# Entanglement entropy in quantum critical systems: from one to two dimensions

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NDNS: from DMRG to TNF
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# Plan

- Introduction: entanglement entropy in quantum many-body systems
  - Definition, how to use it
  - Applications (in particular, in 1D)
- From 1D to 2D
  - Coupled Tomonaga-Luttinger liquids

    S.F. and Y.B. Kim, arXiv: 1009.3016
  - Critical Rokhsar-Kivelson wave functions

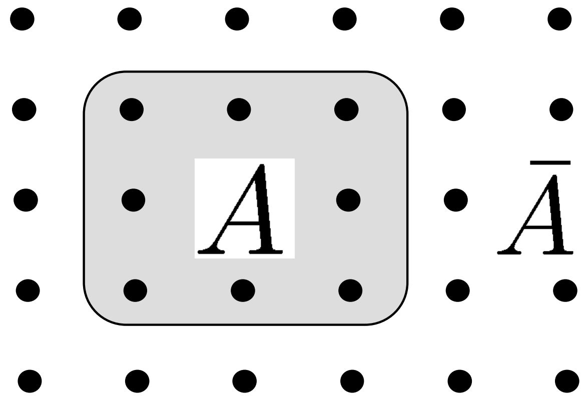
    J.-M. Stephan, S.F., G. Misguich, and V. Pasquier, Phys. Rev. B 80, 184421 (2009)

# Entanglement entropy – How entangled $A$ & $\bar{A}$ are

Many-body state $|\Psi\rangle$

Reduced density matrix

$$\rho_A = \mathrm{Tr}_{\bar{A}} \, |\Psi\rangle\langle\Psi|$$




**Entanglement entropy (von Neumann entropy)**

$$S_A = -\mathrm{Tr}\, \rho_A \log \rho_A \; (= S_{\bar{A}})$$

$$= -\sum_i p_i \log p_i \qquad \{p_i\}\text{:eigenvalues of } \rho_A$$

<u>Two-qubit examples:</u>

- product state $\longrightarrow$ pure state

  $|\Psi\rangle = |00\rangle$ $\qquad$ $\rho_A = |0\rangle\langle 0|$ $\qquad$ $S_A = 0$

- entangled state $\longrightarrow$ mixed state

  $|\Psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ $\qquad$ $\rho_A = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$ $\qquad$ $S_A = \log 2$

# Entanglement entropy in many-body systems

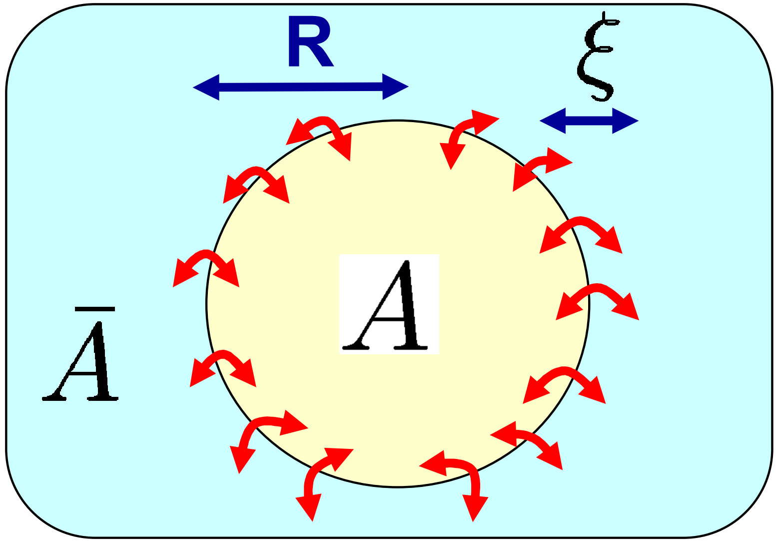
Look at the scaling of $S_A = -\mathrm{Tr}\ \rho_A \log \rho_A$

→ Important (possibly universal) info on the system

- Short-range correlations only

$$S_A \approx \alpha R^{d-1}$$ *boundary law*

Srednicki, PRL, 1993
Wolf,Verstraete,Hastings,Cirac,PRL,2008

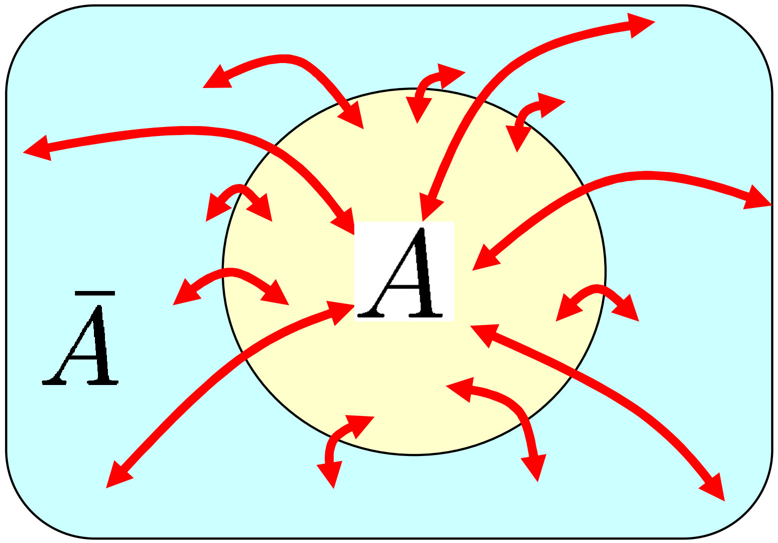



- Power-law decaying correlations

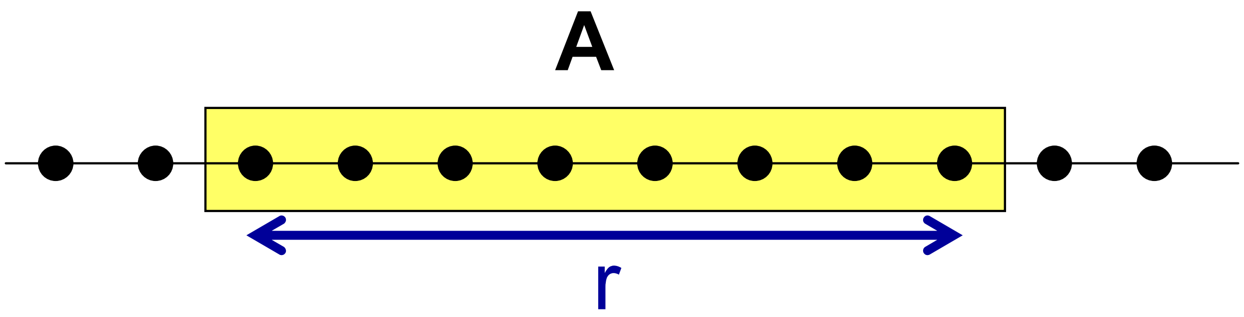
Deviation from boundary law

e.g., free fermion: $S_A \approx \alpha R^{d-1} \log R$

Wolf, PRL, 2006; Gioev & Klich, PRL,2006;
Swingle,PRL,2010



# One dimension



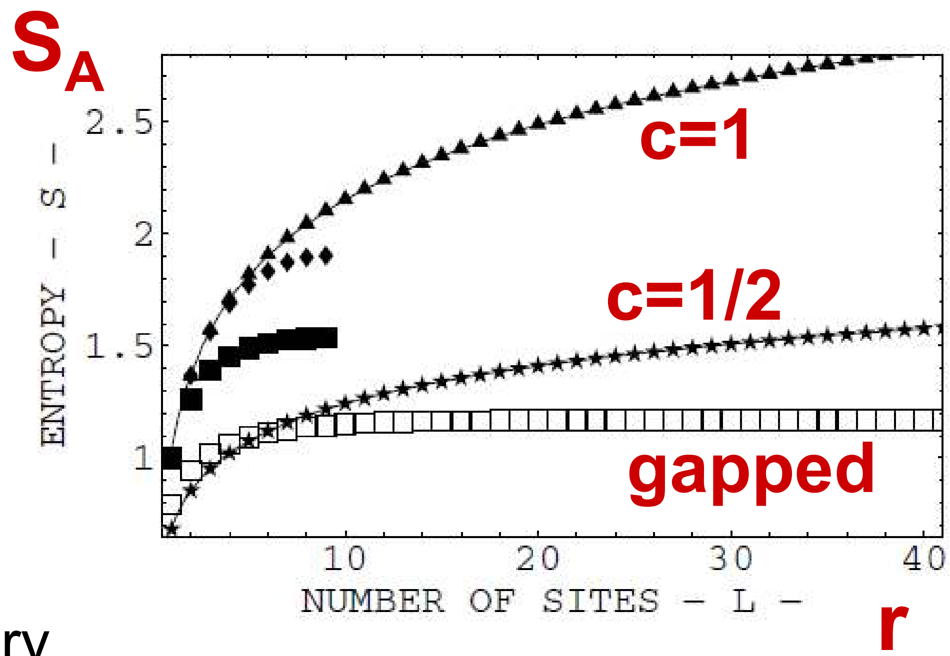

Holzhey, Larsen, & Wilczek, Nucl.Phys.B, 1994
Vidal, Latorre, Rico, & Kitaev, PRL, 2003
Calabresse & Cardy, J.Stat.Mech, 2004

- Gapped (non-critical) system

$$S_A \to \text{const.} \ (r \to \infty)$$

- Critical system

$$S_A \simeq \frac{c}{3} \log r + s_1$$




c: central charge of conformal field theory

$\simeq$ number of gapless modes

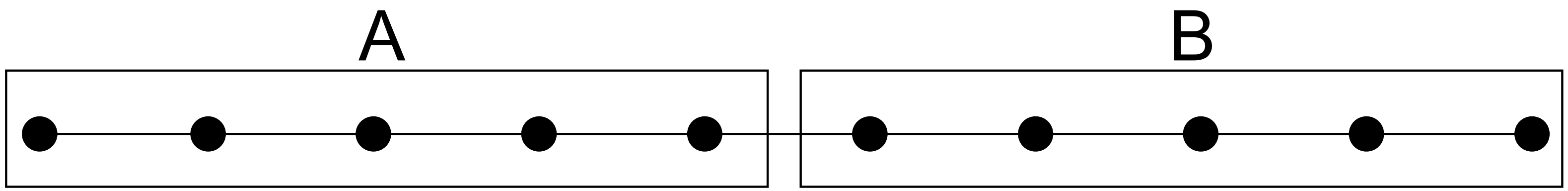
# Why interesting ?

- Unbiased way to determine the central charge

  Detection of spin Bose metal (c=3) in a zigzag ladder

  Sheng,Motrunich,& Fisher,PRB,2009

- Assessment of the efficiency of DMRG




$m \gg e^{S_A}$ = (number of important states)

Vidal, Latorre, Rico, Kitaev, PRL, 2003

- Gapped system: $e^{S_A} \to \mathrm{const.}$
- Gapless system: $e^{S_A} \simeq (\mathrm{const.}) \times L^{c/6}$

  e.g., XX chain (c=1), L=2000 ⟶ $e^{S_A} \simeq 4.7$

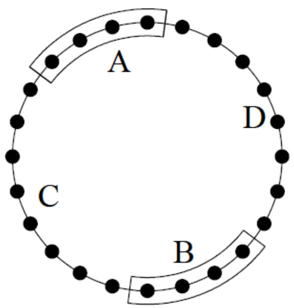
# Further progress …

Detailed info of CFT

Higher-dim critical systems

•Tomonaga-Luttinger liquid parameter is encoded in two-interval entropy

SF, Pasquier,Shiraishi, PRL,2009
Calabrese,Cardy,Tonni,J.Stat.Mech,2010




•Rokhsar-Kivelson wave fn.
2D state described by (1+1)D CFT

Fradkin,Moore,PRL,2006
Hsu,Mulligan,Fradkin,Kim,PRB,2009

•Quantum O(N) model

Metlitski,Fuertes,Sachdev,PRB.2009

•$AdS_{d+2}$ / $CFT_{d+1}$ correspondence

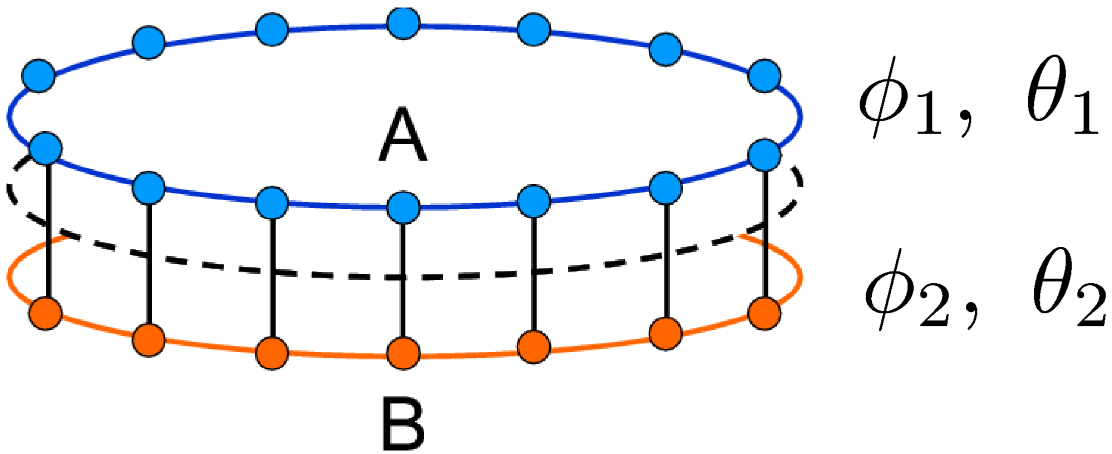
Ryu,Takayanagi,PRL,2006

Our works:

We consider new quantities for 1D which can possibly contain some info of CFT (in particular, TLL parameter).
The obtained results also provide some insights on entanglement entropy in certain 2D systems.

