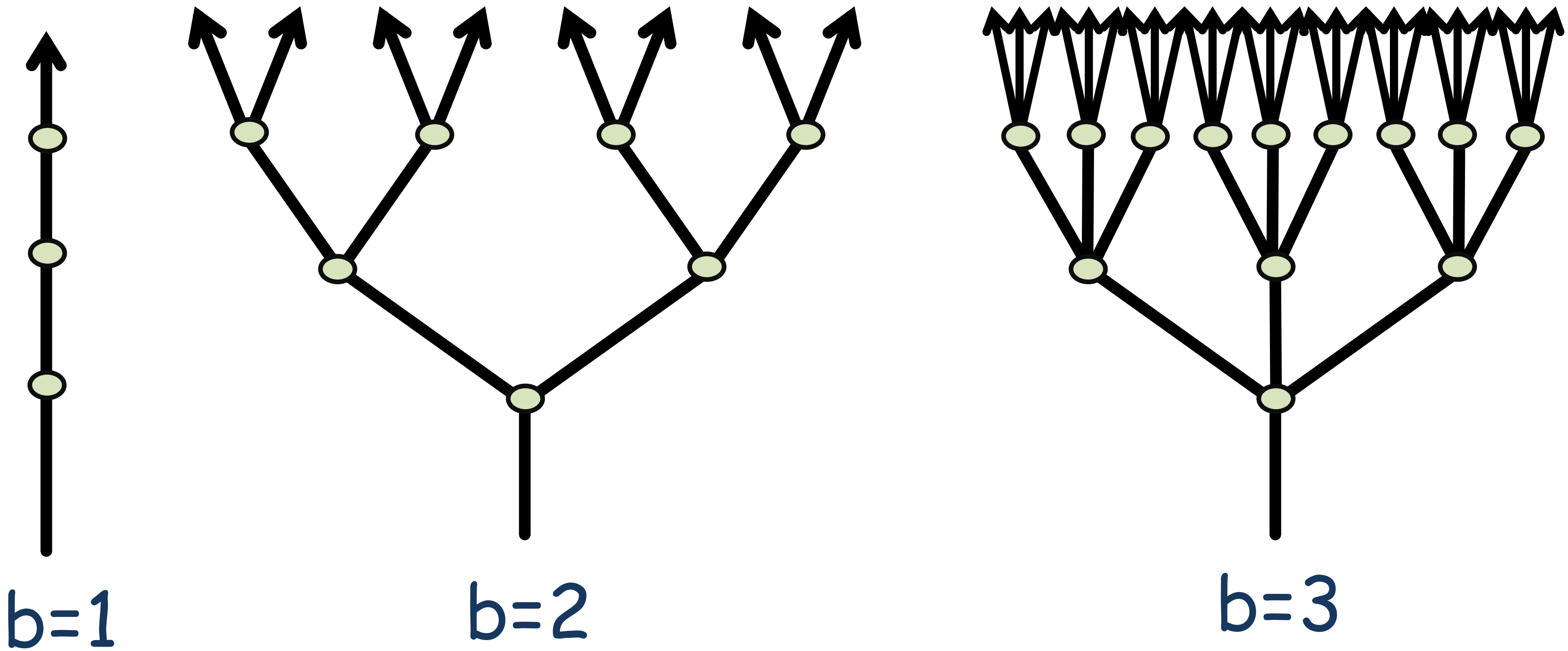


Holographic tree



Holographic Trees

- regular b-ary holographic trees:



