Applying the variational principle to (1+1)-dimensional relativistic quantum field theories

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October 28, 2010
"NDNS: From DMRG to TNF"
YITP, Kyoto University, Japan

Overview

Importance of the variational method:

- NRG and DMRG $\longrightarrow$ variational optimization over TN:
  $\Rightarrow$ very successful in describing low-energy states of (low-dimensional) quantum lattice systems
- Other variational methods in quantum physics:
  - single-particle quantum mechanics (*e.g.* quartic potential)
  - quantum chemistry: Hartree-Fock, Density Functional Theory, ...

Advantages over alternatives:

- non-variational Monte Carlo: sign problem for many interesting systems
- perturbation theory: interesting physical effects are often non-perturbative

Nonperturbative application of the variational approach for relativistic quantum field theory:

long history $\longleftrightarrow$ little success

Feynman @ International Workshop on Variational Calculations in Quantum Field Theory, Wangeroo, West Germany, 1 - 4 September 1987 *:

...it is no damn good at all!

* R. P. Feynman in Proceedings of the International Workshop on Variational Calculations in Quantum Field Theory (L. Polley and D. E. L. Pottinger, eds.), World Scientific Publishing, Singapore, pp. 28–40 (1987).

What are the "Difficulties in Applying the Variational Principle to Quantum Field Theories"?

1. Sensitivity to High Frequencies:
    - the variational principle's only task is to minimize the ground state energy, which is dominated by zero-point fluctuations of high frequencies/UV modes/deepest scale
    - the low frequencies, which generate the interesting observable physics, will be ill-described
2. Only Gaussian Trial States
3. We Still Have To Do a Functional Integral
    - it's hard to find a non-gaussian wave function that respects extensivity and allows for an efficient evaluation of expectation values

Feynman's second and third argument is generally applicable to extended quantum systems and has been solved:

- lattice systems: AKLT (1988) → MPS → tensor networks
- field theories: continuous matrix product states (2010)

→ did Feynman foresee this possibility?

...I think it should be possible some day to describe field theory in some other way than with the wave functions and amplitudes. It might be something like the density matrices where you concentrate on quantities in a given locality ...

The first argument is catastrophique for (relativistic) field theories:

- non-trivial vacuum all the way to the highest scale
- infinite amount of UV modes

⇒ divergences (*e.g.* the ground state energy density)

A continuous Matrix Product State (cMPS) is given by:

$$|\Theta\rangle = \mathrm{Tr}_{\mathrm{anc}}\left[\mathcal{P}\exp\left\{\int_{-\infty}^{+\infty} \mathrm{d}x\; Q(x)\otimes\hat{1} + R_\alpha(x)\otimes\hat{\psi}_\alpha^\dagger(x)\right\}\right]|\Omega\rangle \qquad (*)$$

- Q and R position-dependent $D\times D$ matrices, acting on the D-dimensional ancilla

  $\rightarrow$ we will use translational invariance: constant Q and R
- Trace over ancilla for periodic system or D-dimensional boundary vectors $\langle v_L|$, $|v_R\rangle$ for open system

  $\rightarrow$ will not matter for an infinite size system
- $\hat{\psi}_\alpha^\dagger(x)$ a type α particle (boson/fermion) creation operator acting on an empty vacuum state $|\Omega\rangle$ (*i.e.* $\hat{\psi}_\alpha(x)|\Omega\rangle = 0$)

  $\rightarrow$ we will restrict to fermionic theories

$\Rightarrow$ more info: Frank Verstraete's talk tomorrow!

* F. Verstraete and J.I. Cirac, *Phys. Rev. Lett.* **104**:190405, 2010

Properties of cMPSs for relativistic QFTs:

- they overcome Feynman's second and third objection: the cMPS is an extensive, non-gaussian variational ansatz that allows for an efficient evaluation of expectation values
- they yield 'regularized' expectation values for the relativistic kinetic energy of fermions, as well as of any interaction term in the Hamiltonian, provided that

$$\{R_\alpha, R_\beta\} = \delta_{\alpha,\beta}, \qquad \alpha, \beta \in \{1,2\} \quad (*)$$