Entanglement entropy between two coupled Tomonaga-Luttinger liquids

S.F. and Yong Baek Kim
arXiv: 1009.3016

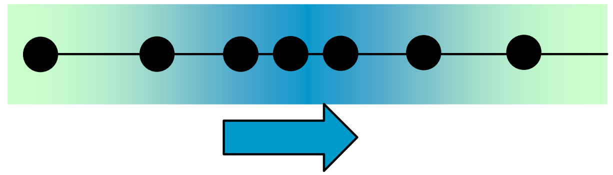


# (spinless) Tomonaga-Luttinger liquid (TLL)

Universal description of interacting 1D systems
in terms of density & phase fluctuations

$$\psi^\dagger(x) \sim [\rho(x)]^{1/2} e^{-i\sqrt{\pi}\theta(x)}$$

$$\phi(x) \sim -\sqrt{\pi}\int^x dx' \rho(x')$$



collective mode
(compressional wave)

Effective Hamiltonian: free boson with c=1

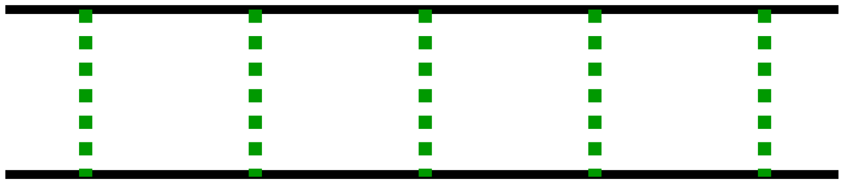
$$H = \int dx \frac{v}{2}\left[K\left(\frac{d\theta}{dx}\right)^2 + \frac{1}{K}\left(\frac{d\phi}{dx}\right)^2\right]$$

$K$ : TLL parameter
$v$ : velocity

| | | | |
|---|---|---|---|
| spinless fermion | repulsive | free | attractive |
| boson | | 1 (hard-core limit) | repulsive |

$K$ →

# Two coupled TLLs on parallel chains



$\phi_1, \ \theta_1 \qquad H_1 \quad v, \ K$

$\phi_2, \ \theta_2 \qquad H_2 \quad v, \ K$

Density-density interaction $\quad H_{12} = \int_0^L dx \, \frac{U}{\pi} \frac{d\phi_1}{dx} \frac{d\phi_2}{dx}$

The total Hamiltonian $H = H_1 + H_2 + H_{12}$ is diagonalized in symmetric/antisymmetric basis

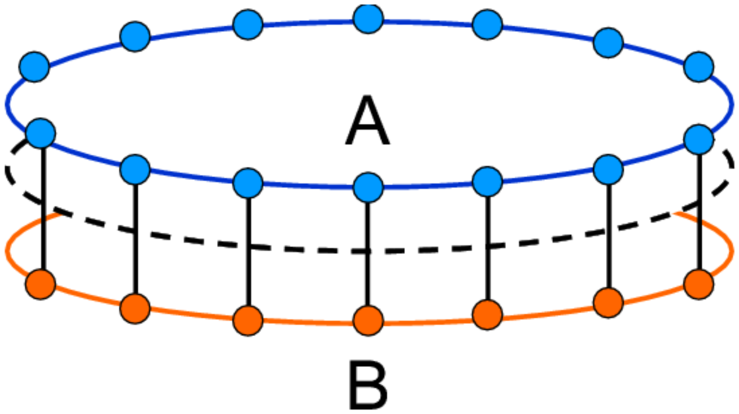
$$\phi_\pm = \frac{1}{\sqrt{2}}(\phi_1 \pm \phi_2), \quad \theta_\pm = \frac{1}{\sqrt{2}}(\theta_1 \pm \theta_2)$$

$$H = H_+ + H_- \qquad H_\pm = \int_0^L dx \, \frac{v_\pm}{2} \left[ K_\pm \left( \frac{d\theta_\pm}{dx} \right)^2 + \frac{1}{K_\pm} \left( \frac{d\phi_\pm}{dx} \right)^2 \right]$$

$$v_\pm = v \left( 1 \pm \frac{KU}{\pi v} \right)^{\frac{1}{2}}, \quad K_\pm = K \left( 1 \pm \frac{KU}{\pi v} \right)^{-\frac{1}{2}}$$

* Underlying mechanism of spin-charge separation

# Our problem




$\phi_1, \ \theta_1$

$\phi_2, \ \theta_2$

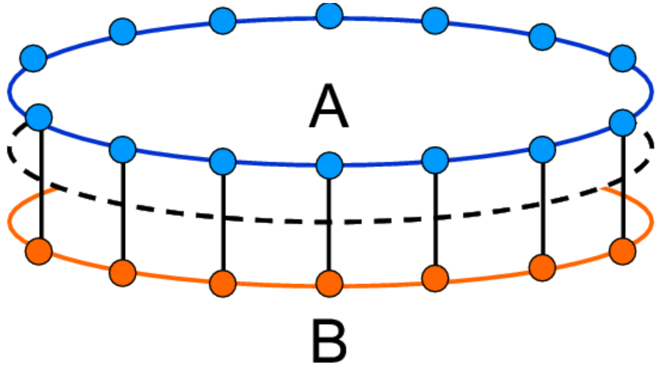
Entanglement entropy S between the two ``rings''

- Entanglement arising from the coupling of two chains

- Different from a block entanglement which detects central charge.
  One can possibly detect a different kind of information?
- Maybe not accessible by DMRG but still useful in ED.

- Starting point for understanding 2D array coupled TLLs (sliding TLL)

# Main result




Renyi entanglement entropy

$$S_n = \frac{-1}{n-1} \log(\mathrm{Tr}\ \rho_A^n) \quad (n \geq 1)$$

$$\begin{cases} S_1 \equiv \lim_{n\to 1} S_n = -\mathrm{Tr}\ \rho_A \log \rho_A & \text{(von Neumann entanglement entropy)} \\ S_\infty = -\log \lambda_{\max} & \text{(single-copy entanglement)} \end{cases}$$

$\lambda_{\max}$ :largest eigenvalue of the density matrix

<u>Scaling with ``ring'' length L</u>

$$S_n = \alpha_n L + \gamma_n + \dots$$

$\alpha_n L$: boundary contribution

$\gamma_n$: universal ⟵ determined by $K_+/K_-$

cf. D. Poilblanc, PRL, 2010
Similar quantity in gapped spin ladder
Constant term was not identified

# Path-integral formulation

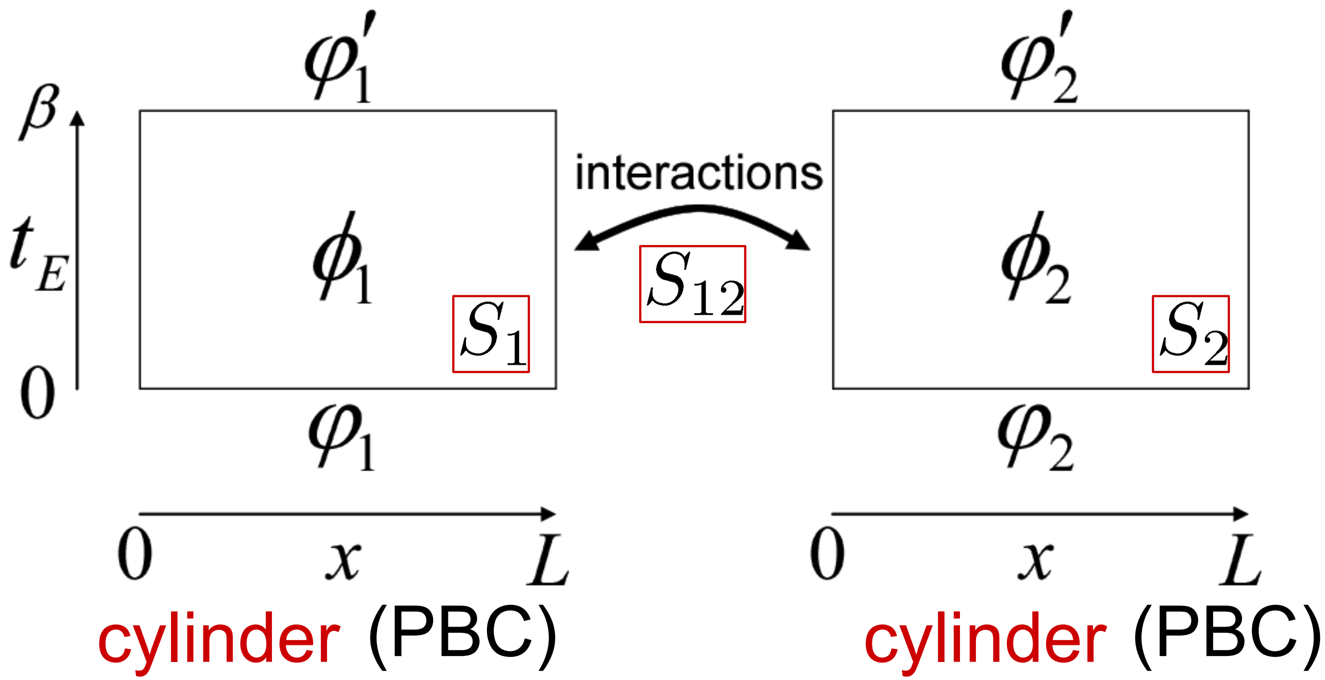
Finite-temperature total density matrix:

$$\rho = \frac{1}{Z} e^{-\beta H} \quad \text{with} \quad Z = \mathrm{Tr}\, e^{-\beta H} \qquad \beta \to \infty$$

Euclidean action: $S = S_1 + S_2 + S_{12} = S_+ + S_-$

Matrix element: $\langle \varphi_1', \varphi_2' | \rho | \varphi_1, \varphi_2 \rangle$

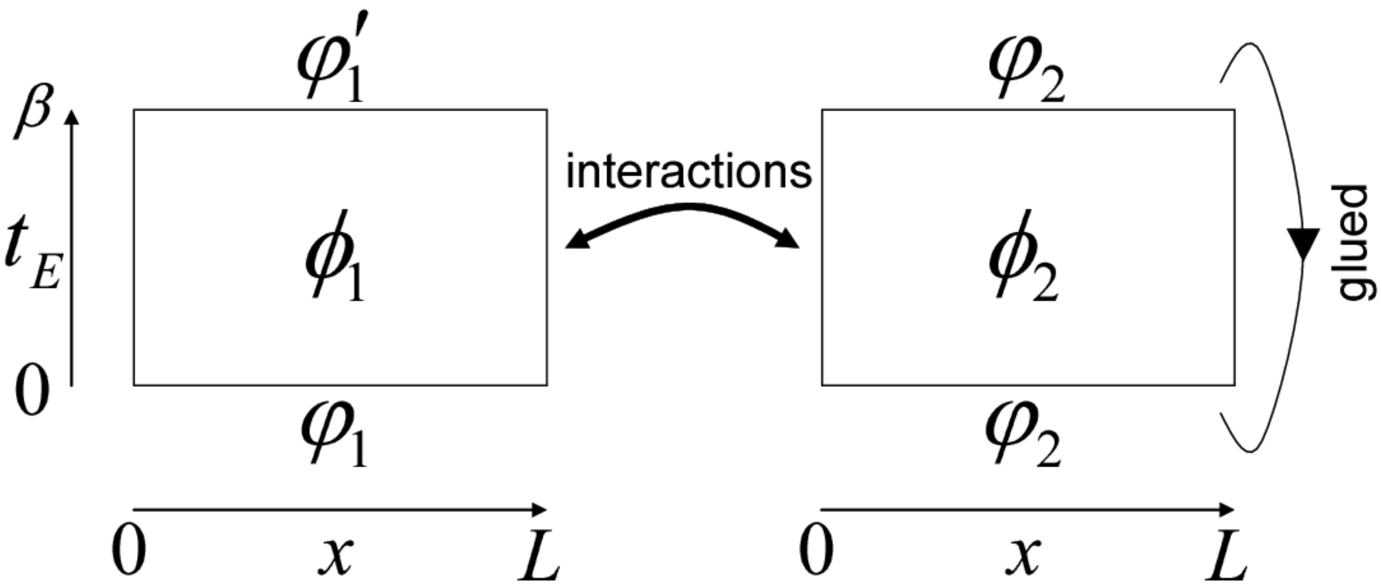
$\varphi_\nu = \{\varphi_\nu(x)\}_{0 \leq x < L}$ : field configuration along a chain



# Path-integral formulation

Reduced density matrix for the 1st chain:

$$\langle \varphi_1' | \rho_A | \varphi_1 \rangle = \int \mathcal{D}\varphi_2 \ \langle \varphi_1', \varphi_2 | \rho | \varphi_1, \varphi_2 \rangle$$

