New Development of Numerical Simulations in Low-Dimensional Quantum Systems:
From Density Matrix Renormalization Group to Tensor Network Formulations
October 27-29, 2010 Yukawa Institute for Theoretical Physics

Geometric Calculation of Entanglement Entropy via AdS/CFT

``Can AdS/CFT be useful to condensed matter physics ?''

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Contents

① Motivations

The main purpose of this talk

⇒ introduce the AdS/CFT correspondence (in string theory) to condensed matter physicists

AdS→ anti-de Sitter space; CFT→conformal field theory

Recently, condensed matter physics interpretations of AdS/CFT in various examples have been intensively discussed in string theory community.

This is sometimes called ``AdS/CMT''.

Q1. What is the idea of AdS/CFT ?

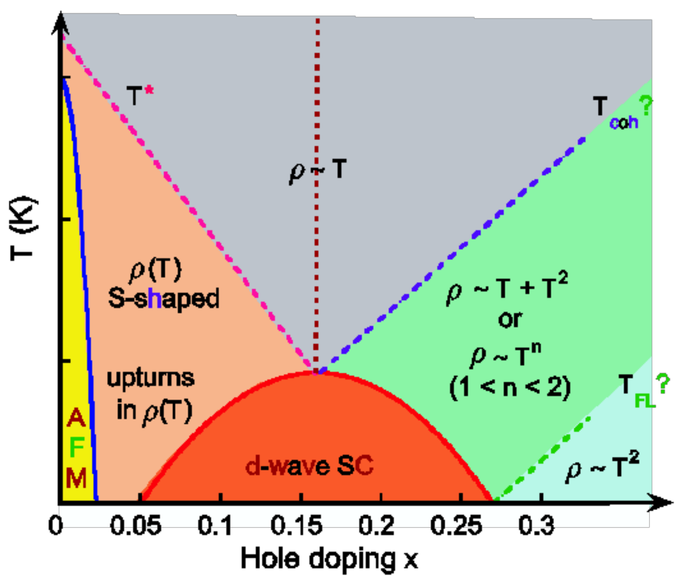
Roughly speaking, it claims the following equivalence:

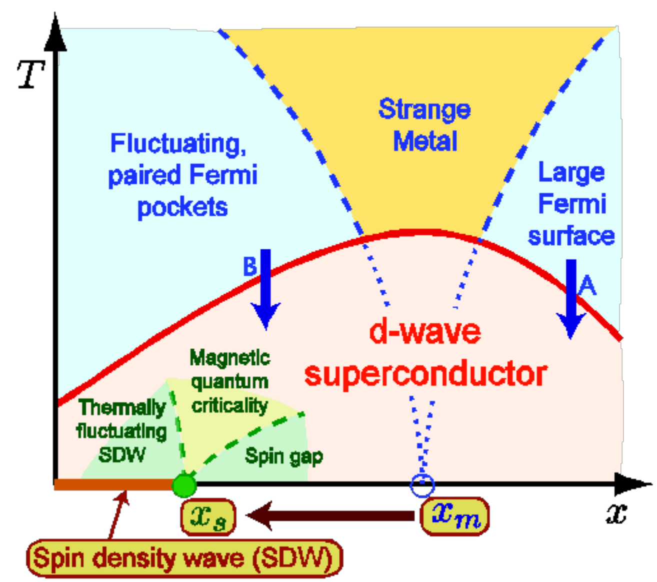
Strongly coupled quantum many-body systems d+1 dim.	=	Gravity (general relativity) with negative Λ d+2 dim.
⬇		⬇
Various systems (or phases)	=	Various classical solutions (typically black holes)
Very complicated		Easy to analyze (Analytical or Laptop computers)

Q2 Why CFT ?

The conformal symmetry makes systems more universal and it often appears in tractable examples in string theory.