Branching Parameter, b

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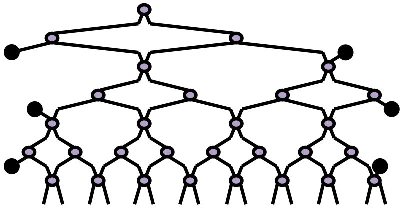
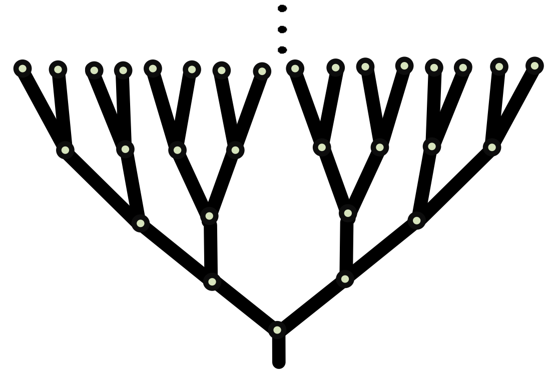
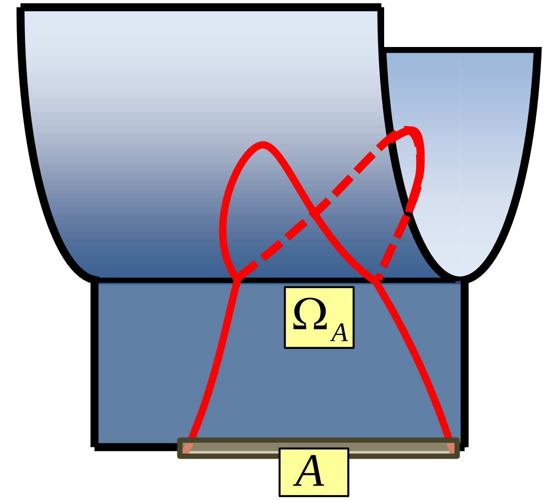
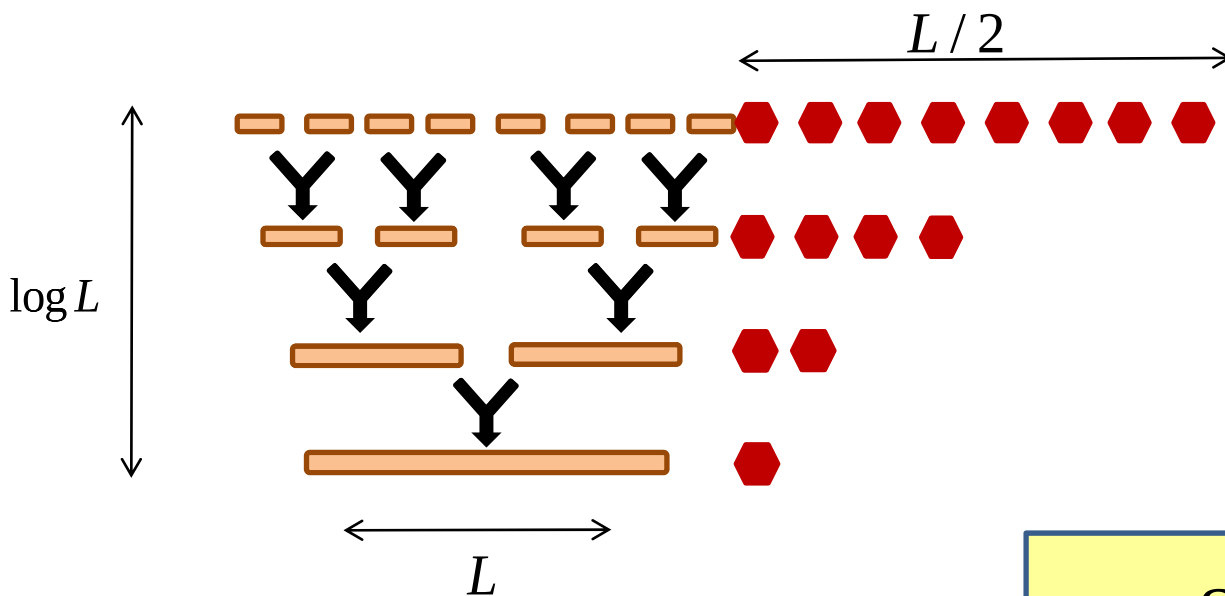