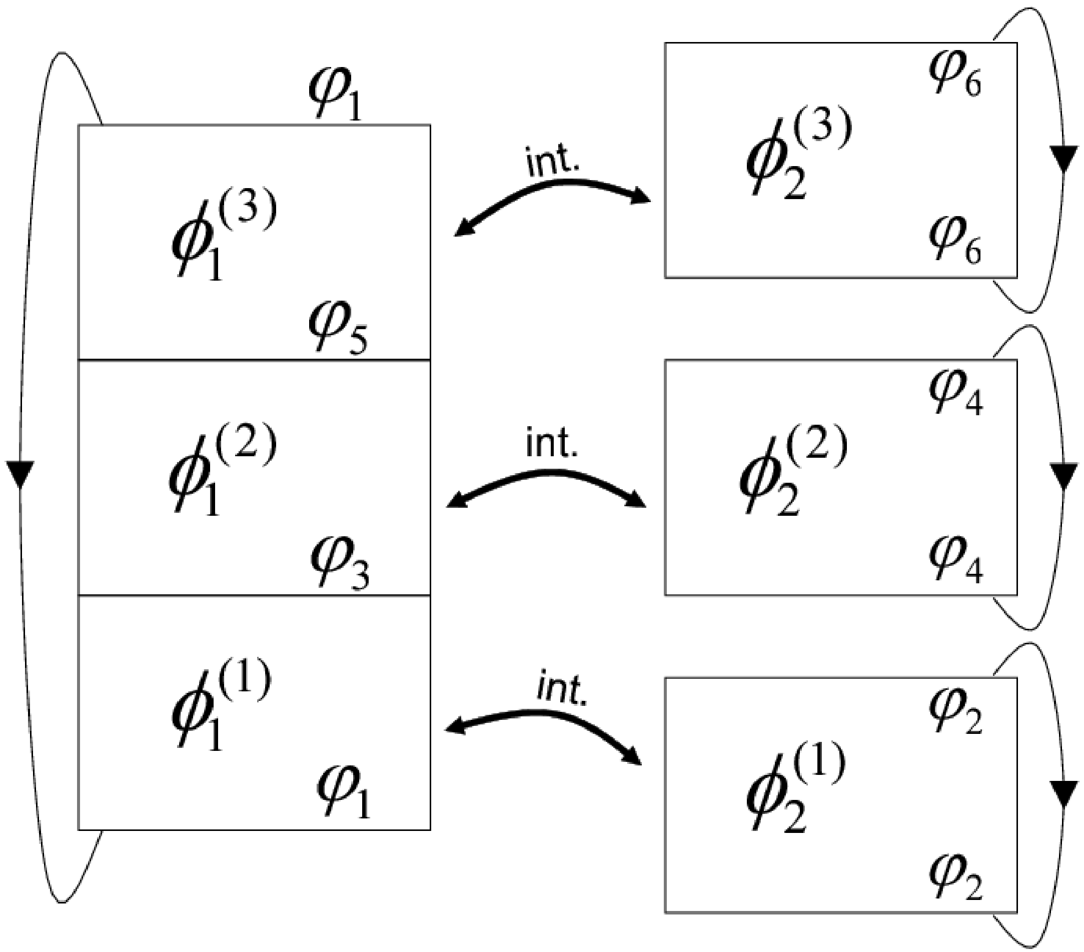



cylinder (PBC) torus (PBC)

# Path-integral formulation

Reduced density matrix moment: $\mathrm{Tr}\ \rho_A^n$

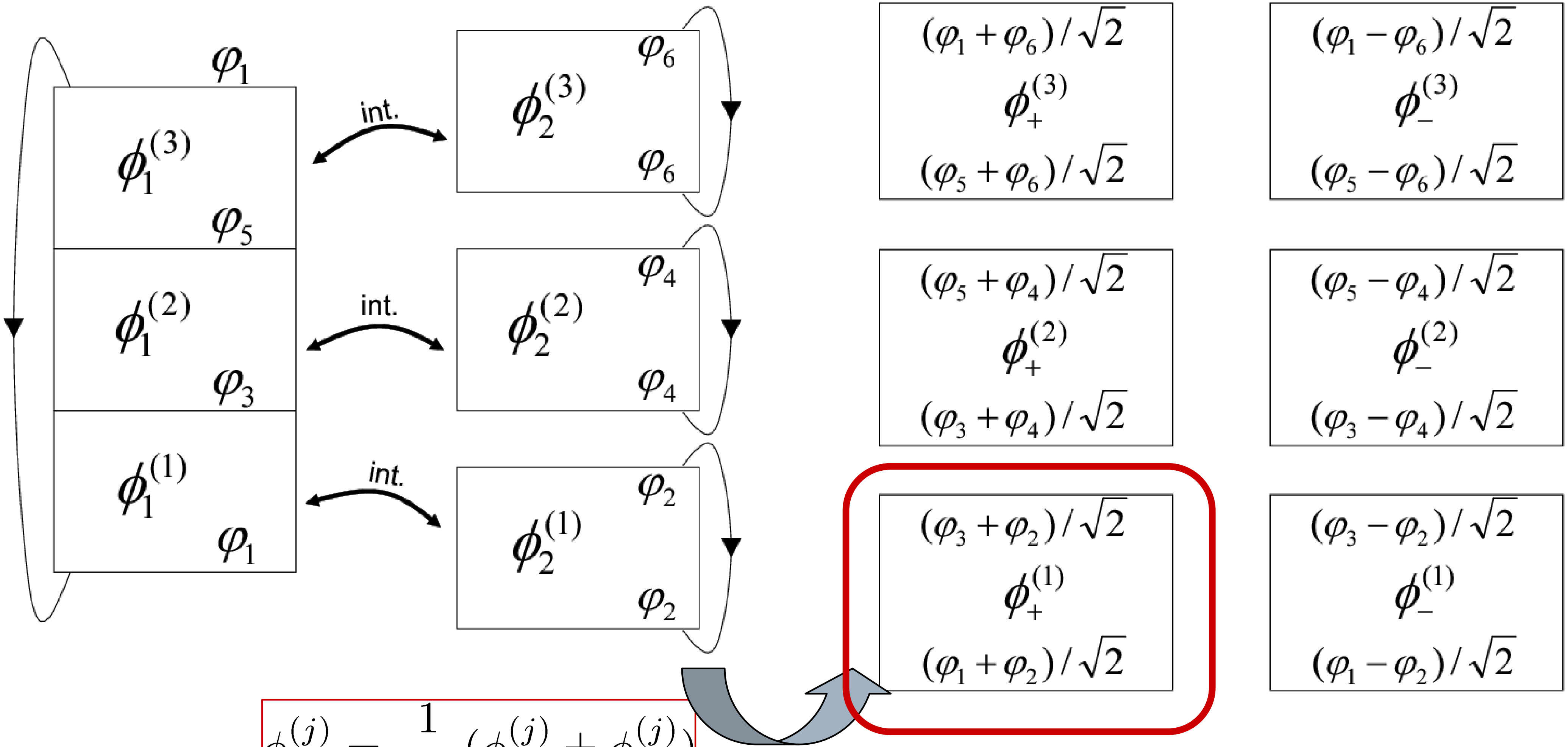
n=3 case




1 large torus

3 small tori

# Change of the basis

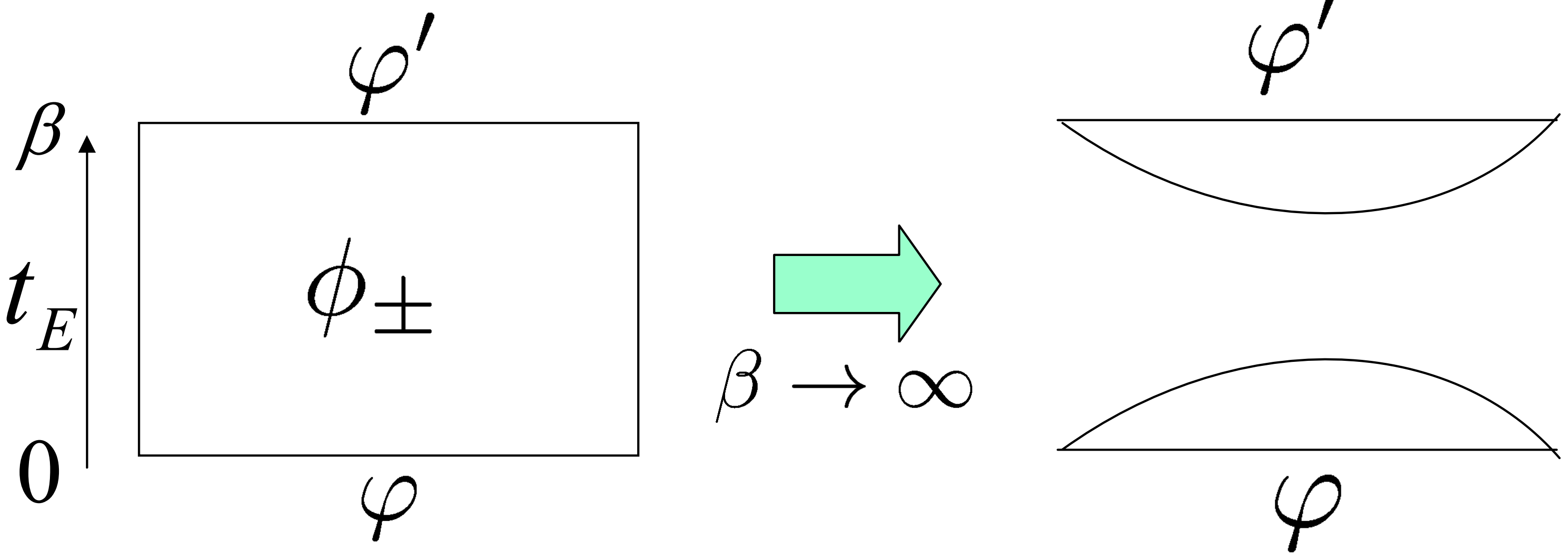



$$\phi_\pm^{(j)} = \frac{1}{\sqrt{2}}(\phi_1^{(j)} \pm \phi_2^{(j)})$$

We first treat each sheet by fixing the boundary configs.

$\mathrm{Tr}\ \rho_A^n$ is calculated by integrating over the boundary configs.

# Bosonic string propagator




$$\langle \varphi' | e^{-\beta H_\pm} | \varphi \rangle \approx \underline{\langle \varphi' | \Psi_\pm \rangle} e^{-\beta E_\pm} \underline{\langle \Psi_\pm | \varphi \rangle}$$

Ground state wave ``functional''

$$\langle \varphi | \Psi_\pm \rangle = \frac{1}{\sqrt{\mathcal{N}_\pm}} e^{-\frac{1}{K_\pm}\mathcal{E}[\varphi]} \qquad \mathcal{E}[\{\tilde{\varphi}_m\}] = \sum_{m=1}^{\infty} k_m |\tilde{\varphi}_m|^2$$

Fradkin,Moreno,& Schaposnik,
Nucl.Phys.B, 1993

Fourier component of $\varphi$

# Derivation of Renyi entanglement entropy

$$\mathrm{Tr}\ \rho_A^n = (\mathcal{N}_+\mathcal{N}_-)^{-n} \int \prod_{j=1}^{2n} \prod_{m=1}^{\infty} (d\tilde{\varphi}_{j,m} d\tilde{\varphi}^*_{j,m})$$

$$\times \exp\left( - \sum_{m=1}^{\infty} \frac{2k_m}{(K_+K_-)^{1/2}} \underline{\tilde{\Phi}^\dagger_m M_n \tilde{\Phi}_m} \right)$$

$$\tilde{\Phi}_m = (\tilde{\varphi}_{1,m}, \tilde{\varphi}_{2,m}, \cdots, \tilde{\varphi}_{2n,m})^t$$

| | |
|---|---|
| $(\varphi_1+\varphi_6)/\sqrt{2}$<br>$\phi_+^{(3)}$<br>$(\varphi_5+\varphi_6)/\sqrt{2}$ | $(\varphi_1-\varphi_6)/\sqrt{2}$<br>$\phi_-^{(3)}$<br>$(\varphi_5-\varphi_6)/\sqrt{2}$ |
| $(\varphi_5+\varphi_4)/\sqrt{2}$<br>$\phi_+^{(2)}$<br>$(\varphi_3+\varphi_4)/\sqrt{2}$ | $(\varphi_5-\varphi_4)/\sqrt{2}$<br>$\phi_-^{(2)}$<br>$(\varphi_3-\varphi_4)/\sqrt{2}$ |
| $(\varphi_3+\varphi_2)/\sqrt{2}$<br>$\phi_+^{(1)}$<br>$(\varphi_1+\varphi_2)/\sqrt{2}$ | $(\varphi_3-\varphi_2)/\sqrt{2}$<br>$\phi_-^{(1)}$<br>$(\varphi_1-\varphi_2)/\sqrt{2}$ |

⇓ Gaussian integration

$$\mathrm{Tr}\ \rho_A^n = \prod_{m=1}^{\infty} (\det M_n)^{-1} = e^{-\alpha L} (\det M_n)^{1/2}$$

infinite product

$\zeta$-fn. regularization

$$S_n = \frac{-1}{n-1} \log(\mathrm{Tr}\ \rho_A^n) = \alpha_n L + \gamma_n$$

$$\gamma_n = \frac{-1}{2(n-1)} \log(\det M_n)$$

# Expression of universal constant

$$\gamma_n = \frac{-1}{2(n-1)} \log(\det M_n)$$

<u>example:</u> $\gamma_2 = -\log\left[\frac{1}{2}\left(\sqrt{\frac{K_-}{K_+}} + \sqrt{\frac{K_+}{K_-}}\right)\right]$

$$M_n := \begin{pmatrix} A & \frac{1}{2}B & & & \frac{1}{2}B \\ \frac{1}{2}B & A & \frac{1}{2}B & & \\ & \frac{1}{2}B & A & \ddots & \\ & & \ddots & \ddots & \frac{1}{2}B \\ \frac{1}{2}B & & & \frac{1}{2}B & A \end{pmatrix}$$