Quantum criticality appears in many cond-mat systems
e.g. Heavy fermion systems, High Tc cuprates, Fractional Quantum Hall Effect, Graphene etc…



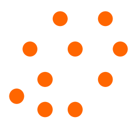


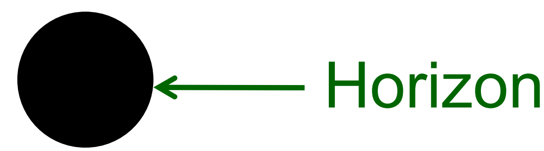
Figs taken from Sachdev 0907.0008

② An Introduction to AdS/CFT

(2-1) What is ``Holography'' ?

In the presence of gravity,

A lot of massive objects in a small region  → Black Holes (BHs)



The information hidden inside BHs is measured by the Bekenstein-Hawking black hole entropy:

$$S_{BH} = \frac{\mathrm{Area(Horizon)}}{4G_N}$$

This consideration leads to the idea of the entropy bound:

$$S(A) \leq \frac{\text{Area}(\partial A)}{4G_N}$$

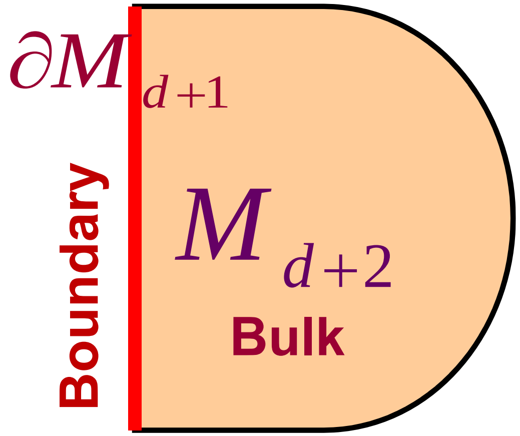
(S(A) = the entropy in a region A)

➡ The degrees of freedom in gravity are proportional to the area instead of the volume !

cf. In non-gravitational theory, proportional to volume.

Motivated by this, holographic principle has been proposed by 't Hooft 93' and Susskind 94':

d+2 dim. Gravity on M
= d+1 dim. non-gravitational theory (e.g. QFT) **on ∂M_{d+1}**



(2-2) AdS/CFT Correspondence

The best established example of holography is known as AdS/CFT correspondence [1997 Maldacena], which is currently one of the most active topics in string theory.

AdS/CFT

$$\text{(Quantum) Gravity on } AdS_{d+2} \quad = \quad \text{CFT on } R^{d+1}$$

$$\text{Isometry of } AdS_{d+2} = SO(d+1,2) \; = \; \text{Conformal Sym.}$$

Note: The AdS/CFT is still a conjecture though so many evidences have been found in the past 10 years.

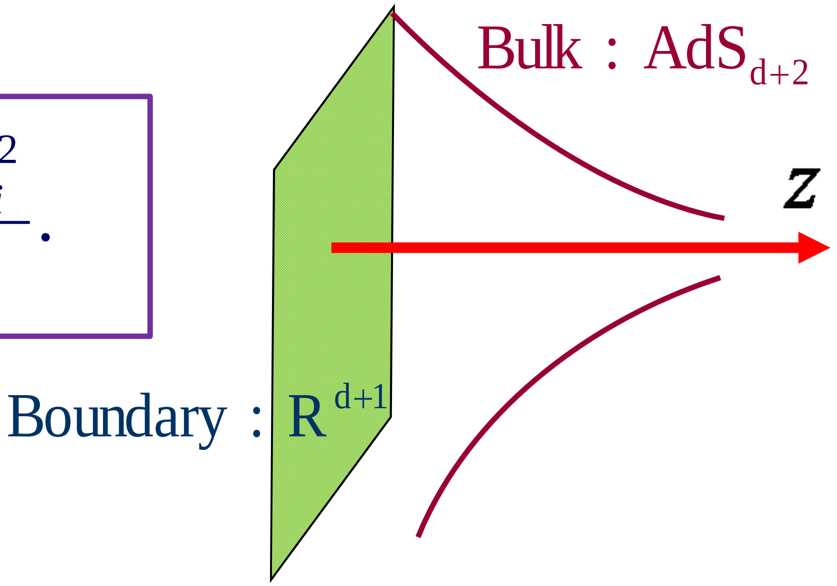
What are AdS spaces ?