Holographic trees and entanglement entropy

b=2 branching MERA in
D=1 spatial dimensions







contributions to entropy

$8 = 2^{\log_2 L - 1} = L/2$

$4 = 2^2 = L/4$

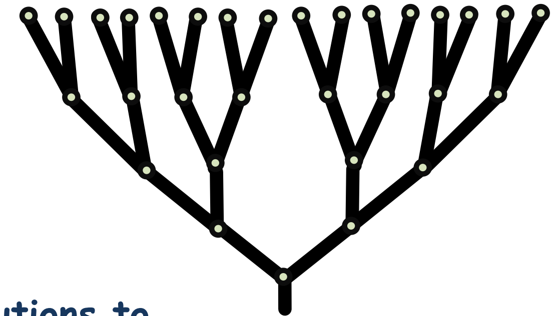
$2 = 2^1 = L/8$

$1 = 2^0 = L/16$

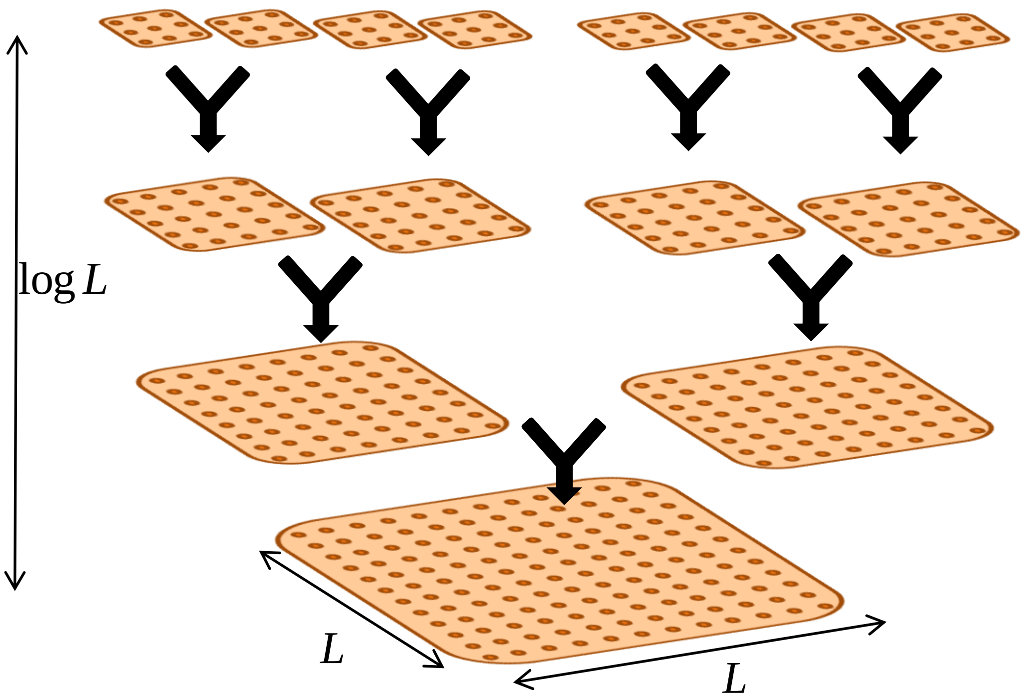
$$S_L \approx L$$

entropic "bulk" law (!)

Holographic tree and entanglement entropy



b=2 branching MERA in D=2 spatial dimensions



$$S_L \approx \underbrace{L + L + \cdots + L}_{\log L}$$

$$S_L \approx L \log L$$

logarithmic violation (!)

Is the (b=2) branching MERA a good ansatz for $S_L = L\log L$ phase??

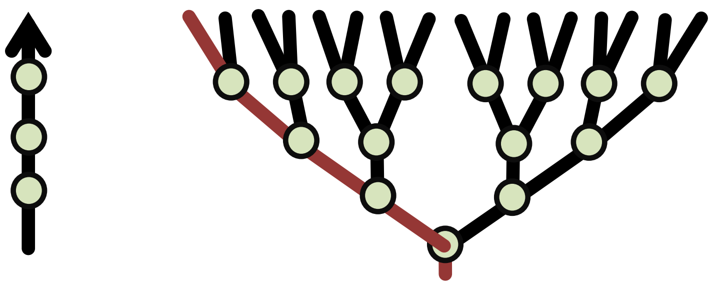
Yes!

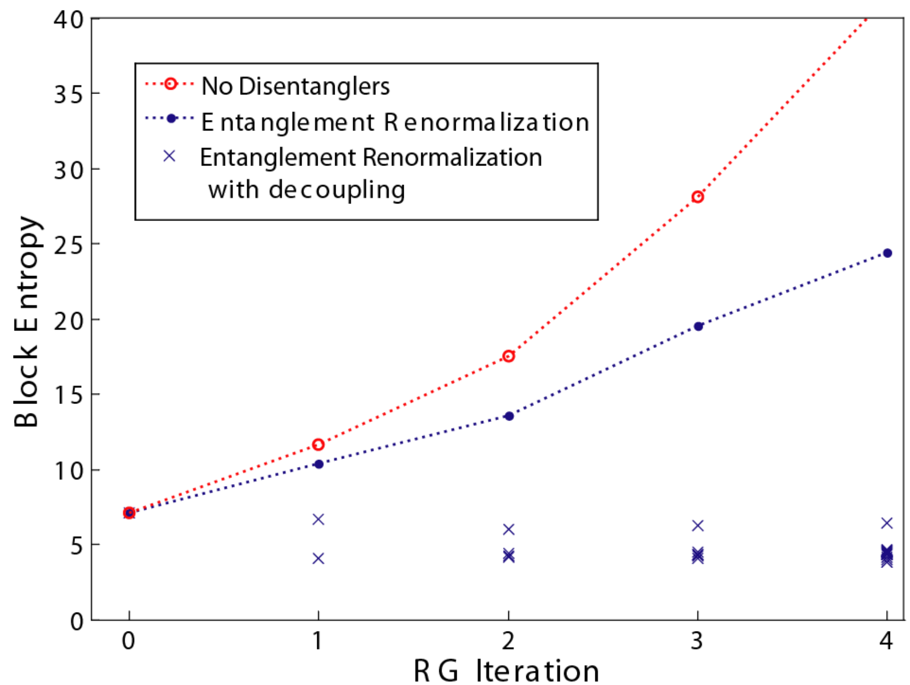
Example: free fermions in 2D

$$H = \sum_{\langle x,y \rangle} (a_x^\dagger a_y + h.c.)$$

critical model type II
(1D Fermi surface)

$$S_L \approx L \log L$$





Branching MERA — Evenbly, Vidal, in preparation

	Gap.	Crit.I	Crit.II
2D S_L	L	L	$L\log(L)$

Guifre Vidal

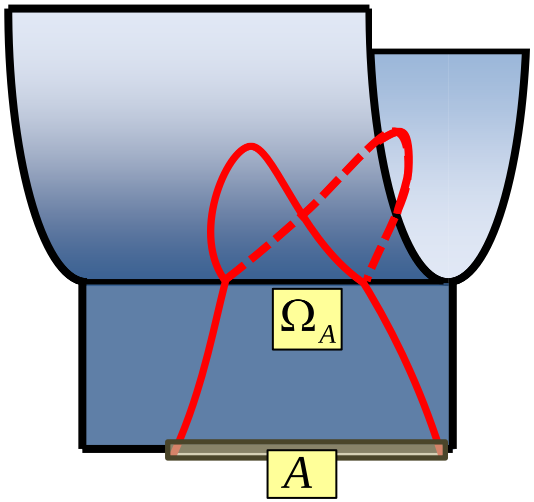
Scaling of entanglement: free fermions vs branching MERA