$$A := \frac{1}{2}\left(\sqrt{\frac{K_-}{K_+}} + \sqrt{\frac{K_+}{K_-}}\right)$$

$$B := \frac{1}{2}\left(\sqrt{\frac{K_-}{K_+}} - \sqrt{\frac{K_+}{K_-}}\right)$$

2n X 2n matrix

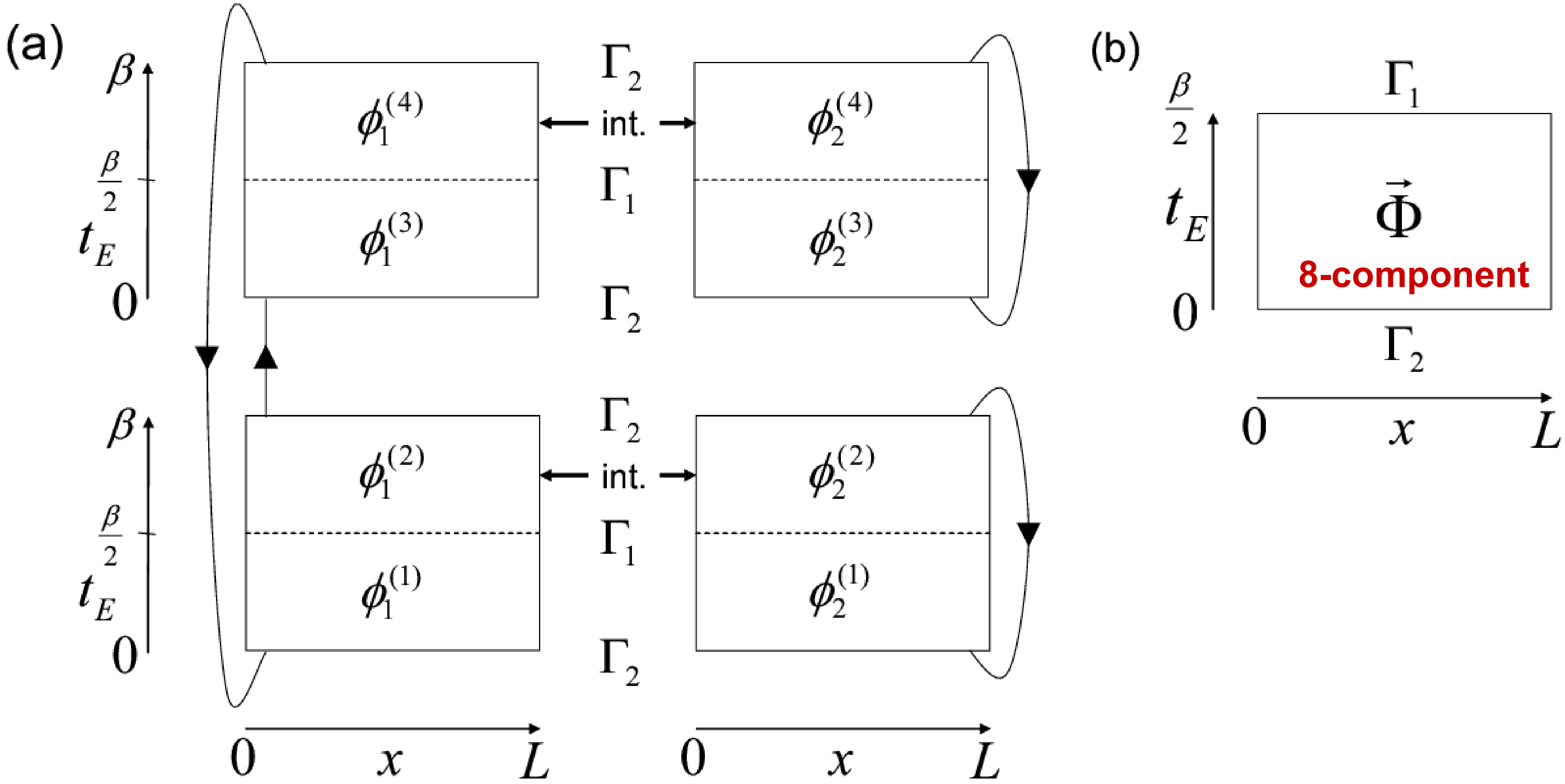
Same form as 1D tight-binding model with 2n sites

It is not obvious how to consider the von Neumann case n->1.
Let us consider expanding around K+=K- (no inter-chain interaction)

$$\gamma_n = -\frac{n}{4(n-1)}\kappa^2 + \mathcal{O}(\kappa^4) \qquad \kappa := \frac{K_- - K_+}{K_- + K_+}$$

Divergent as n->1!  The limit n->1 is not smooth?

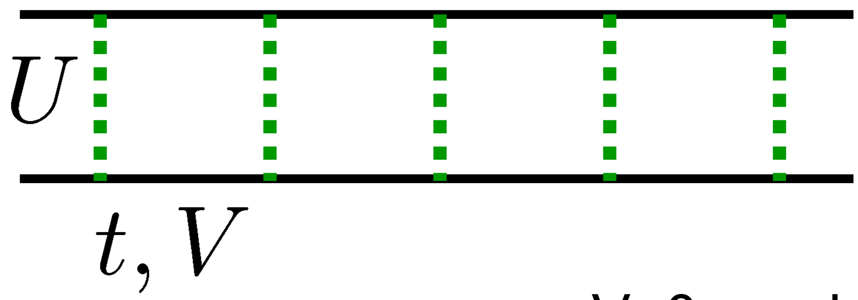
# Another approach: conformal boundary state




$$Z_2 = Z_{\Gamma_1\Gamma_2} = \langle\Gamma_1|e^{-\frac{\beta}{2}\tilde{H}}|\Gamma_2\rangle$$

Systematic approach without regularization procedure.
The same result with the previous one was obtained.

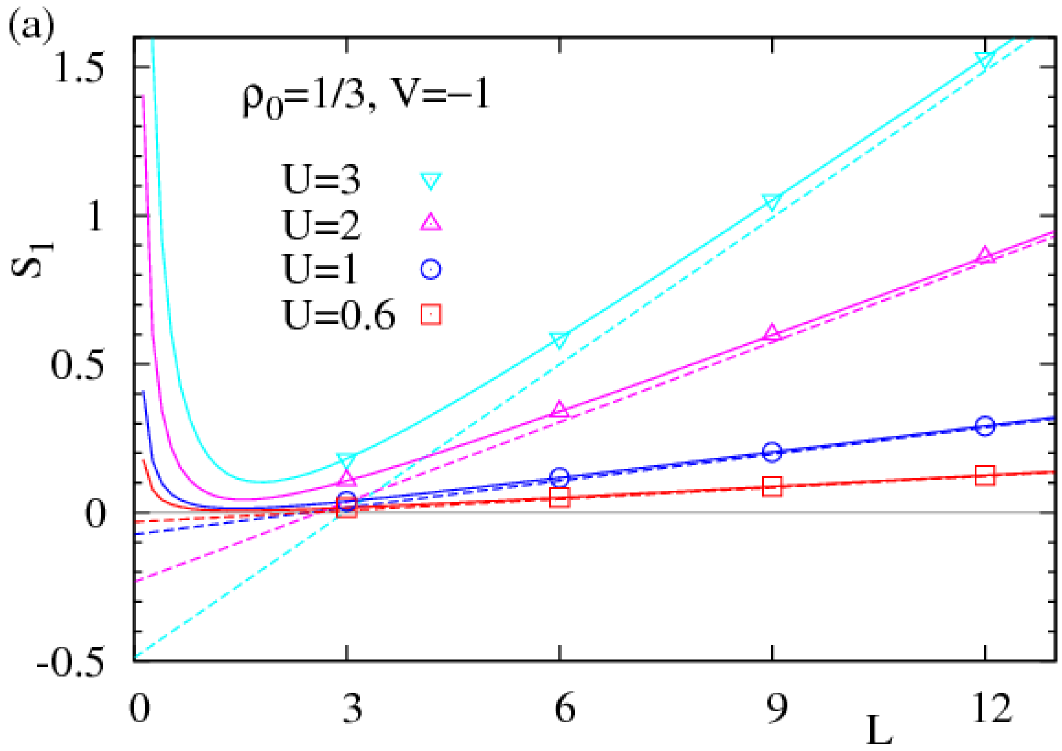
# Numerical check




Hard-core bosonic model on a ladder
Exact diag. up to 12 X 2

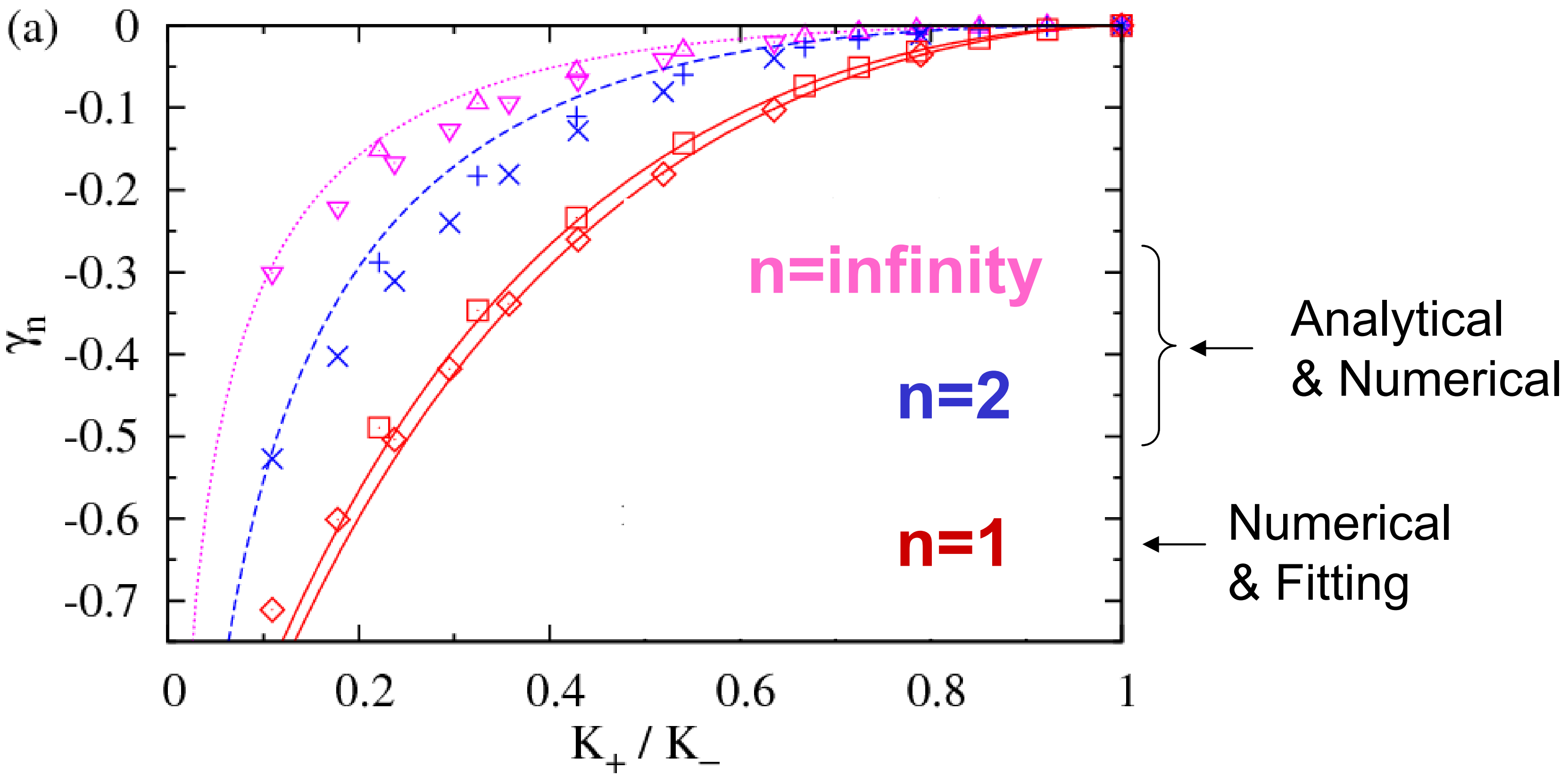
V=0: equivalent to fermionic Hubbard chain
Strong effect of marginally irrelevant perturbation

We examined the case of V<0.




$$S_n = \alpha_n L + \gamma_n + \frac{\delta_n}{L}$$

# Numerical check



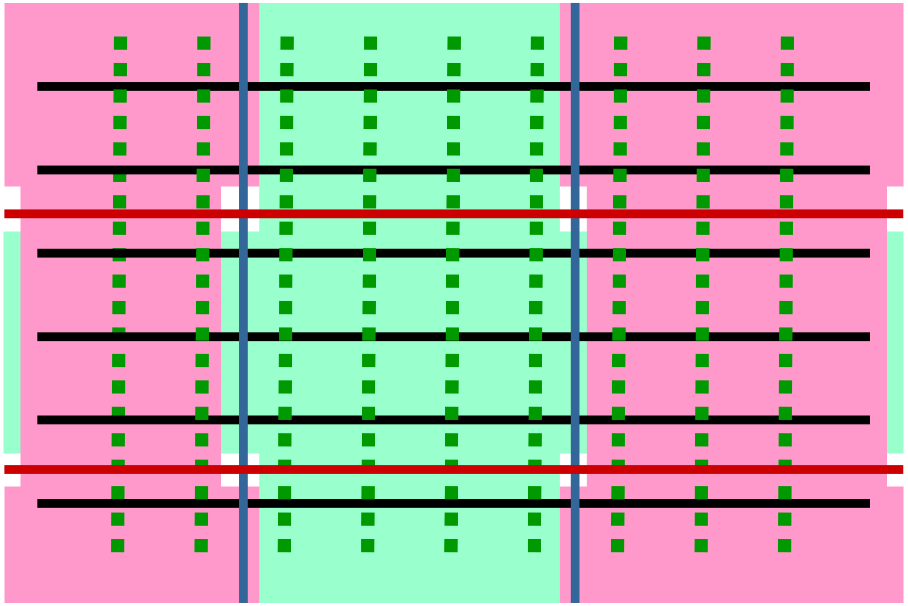

Broad agreement between analytical and numerical results for n=2,inf

Von Neumann case (n->1) also obeys a function of $K_+/K_-$

Fitted well by $\gamma_1 = -a\kappa^b$ $\kappa := \frac{K_- - K_+}{K_- + K_+}$ $b \approx 1.6\text{-}1.7$

cf. $\gamma_n = -\frac{n}{4(n-1)}\kappa^2 + \mathcal{O}(\kappa^4)$ n=2,3,..,infty