They are homogeneous solutions to the vacuum Einstein equation with a negative cosmological constant:

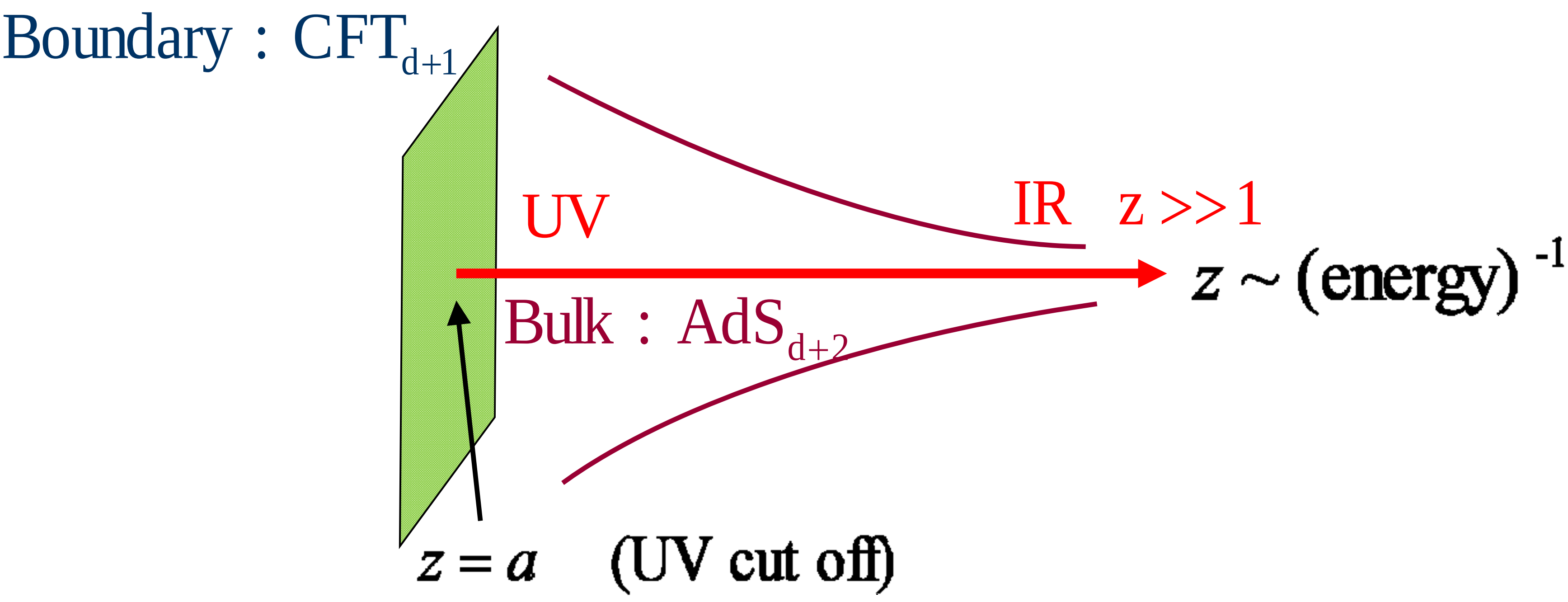
$$S_g = \frac{1}{16\pi G_N}\int dx^{d+2}\sqrt{-g}\left[R-\Lambda\right], \qquad \Lambda \equiv -\frac{(d+1)d}{R^2}.$$

The metric of AdSd+2 (in Poincare coordinate) is given by

$$ds^2_{AdS_{d+2}} = R^2\,\frac{dz^2 - dx_0^2 + \Sigma_{i=1}^{d} dx_i^2}{z^2}.$$



A Sketch of AdS/CFT



The radial direction z corresponds to the length scale in CFT under the RG flow.