$\Rightarrow$ regularization through intrinsic soft momentum cutoff Λ: momentum occupation $n_{\alpha,\beta}(k)$ goes as $\mathcal{O}(k^{-4})$ for $k \gtrsim \Lambda$

- they easily allow scale transformations: $Q \rightarrow cQ, R_\alpha \rightarrow \sqrt{c}R_\alpha \quad \Rightarrow \quad n_{\alpha,\beta}(k) \rightarrow n_{\alpha,\beta}(k/c), \Lambda \rightarrow c\Lambda$

* $\alpha = 1,2$: two components of the Dirac spinor in $(1+1)$ dimensions

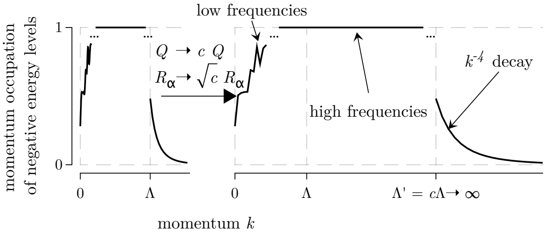
The exact ground state of a relativistic Hamiltonian is non-trivial/non-empty all the way up to the deepest scales

Manifestation of Feynman's sensitivity to high frequencies

- Momentum space interpretation:
  - energy can most strongly be lowered by having particles with very large momentum in the ground state $(*)$
  - variational principle will send $\Lambda \to \infty$ by letting $c \to \infty$,
- Real space interpretation:
  - if $\langle \hat{T} \rangle < 0$, then the (kinetic) energy density can be lowered with $c \to \infty$ since $\hat{t} \to c^2 \hat{t}$
  - for comparison: non-relativistic $\hat{T}$ is positive definite

* free Hamiltonian: filling negative energy levels,

interacting Hamiltonian: some other non-trivial configuration



Why is this problematic

- a cMPS will be able to accurately describe a ground state if $D \sim \mathcal{O}(e^S)$ with $S \sim \log(\Lambda/\Delta)$ with Δ the energy gap
- if D is too small, or Λ too large, compromises will be made in the description of long range behavior (IR scale)
- if the variational algorithm can push $c \to \infty$: behavior at any observationally accessible scale will be totally wrong for any finite D

Renormalizability: physical quantities are nearly independent to the UV specifics of a renormalizable QFT

$\Rightarrow$ We are free to 'regularize' the field theory by modifying the Hamiltonian such that

- the low energy dynamics are unchanged
- the high energy dynamics are such that the asymptotic solution for $k \to \infty$ is the the empty vacuum $|\Omega\rangle$ *

* Any other unentangled vacuum would do as well.

Regularizing relativistic fermions: one of many possibilities

$$\hat{H} \to \hat{H} + \frac{1}{\Lambda} \int_{-\infty}^{+\infty} \mathrm{d}x \left(\frac{\mathrm{d}\psi_\alpha^\dagger}{\mathrm{d}x}(x) \right) \left(\frac{\mathrm{d}\psi_\alpha}{\mathrm{d}x}(x) \right)$$

$\Rightarrow$ dispersion relation of free fermions will change to

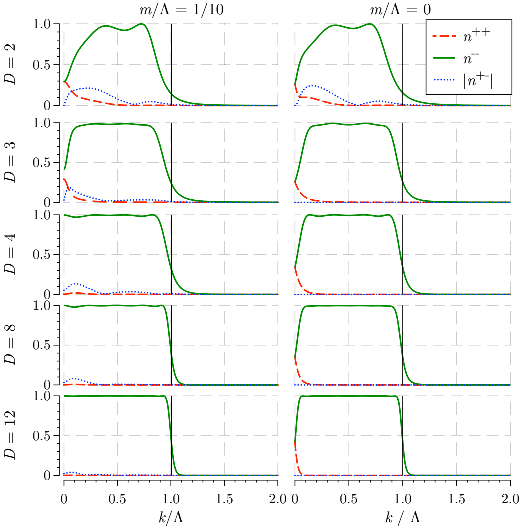
$$\omega(k) = \sqrt{m^2 + k^2 + k^4/\Lambda^2} = \sqrt{m^2 + k^2} + \mathcal{O}(k^4/\Lambda^2)$$

$\Rightarrow$ breaks relativistic invariance

$\Rightarrow$ ground state of the free theory will be completely empty (sharp cutoff / Fermi surface) for $|k| \gtrsim \Lambda$
ground state of an interacting theory will quickly become empty for $|k| > \Lambda$ (no sharp cutoff / Luttinger liquid)

$\Rightarrow$ note: no fermion doublers in this regularization scheme!