# Implications on 2D sliding TLL



2D array of coupled TLLs

torus of $L_x \times L_y$

Independent TLL for each Fourier component in y direction

Stable critical phase

Emery et al.,PRL,2000
Vishwanath & Carpentier,PRL,2001

## Horizontal cutting

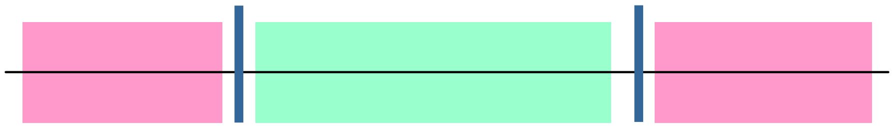
$$S_n = \alpha_n L_x + \gamma_n$$

universal constant determined by a multiple of TLL parameters

## Vertical cutting

analogous to a block entanglement in 1D gapless system with $c=L_y$

$$S = \frac{L_y}{3} \log L_x + \text{const.}$$



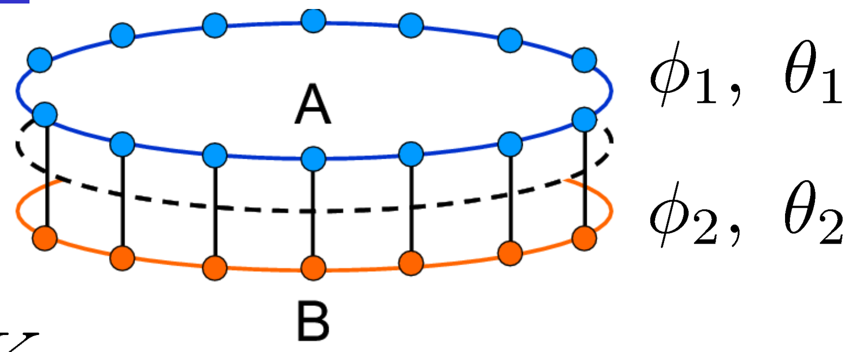
# Summary: two coupled Tomonaga–Luttinger liquids

$$S_n = \underbrace{\alpha_n L}_{\text{boundary contribution}} + \underbrace{\gamma_n}_{\text{universal}} + \dots$$

boundary contribution

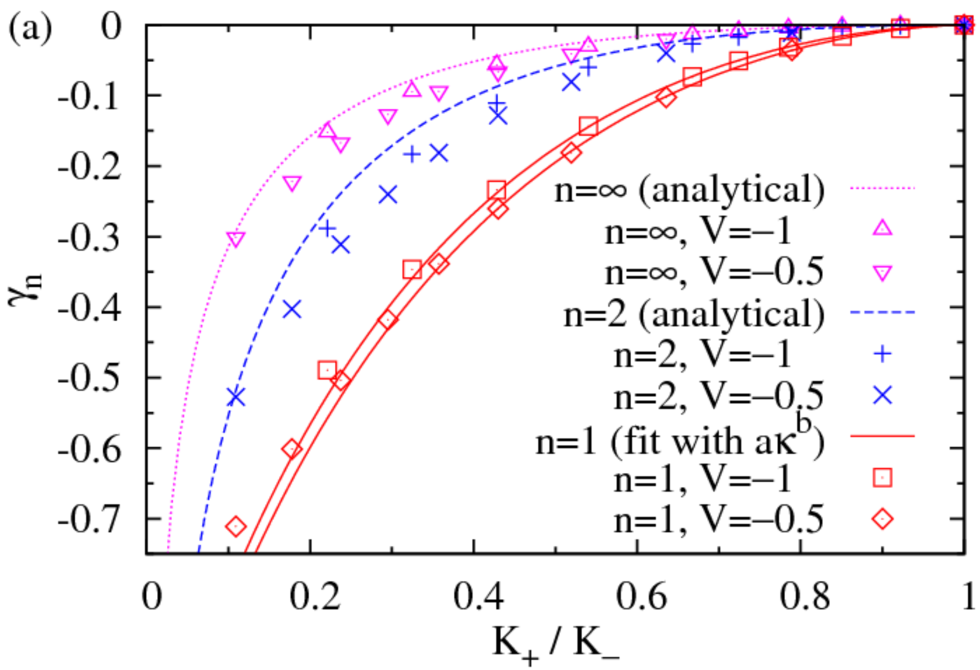
universal

determined by $K_+/K_-$




- Entanglement entropy arising from the coupling of TLLs
- Path integral + wave functional
- Another appraoch: Conformal boundary state

  ⟶ the same result
- Implications: highly anisotropic entanglement in sliding TLL



Entanglement entropy in Rokhsar-Kivelson wave functions:
transfer matrix approach

Jean-Marie Stephan, S.F., Gregoire Misguich,
& Vincent Pasquier

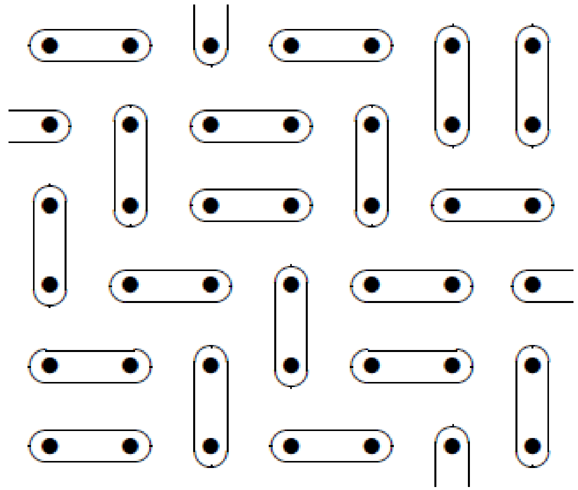
Physical Review B 80, 184421 (2009)
[Editor's suggestion]

# Generalized Rokhsar–Kivelson (RK) wave function

2D classical system with Boltzmann weight $e^{-E(c)}$

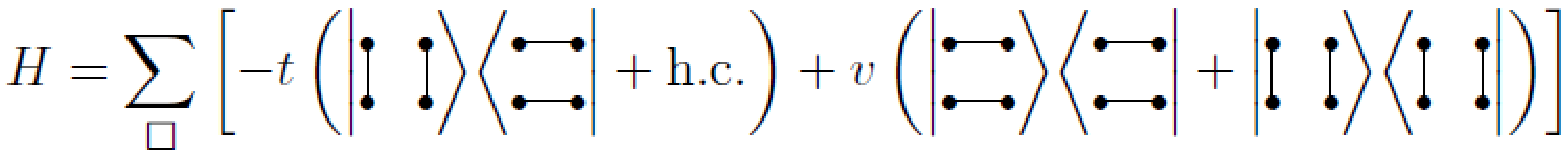
$$|\mathrm{RK}\rangle = \frac{1}{\sqrt{\mathcal{Z}}}\sum_c e^{-\frac{1}{2}E(c)}|c\rangle \qquad \mathcal{Z} = \sum_c e^{-E(c)}$$

Rokhsar,Kivelson, PRL, 1988
Henley, J.Phys. Condens.Matter, 2004
Castelnovo,Chamon,Mudry,Pujol, Ann.Phys.,2007



- Dimer model

$$\begin{cases} E(c) = 0 & \text{if c is physical} \\ E(c) = \infty & \text{if c is unphysical} \end{cases}$$



|RK> is an exact ground state of H at t=v.

- Correlation fn of diagonal operators $\langle \mathrm{RK}|\mathcal{O}_1\mathcal{O}_2|\mathrm{RK}\rangle$

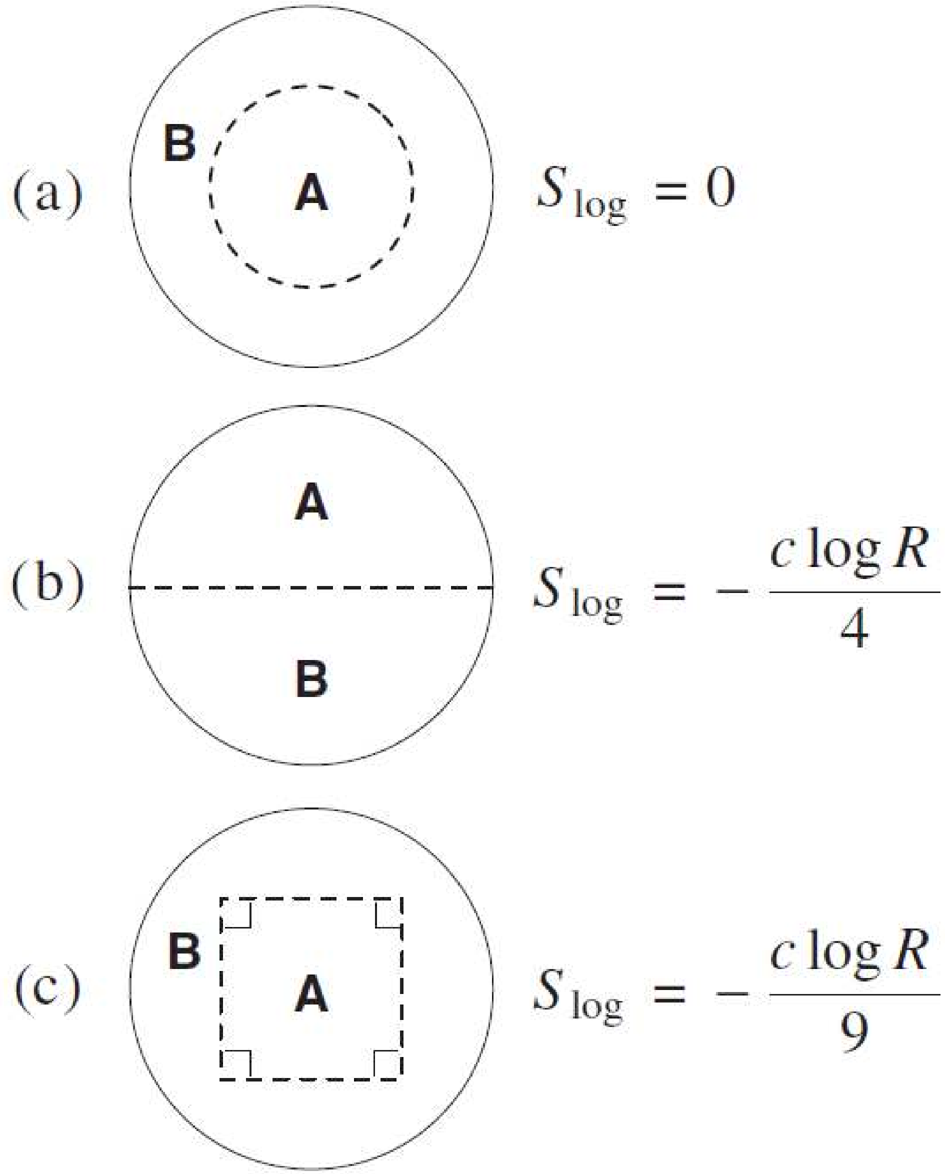
→ same as the classical system

# Fradkin and Moore, PRL, 2006

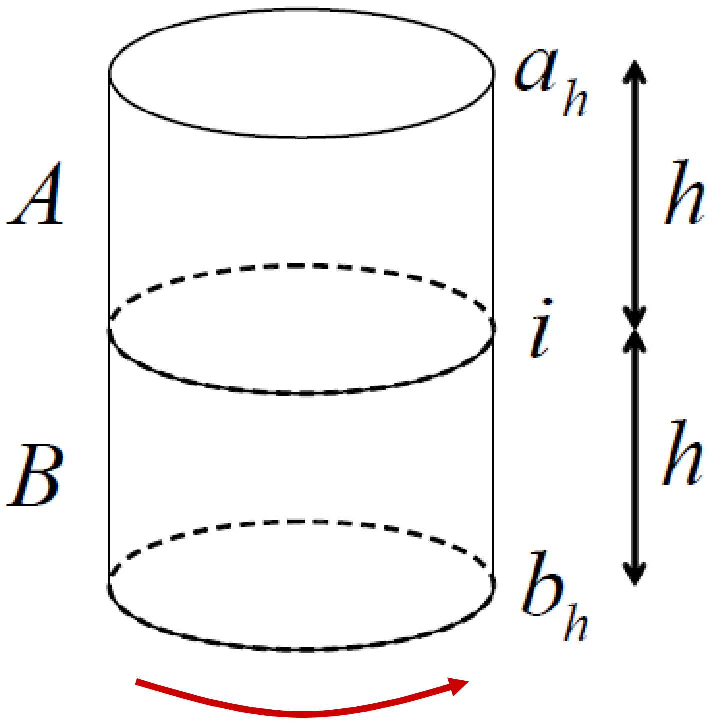
$$S = \alpha L + S_{\log}$$

Additive logarithmic contribution depending on the geometry

Detection of the central charge



# Our setting




$|\mathrm{RK}\rangle$ on a cylinder

Entanglement entropy $S_A$ for a half-cylinder

- $S_{\log} = 0$ for this geometry
- But we found additional constant term which is universal.

$$S(L) = \alpha L + \gamma$$

- Our strategy

$S_A$ → 2D classical problem

→ 1D quantum problem

transfer matrix

# Schmidt decomposition (formulation as a 2D classical problem)

$$|\mathrm{RK}\rangle = \frac{1}{\sqrt{\mathcal{Z}}}\sum_c e^{-\frac{1}{2}E(c)}|c\rangle$$

$$E(c) = E_A(a,i) + E_B(b,i)$$

$$= \frac{1}{\sqrt{\mathcal{Z}}}\sum_i \left[\sum_{a\in\mathcal{E}_i^A} e^{-\frac{1}{2}E_A(a,i)}|a,i\rangle\right] \times \left[\sum_{b\in\mathcal{E}_i^B} e^{-\frac{1}{2}E_B(b,i)}|b\rangle\right]$$