Note: In string theory, the spacetime dimension is actually 10. Thus the AdS spaces appears with compact spaces such as $AdS_5 \times S^5$.

What are CFTs ?

They are typically SU(N) gauge theories in the large N limit.

e.g. Type IIB String on $AdS_5 \times S^5$

= N=4 U(N) Super Yang-Mills in 4 dim.

# of supersymmety

Gauge field + 6 Scalar fields + 4 Fermions

(A_μ) $(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6)$ $(\psi_1, \psi_2, \psi_3, \psi_4)$

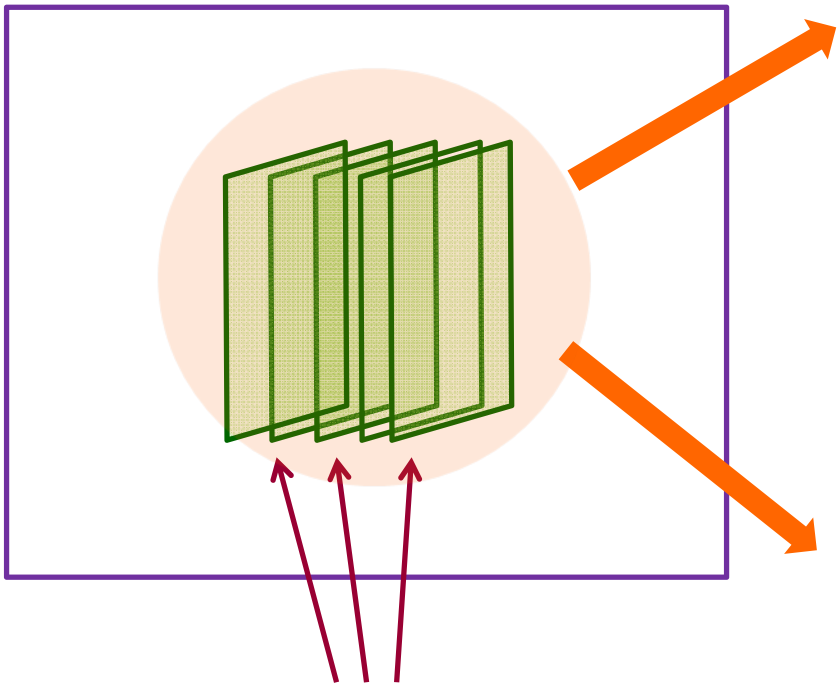
Symmetry of S^5 ⇔ SO(6) R symmetry

In AdS/CFT, a geometric sym. in gravity

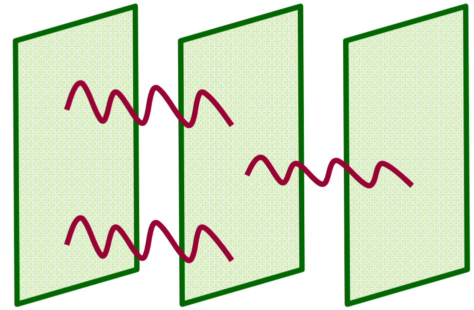
= a global sym. in dual YM gauge theory.

How is AdS/CFT discovered in String Theory ex. AdS5/CFT4

10 dimensional type IIB string theory
with N **D3-branes**



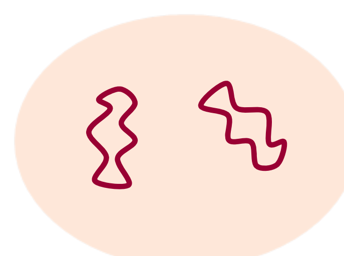
N D3-branes
= (3+1) dimensional sheets



Open Strings between D-branes
→ SU(N) gauge theories

Equivalent !