- Free Fermions:

Dimension of Fermi Surface, Γ

Spatial dimension

	Γ=0	Γ=1	Γ=2	
1D	$\log(L)$			
2D	L	$L\log(L)$		
3D	L^2	L^2	$L^2\log(L)$	

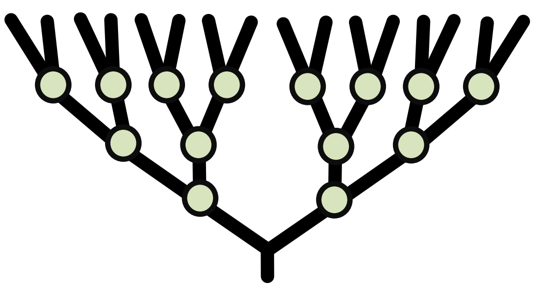


- Regular branching MERA:

Branching Parameter, b

Spatial dimension

	b=1	b=2	b=4	b=8
1D	$\log(L)$	L		
2D	L	$L\log(L)$	L^2	
3D	L^2	L^2	$L^2\log(L)$	L^3



- Proposed relation between dimensionality of Fermi surface and branching parameter:

$$b = 2^{\Gamma}$$

Corrections to the Boundary Law for Entanglement Entropy

$$S_L = L^{D-1} f(L)$$

Boundary Law　　Correction

Arbitrary Corrections!

$f(L) = 1$

Boundary Law

$b < 2^{D-1}$

$f(L) = \log L$

Logarithmic Violation

$b = 2^{D-1}$

$f(L) = L$

Bulk Law

$b = 2^{D}$

Sub-logarithmic

e.g. $f(L) = \log(\log L)$

Super-logarithmic

e.g. $f(L) = (\log L)^{\alpha}$

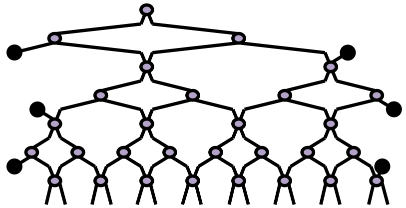
$f(L) = L^{\alpha}$

Example: 1D branching MERA with $S_L \approx (\log L)^2$

Outline

- Entanglement and tensor network methods
  - Scaling of entanglement entropy in ground states
  - Scaling of entanglement entropy in tensor network ansatz
    - physical geometry vs holographic geometry
  - Comparison of entropy scaling:
    - ground states vs tensor network ansatz