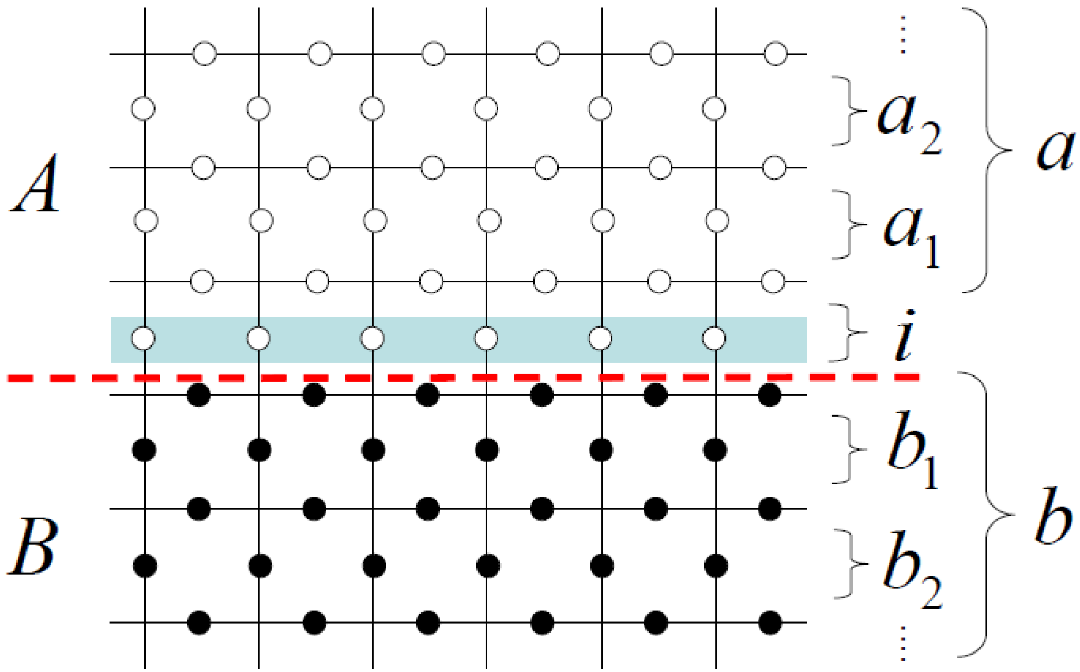
$$= \sum_i \sqrt{p_i}\,|\mathrm{RK}_i^A\rangle|\mathrm{RK}_i^B\rangle \quad \text{with} \quad p_i := \frac{\mathcal{Z}_i^A\mathcal{Z}_i^B}{\mathcal{Z}}$$

partition function with boundary spins fixed

$$\rho_A = \sum_i |\mathrm{RK}_i^A\rangle p_i \langle \mathrm{RK}_i^A|$$

$\{p_i\}$ gives the spectrum of the density matrix.

$$S_A = -\sum_i p_i \log p_i$$



# Transfer matrix – reduction to 1D

$$p_i := \frac{Z_i^A Z_i^B}{Z}$$

$$= \frac{\langle a_h|\mathcal{T}^h|i\rangle\langle i|\mathcal{T}^h|b_h\rangle}{\langle a_h|\mathcal{T}^{2h}|b_h\rangle}$$

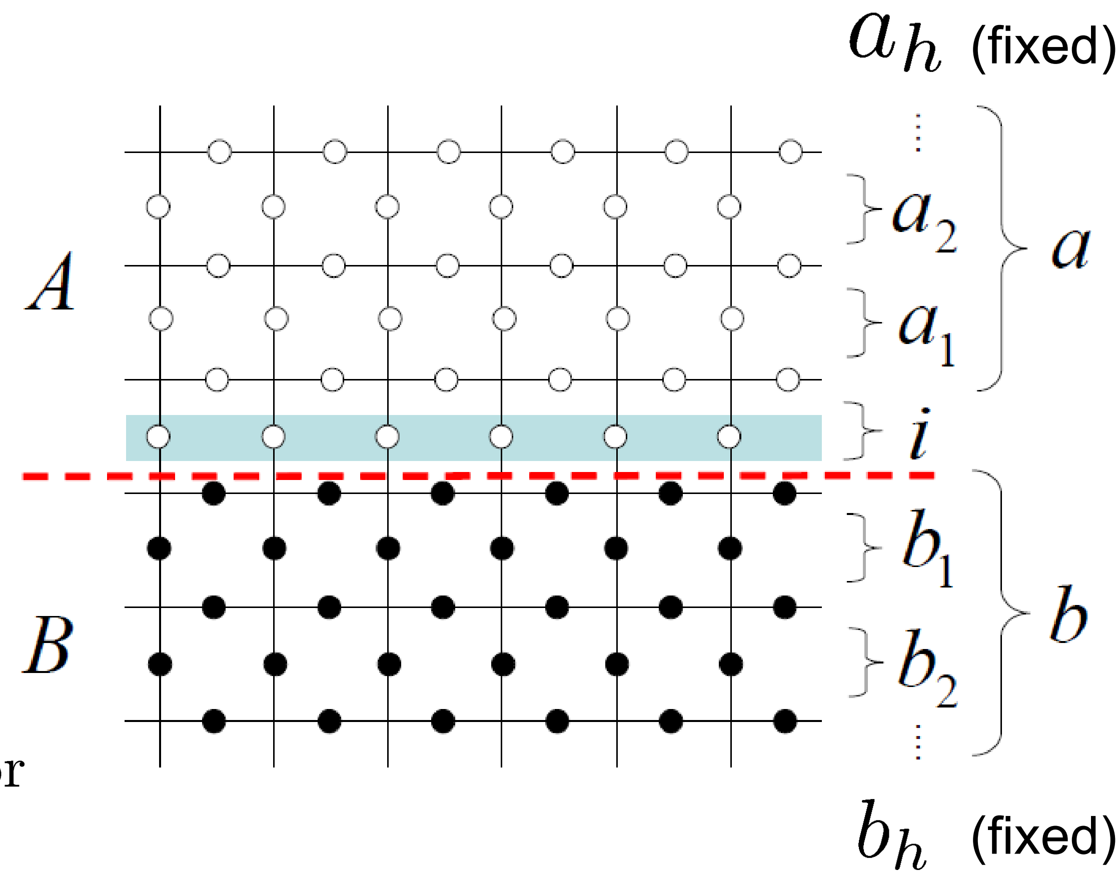
$h \gg L$

$\mathcal{T}^h \approx |g\rangle m_0^h \langle g|$

$m_0$: dominant eigenvalue

$|g\rangle$: corresponding eigenvector

$$p_i = |\langle i|g\rangle|^2$$




density matrix spectrum of |RK> = configuration weights of |g>

entanglement entropy of |RK>: $S_A = -\sum_i p_i \log p_i$

``Shannon entropy of config. weights''

# ``Configuration'' entropy in 1D

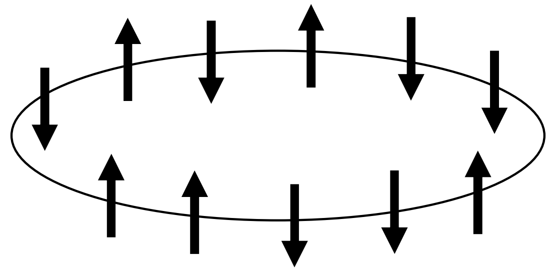
$$S = -\sum_i p_i \log p_i \qquad p_i = |\langle i|g\rangle|^2$$

$|g\rangle$ : ground state of a 1D quantum Hamiltonian H

$\{|i\rangle\}$ : configurations (basis of the Hilbert space)

Typically, $i = \{n_j\}$ or $\{\sigma_j^z\}$

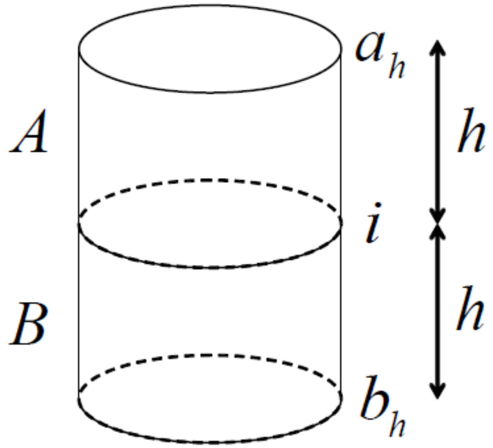
for U(1)-symmetric systems



- S is small if |g> is dominated by a particular crystal state $|i_0\rangle$. and becomes larger as more configs contribute due to quantum fluctuations.
- Measure of quantum fluctuations (or entanglement) occurring in a given basis

# Main results

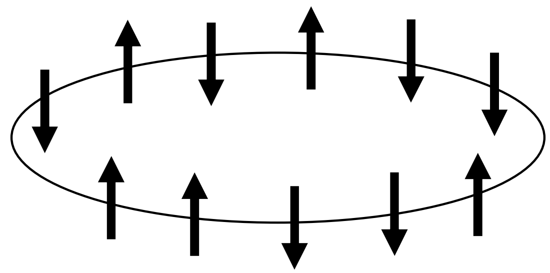
The scaling of configuration/entanglement entropy as a function of L.




$$S(L) = \alpha L + \gamma + \dots$$

$\alpha L$: boundary contribution

$\gamma$: universal



ring of length L

- If 1D quantum/2D classical system is described by a c=1 bosonic field theory (Tomonaga-Luttinger liquid),

$$\gamma = -\frac{1}{2}\log K - \frac{1}{2}$$

K:TLL parameter

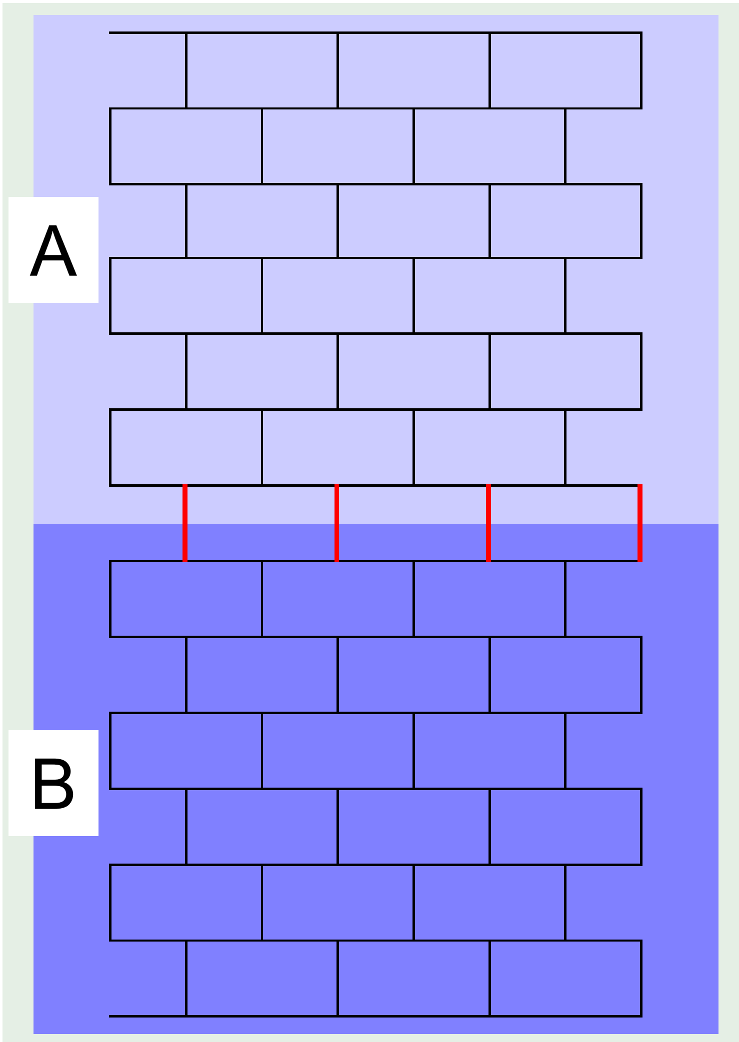
- Gapped crystal (ordered) phase with d-fold GS degeneracy