Type IIB closed string on AdS5 × S5
→ Gravity on AdS5 spacetime



(2-3) Bulk to boundary relation

The basic principle in AdS/CFT to calculate physical quantities is the bulk to boundary relation (GKPW relation 98'):

$$\boxed{Z_{Gravity}(M) = Z_{CFT}(\partial M).}$$

In gravity theories, there are various fields

e.g. metric, scalar fields, gauge fields, fermions,

$$Z_{Gravity} = \int Dg_{\mu\nu} D\phi \; e^{-S(g(x,z),\phi(x,z))} \cong e^{-S(g,\phi)}\Big|_{\substack{\text{Equation}\\ \text{of motion}}} \cdot$$

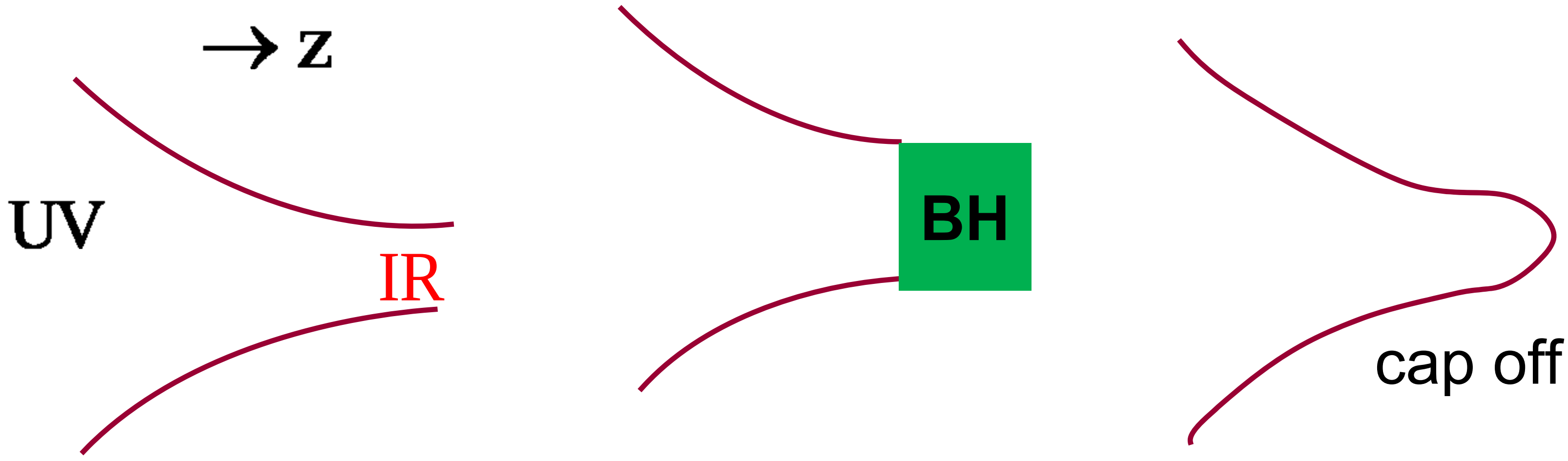
$$Z_{CFT} = \left\langle e^{\int dx^{d+1}[\delta g^{(0)}{}_{\mu\nu}(x)T^{\mu\nu}(x)+\phi^{(0)}(x)O(x)]} \right\rangle \Rightarrow \text{Correlation functions}$$

$$\langle O(x_1)O(x_2)\cdots O(x_n)\rangle$$

(2-4) Black holes and AdS/CMT

AdS/CFT can be naturally generalized to the duality:

asymptotically AdS spaces ⇔ QFTs with UV fixed points .