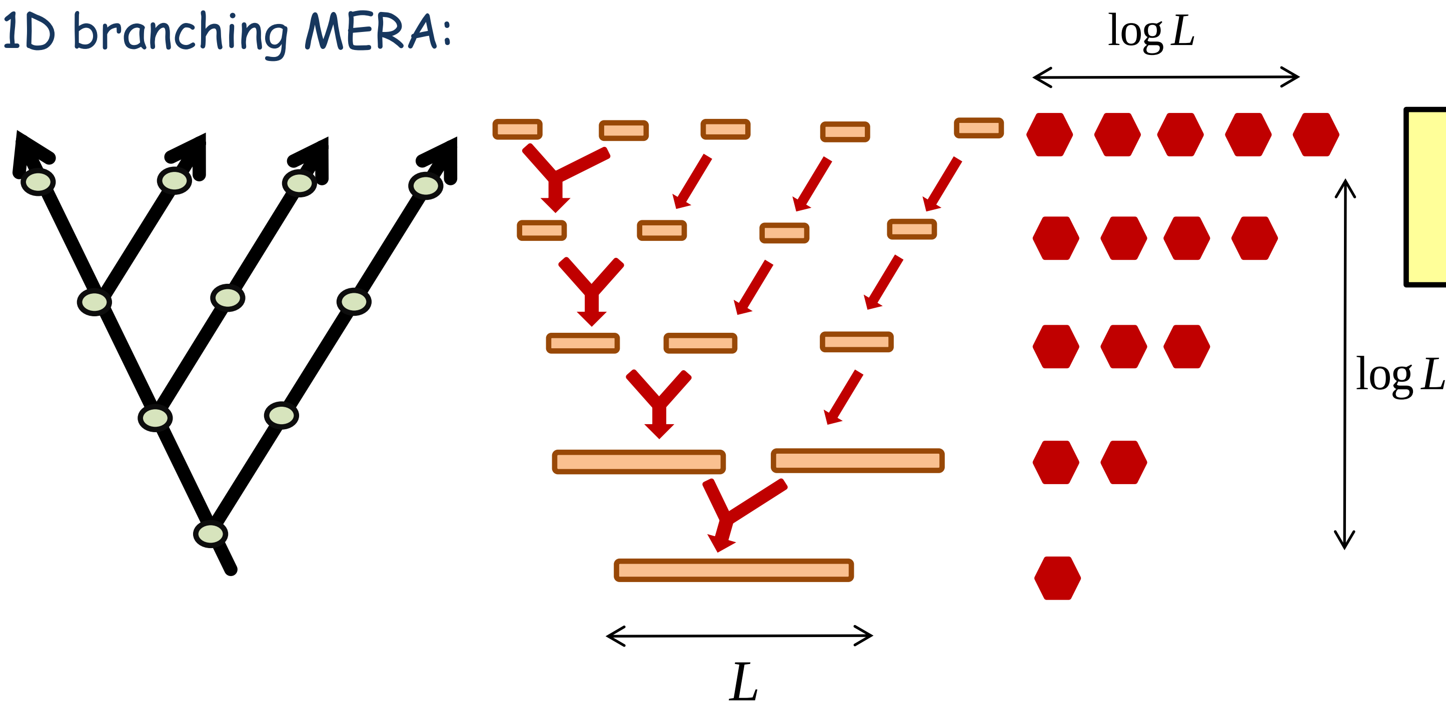
- Introduction of branching MERA



  - Decoupling a many-body theory
  - Holographic trees
  - Scaling of entropy in the branching MERA
  - Example: $S_L = L \log L$ entropy scaling in 2D fermions
  - Example: $S_L = (\log L)^2$ entropy scaling in 1D fermions

Branching MERA beyond Regular Holographic Trees

1D branching MERA:



$$S_L \approx (\log L)^2$$

- Can we find a Hamiltonian that has this ground state entropy scaling?

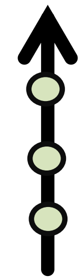
Yes!

$$H = \sum_{r=-\infty}^{\infty}\left(\sum_{\substack{d=-\infty \\ d\neq 0}}^{\infty}\frac{\phi(d)}{d^2}\left(\hat{a}_{r+d}^{\dagger}\hat{a}_r + \hat{a}_r^{\dagger}\hat{a}_{r+d}\right)\right) - \mu\sum_r \hat{a}_r^{\dagger}\hat{a}_r$$

$$\phi(d) \approx \cos\left(\log_2 |d|\right)$$

Branching MERA beyond Regular Holographic Trees

Holographic Tree:



Hamiltonian:

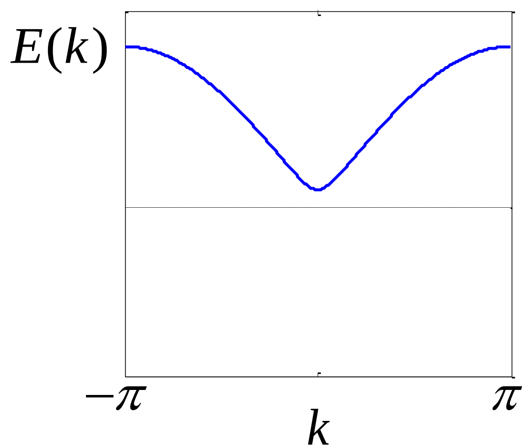
Gapped Ising

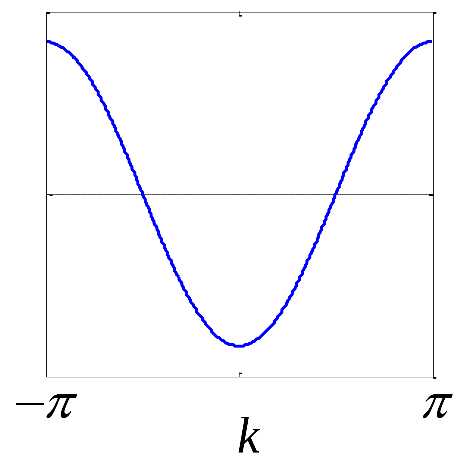
$$H = \frac{1}{2}\sum_r \left(\hat{a}^\dagger_{r+1}\hat{a}_r + \hat{a}^\dagger_r \hat{a}^\dagger_{r+1} + h.c. \right) - \lambda \sum_r \hat{a}^\dagger_r \hat{a}_r$$

Critical XX

$$H = \frac{1}{2}\sum_r \left(\hat{a}^\dagger_{r+1}\hat{a}_r + \hat{a}^\dagger_r \hat{a}_{r+1} \right)$$

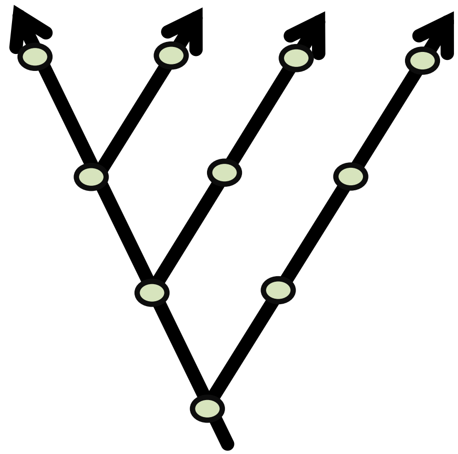
Dispersion:





Branching MERA beyond Regular Holographic Trees

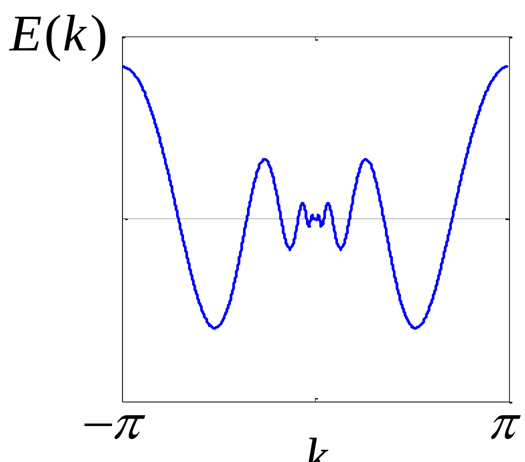
Holographic Tree:



Hamiltonian:

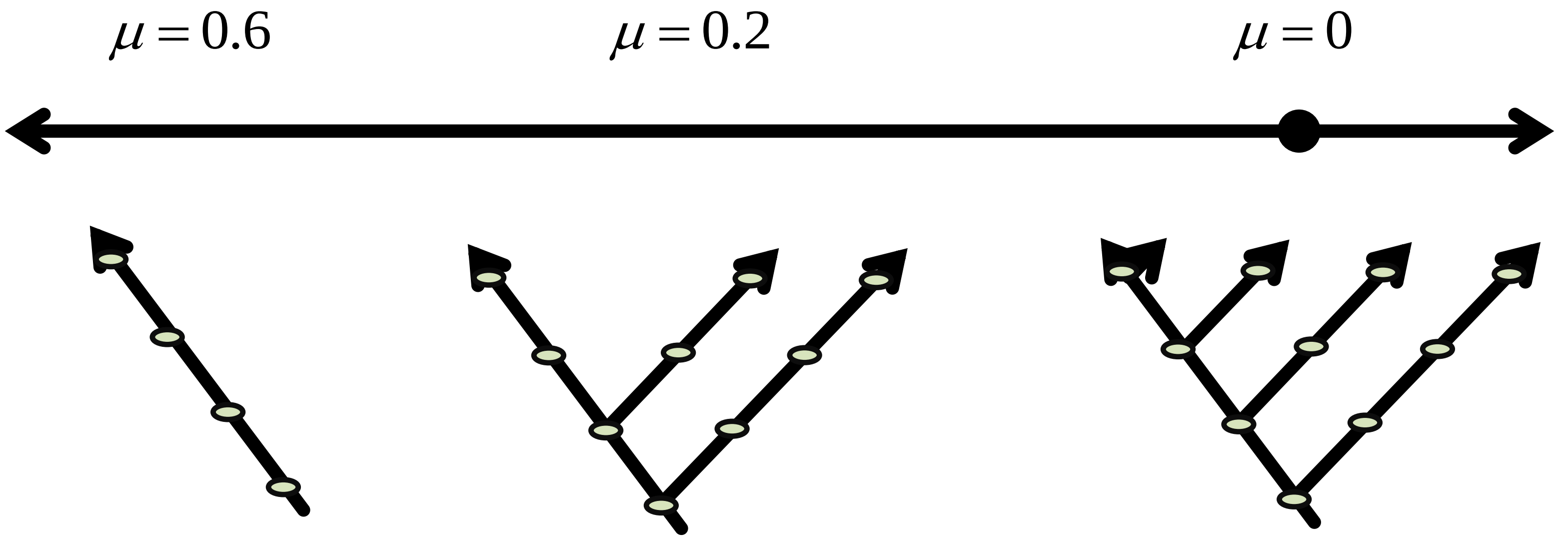
$$H = \sum_{r=-\infty}^{\infty} \left(\sum_{\substack{d=-\infty \\ d\neq 0}}^{\infty} \frac{\phi(d)}{d^2} \left(\hat{a}_{r+d}^{\dagger} \hat{a}_r + h.c. \right) \right) - \mu \sum_r \hat{a}_r^{\dagger} \hat{a}_r$$

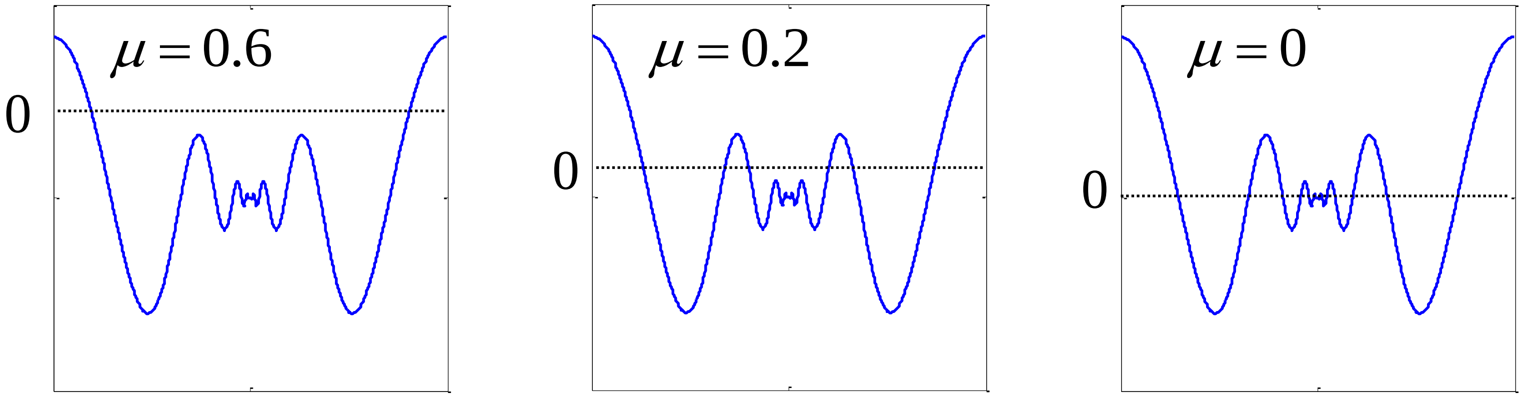
Dispersion:



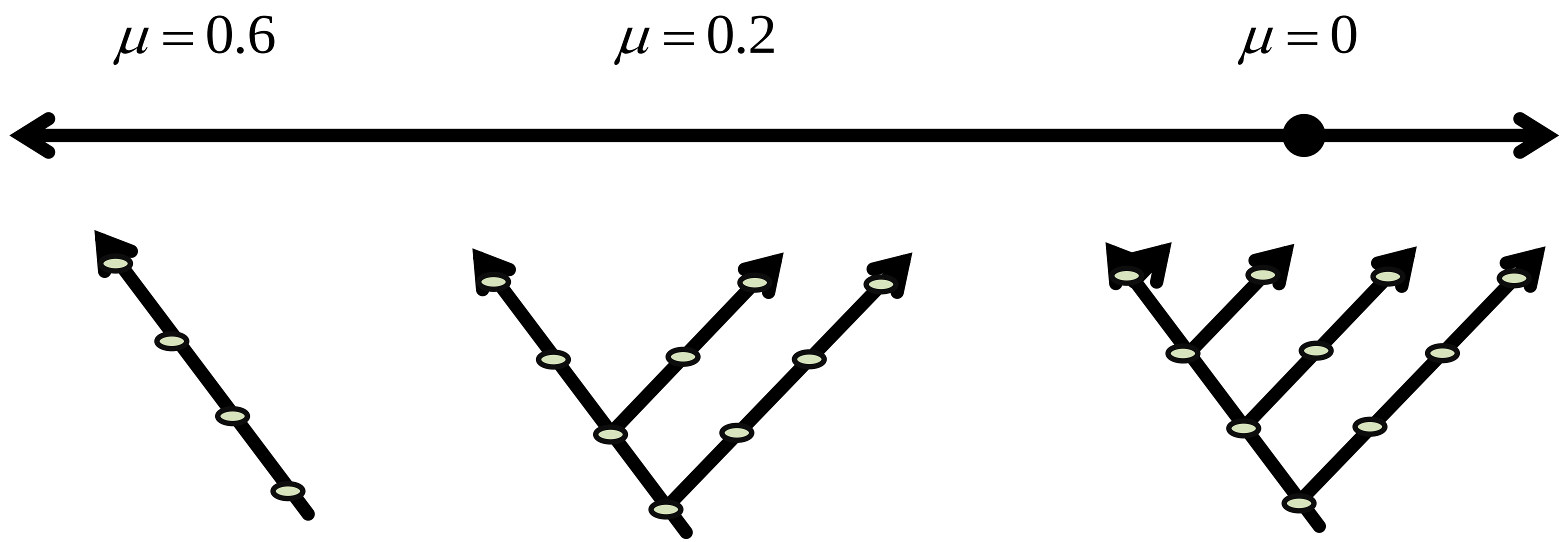
$$E(k) = \left| \sin\left(\frac{k}{2}\right) \cos\left(\pi \log_2 \left|\frac{\pi}{k}\right|\right) \right|$$

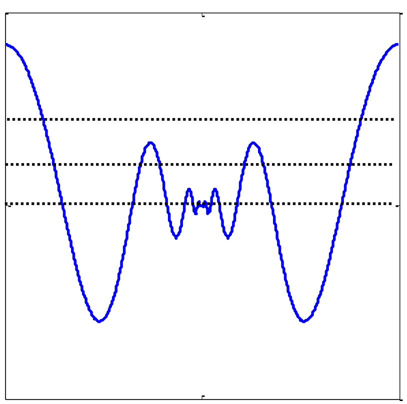
Chemical Potential:

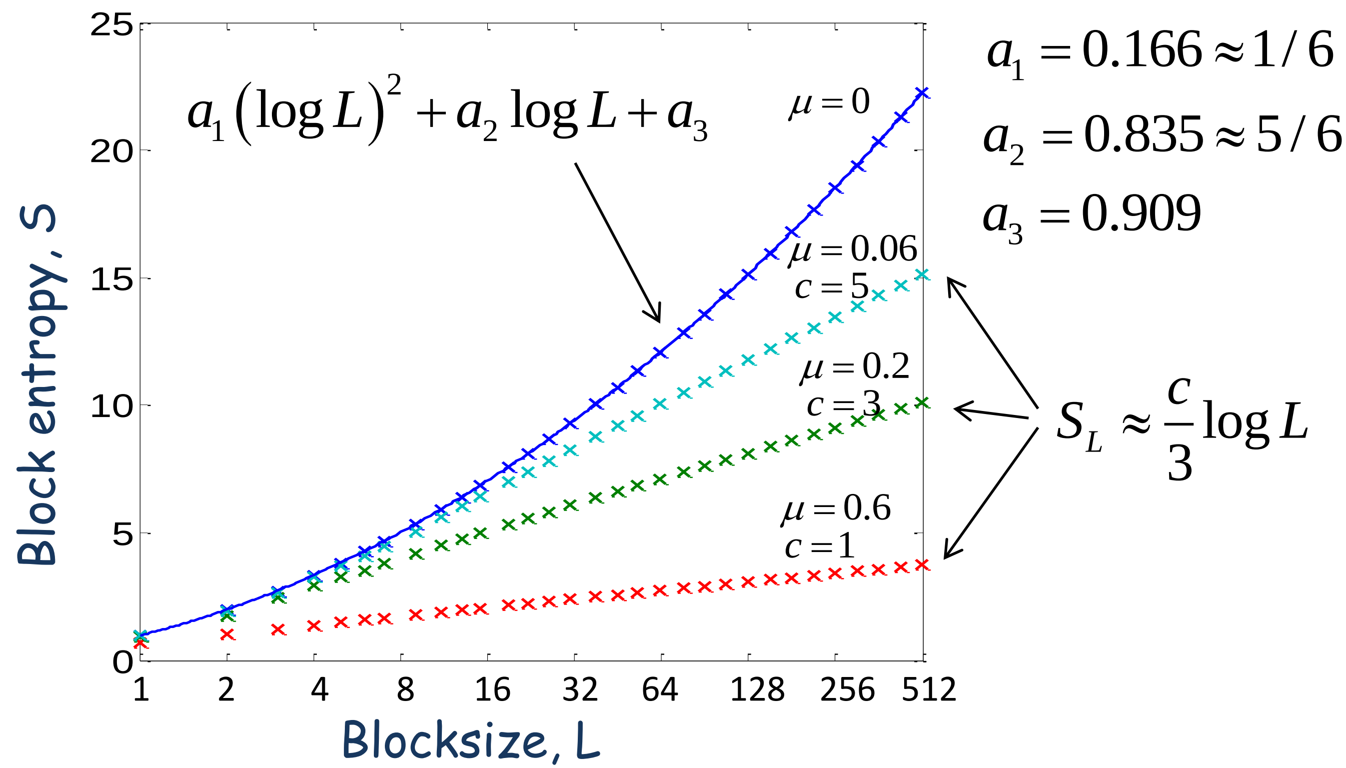




Chemical Potential:





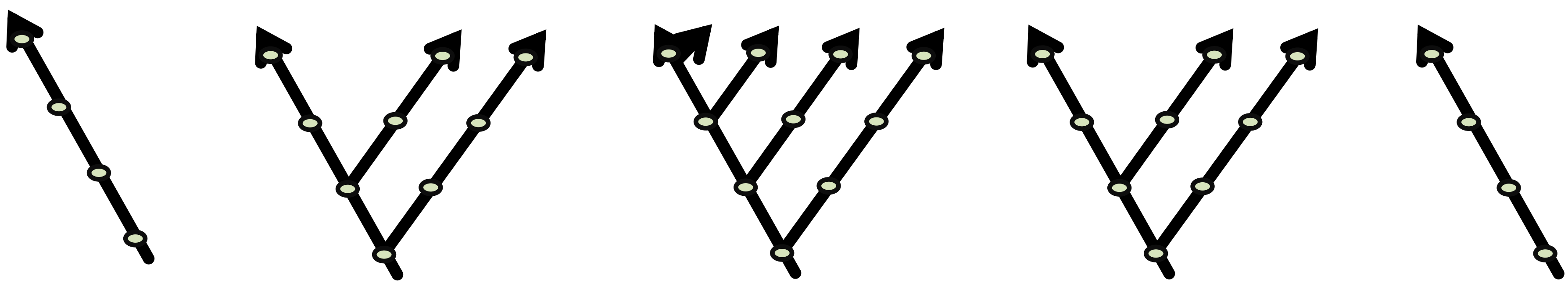


Phase Transition in D=1 Free Fermions

Chemical Potential:

$\mu < 0$	$\mu = 0$	$\mu > 0$
Crit:	Crit.II:	Crit:
$S_L \approx \log L$	$S_L \approx (\log L)^2$	$S_L \approx \log L$

Holographic Geometry:

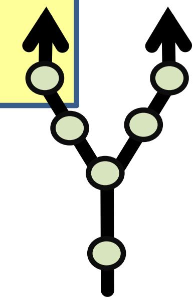


Summary/conclusions

entanglement renormalization (coarse graining transform) → MERA ⟷ holographic geometry (=spatial pattern of entanglement)

↓ new!

decoupling → branching MERA ⟷ holographic tree



entanglement renormalization, when applied to a theory that decouples into a collection of independent theories at low energy, provides:

- a formalism to explicitly factorise the theory into several theories
- new notions of scale invariance, RG flow, RG fixed points,...

branching MERA:

- admits a holographic interpretation of entropy scaling
- an efficient ansatz for critical phases beyond reach of MPS/PEPS
- (further on...) basis of an algorithm to simulate highly entangled critical phases of matter