$$\gamma = \log d$$

# Dimer model on the hexagonal lattice

$$|\mathrm{RK}\rangle = \frac{1}{\sqrt{\mathcal{Z}}}\sum_c |c\rangle$$

sum of all dimer covering states

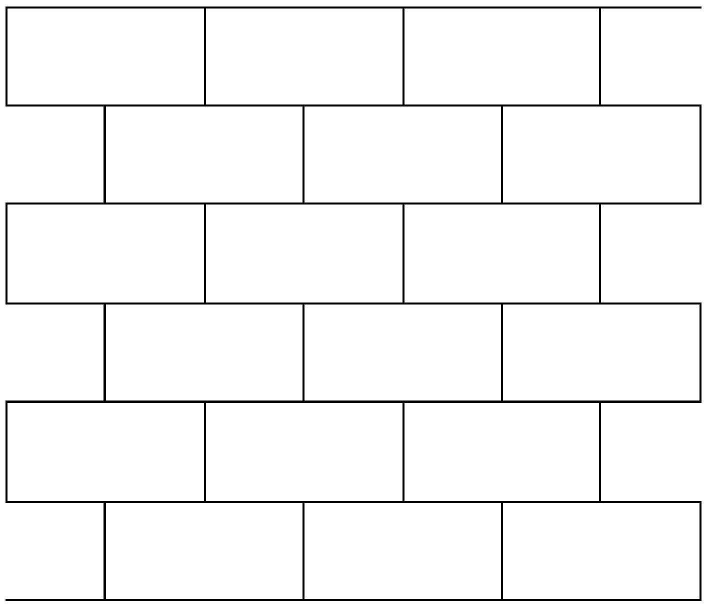


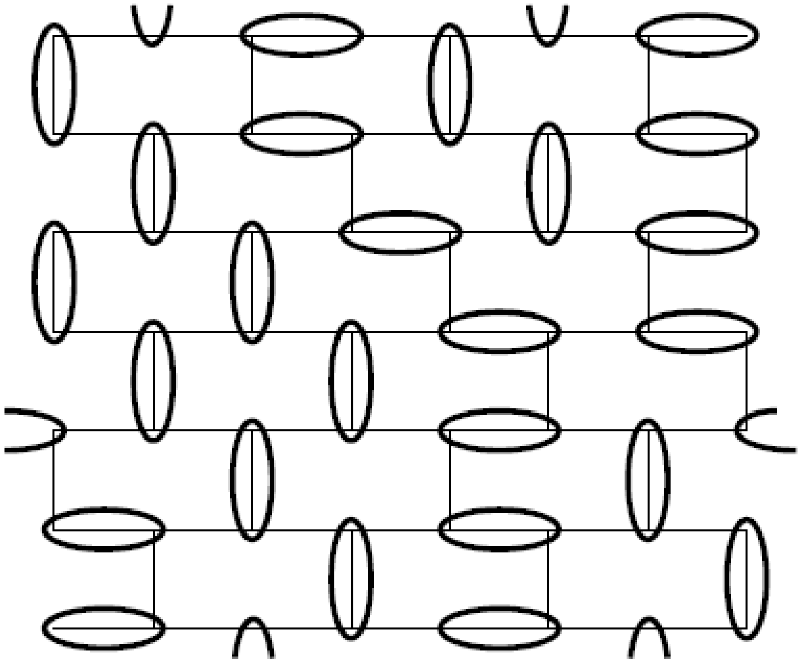

hexagonal lattice

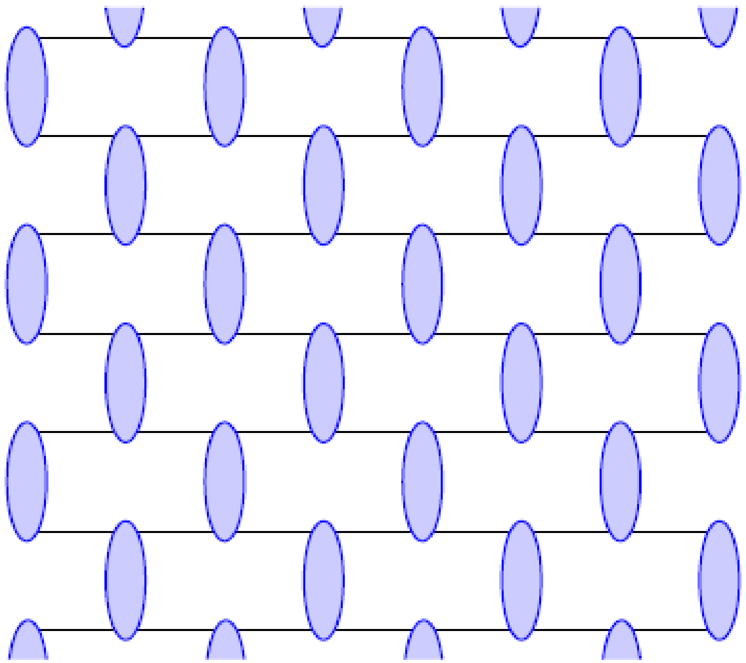
↓

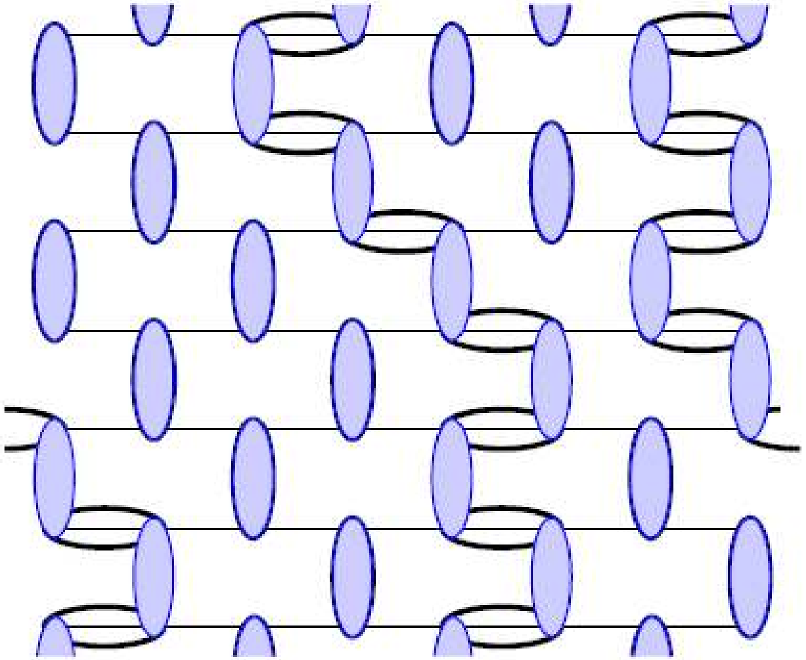
brick wall

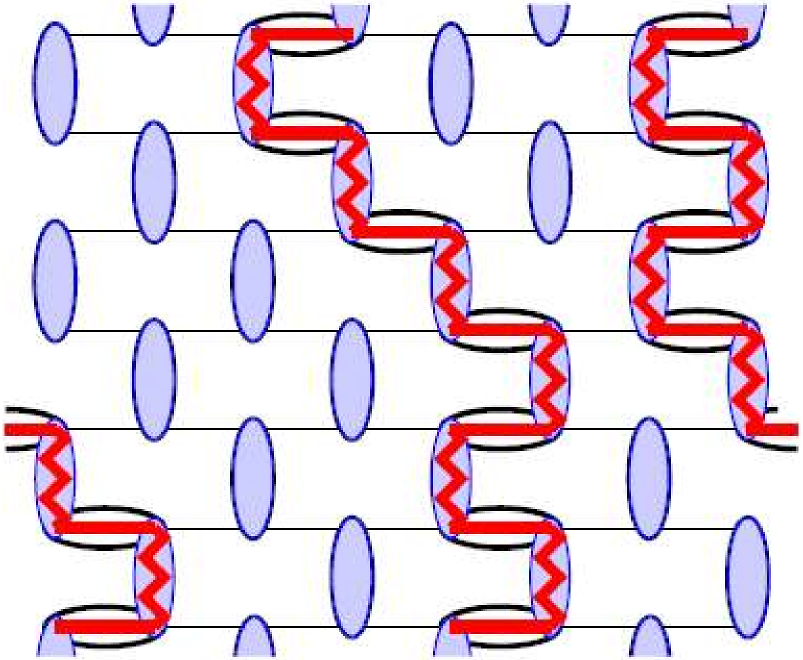
# Mapping onto a free fermion



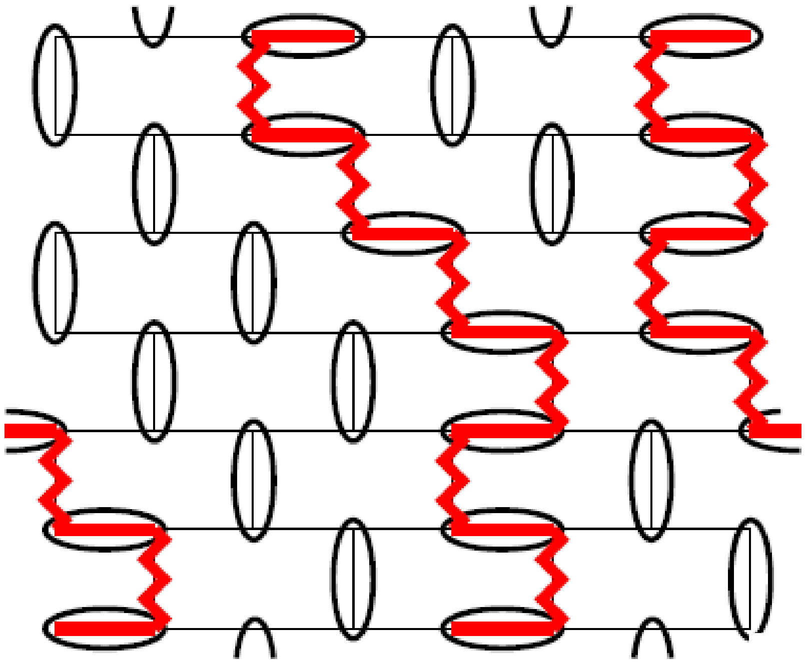








# Mapping onto a free fermion



<u>Transfer matrix</u> $\mathcal{T}$

- Time evolution of fermions
- A fermion hops to left or right:

$$\mathcal{T} c_j^\dagger \mathcal{T}^{-1} = c_j^\dagger + c_{j+1}^\dagger$$

$$\Rightarrow \quad \mathcal{T} = \prod_k \left(1 + e^{ik} c_k^\dagger c_k\right)$$

Dominant eigenvector: $|g\rangle = \left(\prod_{k \in \Omega} c_k^\dagger\right) |0\rangle$

degenerate Fermi sea

$\Omega = [-2\pi/3, 2\pi/3]$

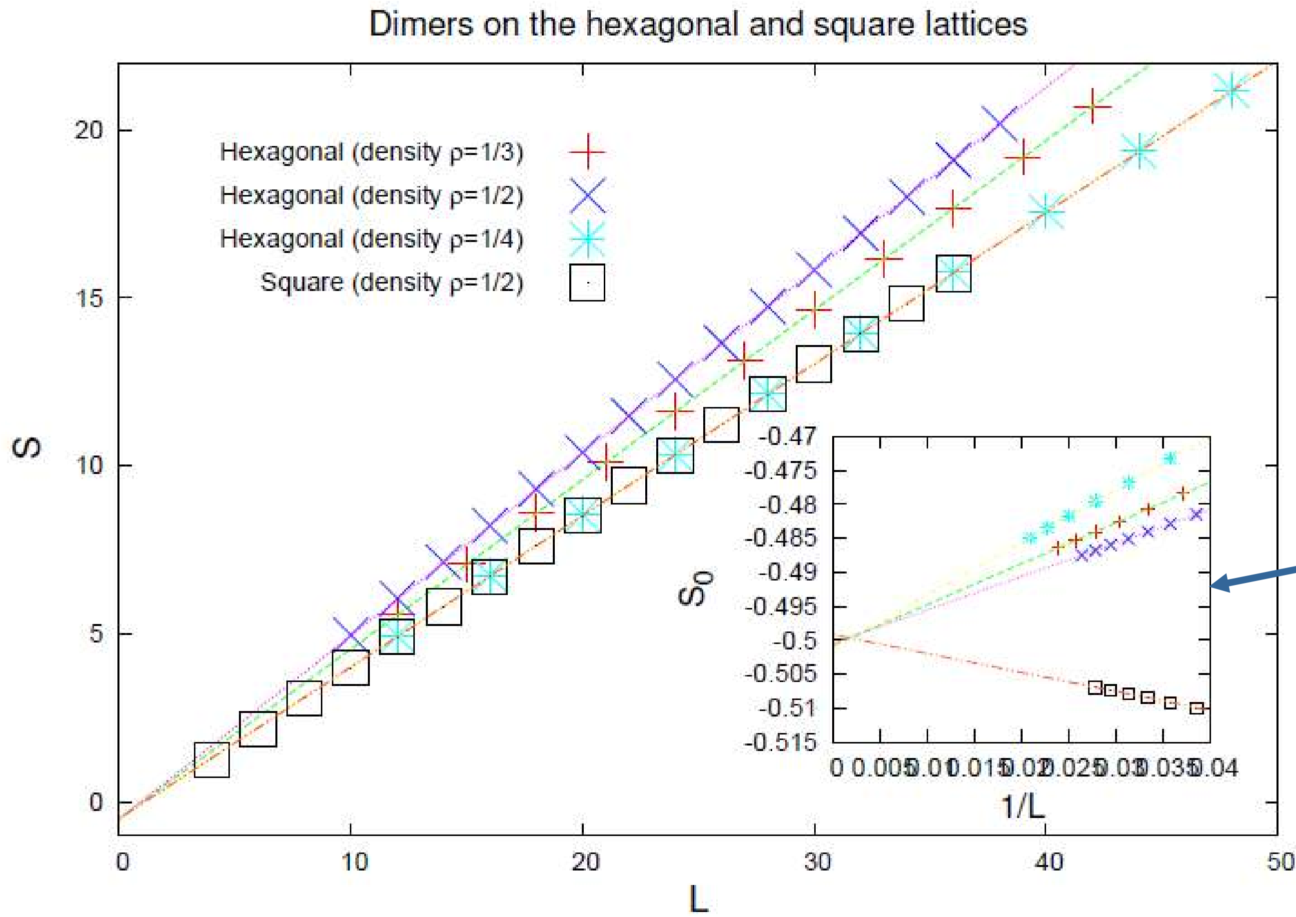
Configuration: $|i\rangle = c_{\alpha_1}^\dagger \dots c_{\alpha_n}^\dagger |0\rangle$

$\alpha_1, \ \alpha_2, \dots$: positions of fermions

$p_i = |\langle i|g\rangle|^2$ = (Vandermonde's determinant)

$$= \frac{1}{L^n} \prod_{1 \le j < j' \le n} 4 \sin^2 \left(\frac{\pi}{L}(\alpha_j - \alpha_{j'})\right)$$

# Scaling of entropy




result of fitting using [12,L]

Linear scaling: $S(L) = \alpha L + \gamma + \ldots$

$\gamma = -\frac{1}{2}$ universal value for dimer RK states on bipartite lattice / 1D free fermions

# Wave functional

$$|\langle\varphi|\Psi\rangle|^2 = \frac{1}{\mathcal{N}} e^{-\frac{2}{K}\mathcal{E}[\varphi]}$$

$$\mathcal{E}[\{\tilde{\varphi}_m\}] = \sum_{m=1}^{\infty} k_m |\tilde{\varphi}_m|^2$$

Fourier component of $\varphi$

Fradkin,Moreno,& Schaposnik, Nucl.Phys.B, 1993

$$Z(\beta) = \int \mathcal{D}\varphi e^{-\beta\mathcal{E}[\varphi]} = \prod_{m=1}^{\infty} \frac{2\pi}{k_m \beta} = e^{-\alpha L}\sqrt{\frac{\beta}{2}}$$