Evaluating results for Dirac fermions



Ground state energy density is not useful.

$\Rightarrow$ Look at occupation of exact energy levels:

- for a gapped system ($m > 0$), low energy behavior is very well reproduced
- sharp cutoff is not reproduced (but this does not matter)
- chiral symmetry is respected for $m = 0$

Note: ancilla is $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^D$.

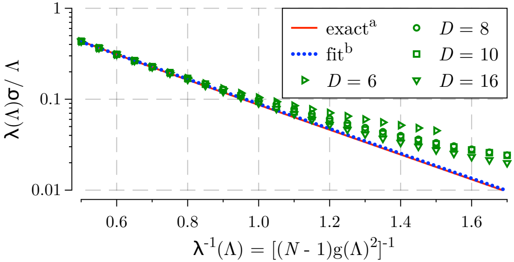
Gross Neveu

- N flavors of massless fermions interacting through a quartic potential (exactly solvable for $N \to \infty$)

$\Rightarrow$ spontaneous breaking of chiral symmetry

$\Rightarrow$ dynamical mass generation

Order parameter of chiral symmetry breaking



Order parameter:

$$\sigma = \langle \hat{\psi}_1^\dagger \hat{\psi}_1 - \hat{\psi}_2^\dagger \hat{\psi}_2 \rangle$$

Exact solution ($N = \infty$):

$$\lambda\sigma = \Lambda e^{-\frac{\pi}{\lambda}}$$

with $\lambda = (N-1)g^2$

Conclusions

We have taken a first step in showing that

- the variational approach towards relativistic QFT might be not so bad after all
- continuous matrix product states are well suited to capture the low energy dynamics of relativistic field theories too

But much more still has to be done!

Outlook

In the near future, we expect to be able to

- treat bosonic theories
- have an ansatz for creating excitations

In the far future, we might be able to

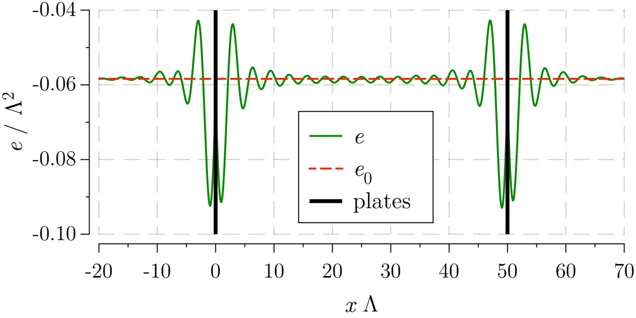
- treat (relativistic) field theories in higher dimensions
- deal with gauge theories
- solve QCD, prove confinement, win one million dollar

Thank you for your attention!

Questions ?

Casimir energy for Dirac fermions

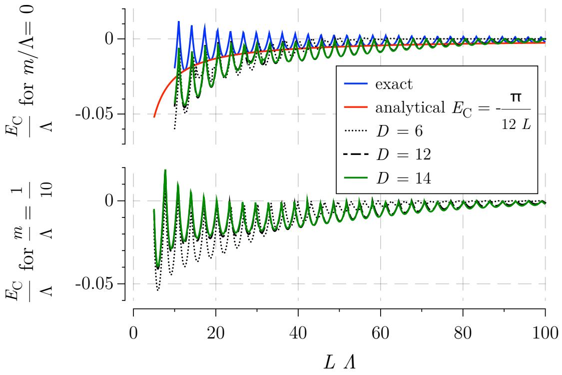
Energy density between two plates:



$\Rightarrow$ Friedel oscillations due to presence of cutoff (Fermi surface).

Casimir energy for Dirac fermions

Total energy as function of plate separation



⇒ Friedel oscillations lead to resonances