Pure AdS	AdS BH	AdS Soliton
⇕	⇕	⇕
CFT at T=0	Finite temp. CFT	QFT with Mass gap
	$S_{BH} \propto N^2$	(confinement)

Condensed matter systems ⇒ Temperature T
Charge density ρ
(Chemical potential μ)

Metallic systems ⇔ Charged AdS BH $T_{BH}>0$, $\rho_{BH}>0$.

Superconductors or Superfluids ⇔ Charged AdS BH with charged scalar hair $\langle \phi \rangle \neq 0.$

Universal Viscosity/Entropy Ratio [Kovtun-Son-Starinets 01']

$$\frac{\eta}{s} = \frac{\hbar}{4\pi k_B}.$$

⇒ Very close to observed values in both quark-gluon plasmas and cold atoms experiments.

Extremely small viscosity

AdS/CFT predicts that this is true for any strongly coupled theory !

(2-5) Calculation of conductivity

As a specific example of AdS/CFT calculations, we would like to compute the holographic conductivity for CFT3.

Solving Maxwell eq. $\partial_\mu(\sqrt{g}F^{\mu\nu})=0$ in the AdS4 charged BH,

$$A_\mu(z,x) = \underbrace{A_\mu^{(0)}(x)}_{\text{External ele-mag field}} + z\cdot A_\mu^{(1)}(x) + ... \quad (z\to 0),$$

$$A_\mu(z,x) \approx \underbrace{(z-z_H)^{-i\gamma\omega}}_{\text{In-going boundary condition}} \cdot F(x) \qquad (z\to z_H \text{ : near horizon limit}),$$

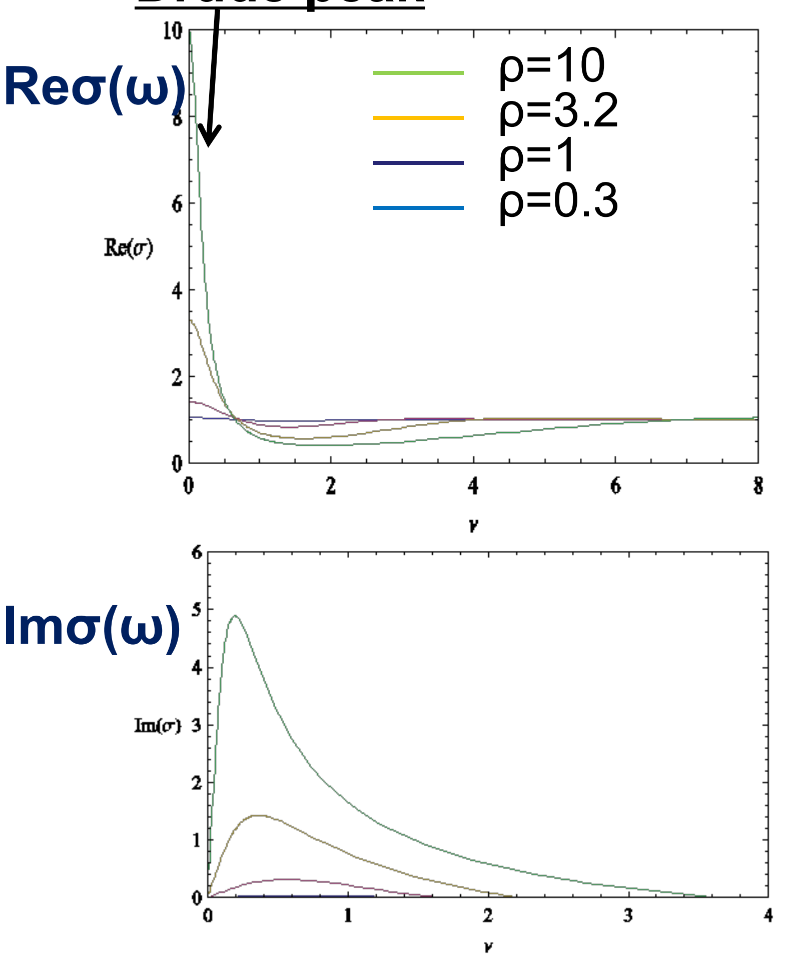
$$\text{Current : } J^\mu(x) = \left.\frac{\delta S}{\delta A_\mu(z,x)}\right|_{z\to 0} = A_\mu^{(1)}(x),$$