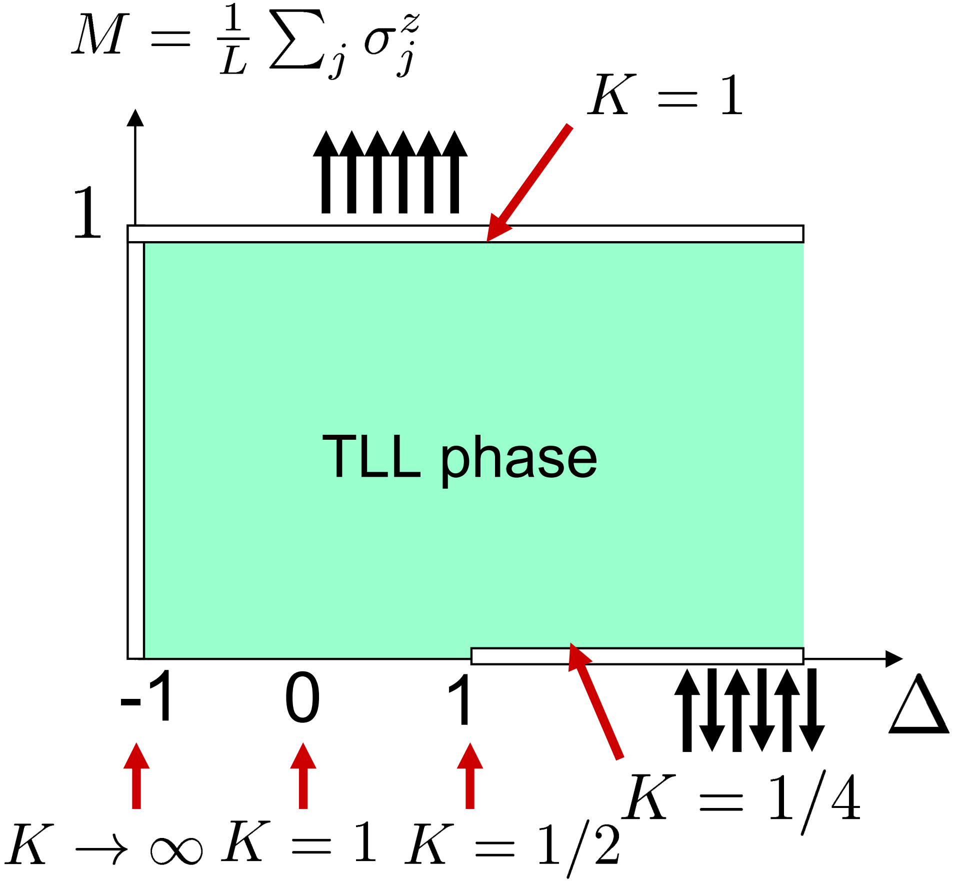
$\zeta$-fn. regularization

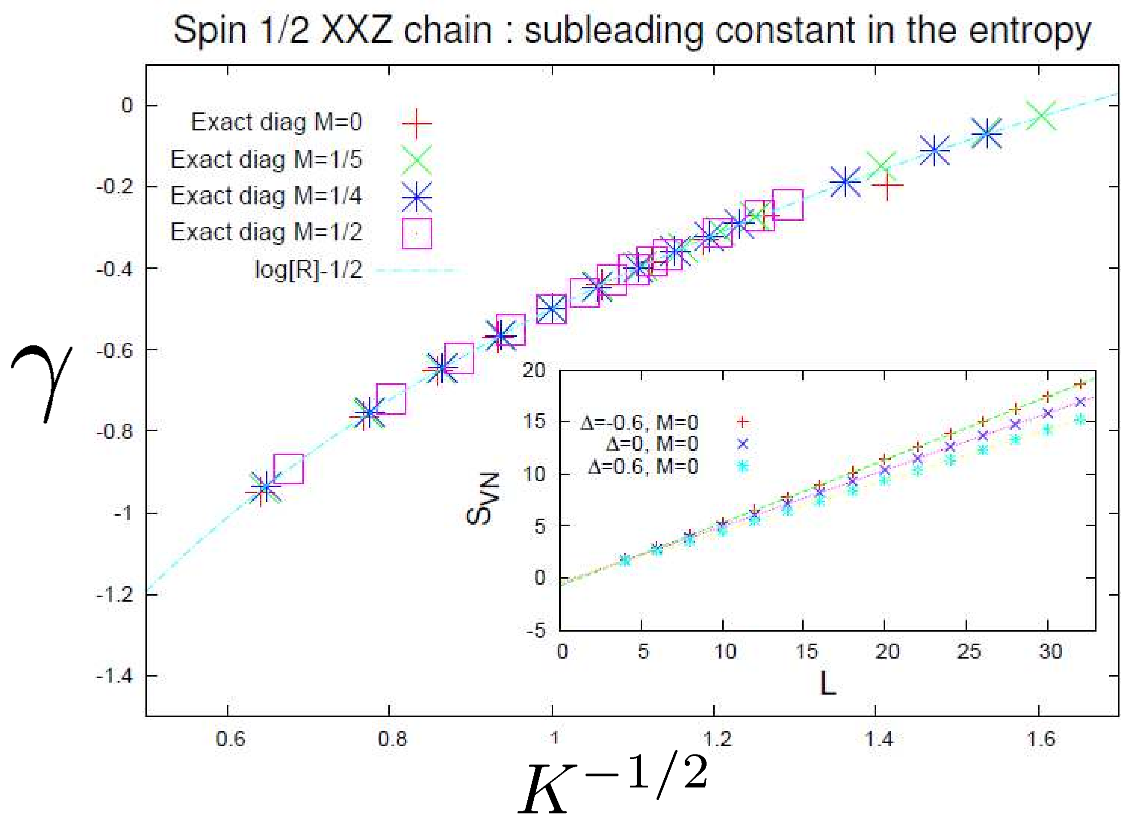
$$S = \left(1 - \beta\frac{\partial}{\partial\beta}\right) \log Z(\beta)\Big|_{\beta=2/K} = \alpha L - \underbrace{\frac{1}{2}\log K - \frac{1}{2}}_{\text{Universal constant } \gamma}$$

The same result was obtained in conformal boundary state approach: Oshikawa,arXiv,2010; Hsu & Fradkin,J.Stat.Mech.,2010

# spin-1/2 XXZ chain ⟷ six-vertex RK state

$$\mathcal{H} = \sum_j \left(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z\right) - h \sum_j \sigma_j^z$$

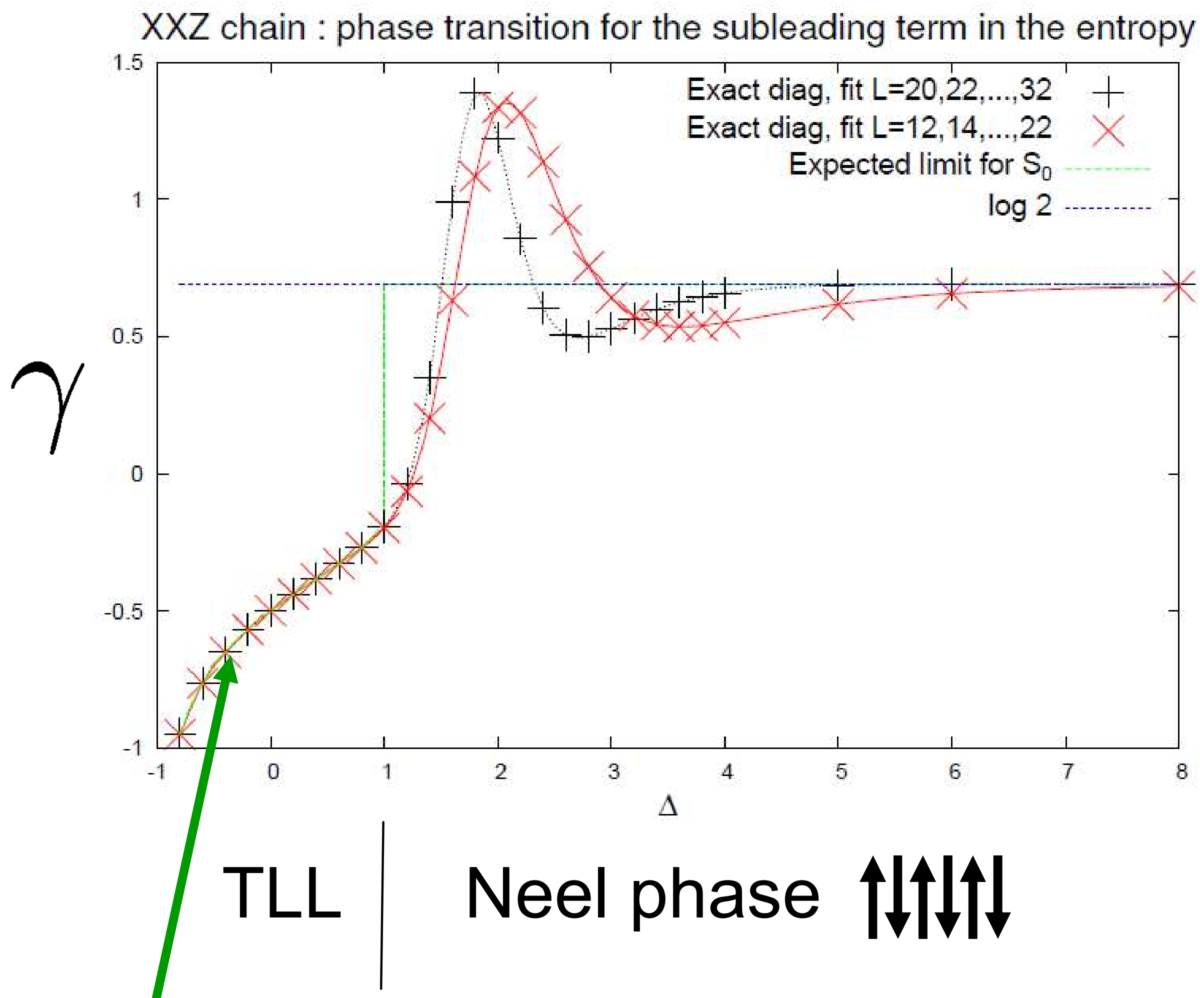



Nice agreement with

$$\gamma = -\frac{1}{2} \log K - \frac{1}{2}$$

# Phase transition to an ordered (crystal) phase




$$\gamma = -\frac{1}{2}\log K - \frac{1}{2}$$

**Interpretation:**

$$\Delta \rightarrow \infty$$

$$|g\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\ldots\rangle + |\downarrow\uparrow\ldots\rangle)$$

$$\Rightarrow \quad S = \log 2$$

Fluctuations produce only extensive contributions.

General case: d-fold degeneracy

$$S_0 = \log d$$

# Summary – Rokhsar–Kivelson wave functions

- We have studied the scaling of two entropies:
  - entanglement entropy of 2D RK wave functions
  - ``configuration'' entropy of 1D wave functions
- General scaling: $S(L) = \alpha L + \gamma + \ldots$

  $\alpha L$: boundary contribution

  $\gamma$: universal
- In c=1 critical systems (TL liquids), $\gamma = -\frac{1}{2}\log K - \frac{1}{2}$
- Gapped crystal (ordered) phase with d-fold GS degeneracy

$$\gamma = \log d$$

- New tool for determining K, detecting phase transitions, etc.

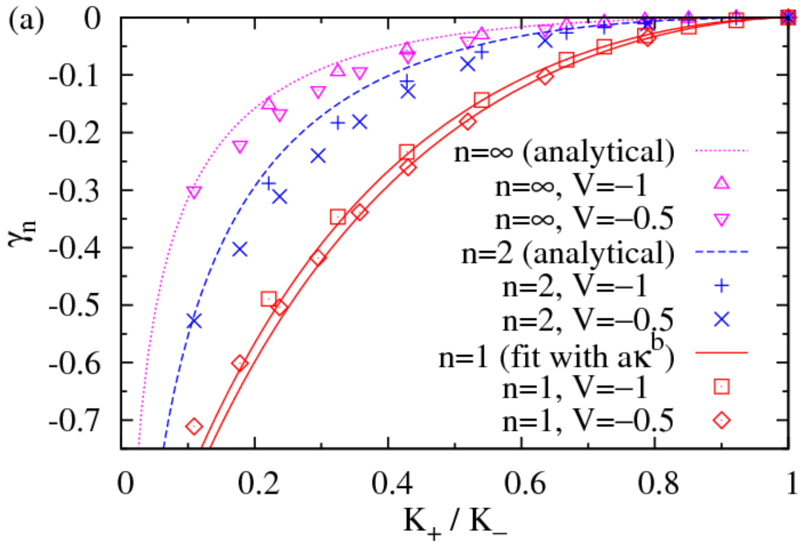
# Summary

$$S(L) = \underbrace{\alpha L}_{} + \underbrace{\gamma}_{} + \dots$$

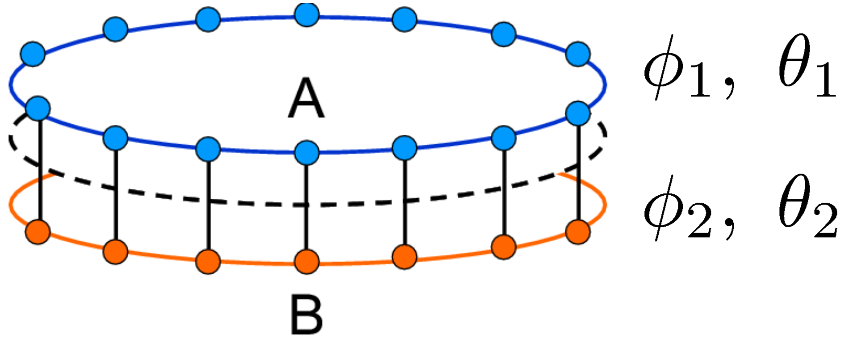
boundary contribution

universal

- Two coupled Tomonaga-Luttinger liquids




determined by $K_+/K_-$




- 2D Rokhsar-Kivelson wave functions

$S_A$ → 1D quantum problem

transfer matrix

$$\gamma = -\frac{1}{2}\log K - \frac{1}{2}$$

(K: TLL parameter)