$$\text{AC Conductivity : } \sigma_{\mu\nu}(\omega) = \frac{A_\mu^{(1)}(\omega)}{-i\omega\cdot A_\nu^{(0)}(\omega)}.$$

2+1 dim. result via AdS/CFT

[Das-Nishioka-TT 10']

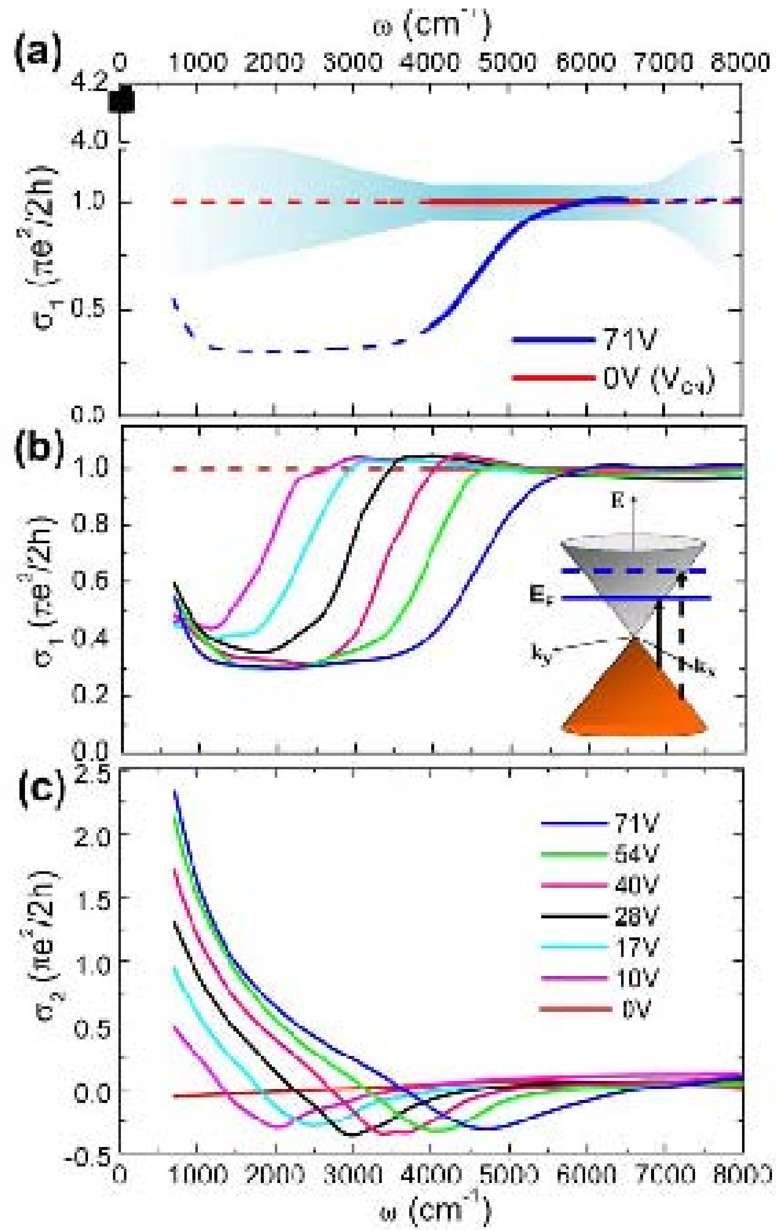
Drude peak



Note: This is based on the probe D3-brane spinning in $AdS_5 \times S^5$.

Optical conductivity of Graphene

[Z.Li et.al. Nature Physics 4, 532 (2008)]



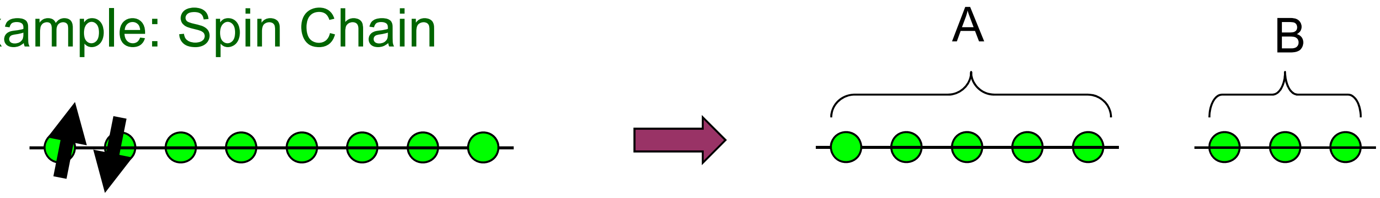
③ Entanglement Entropy from AdS/CFT

(3-1) Entanglement Entropy

Divide a given quantum system into two parts A and B.
Then the total Hilbert space becomes factorized

$$H_{tot} = H_A \otimes H_B \quad .$$

Example: Spin Chain



We define the reduced density matrix ρ_A for A by

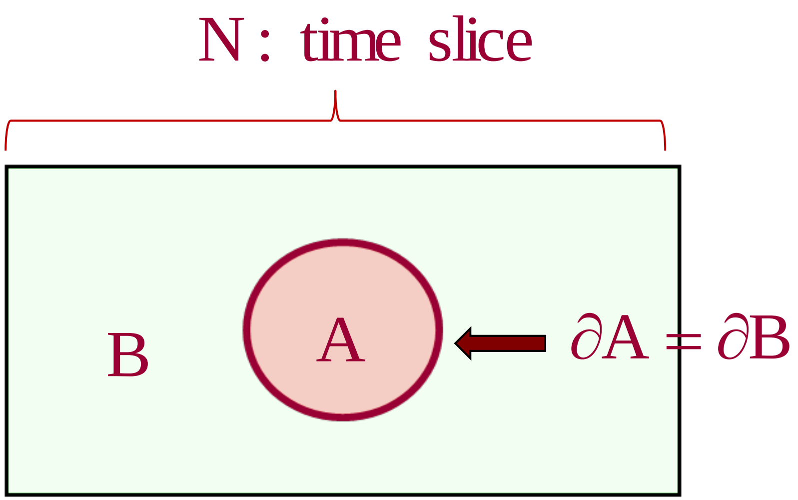
$$\rho_A = \mathrm{Tr}_B \rho_{tot} \quad ,$$

taking trace over the Hilbert space of B .

Now the entanglement entropy S_A is defined by the von-Neumann entropy

$$S_A = -\mathrm{Tr}_A\, \rho_A \log \rho_A \quad .$$

In QFTs, it is defined geometrically (called geometric entropy).



Various Applications

- **Quantum Information and Quantum Computing**

EE = the amount of quantum information

[see e.g. Nielsen-Chuang's text book 00']

- **Condensed Matter Physics**

EE = Efficiency of a computer simulation (DMRG) [Gaite 03',…]

➡ This gets divergent at phase transition point !

➡ A new quantum order parameter !

[Topological entanglement entropy: Kitaev-Preskill 06', Levin-Wen 06']

- **Quantum Gravity and String Theory**

An origin of black hole entropy ?

A useful physical quantity in quantum gravity ?

Basic property: Area law

EE in d+1 dim. QFTs (in the ground states) includes UV divergences.

Its leading term is proportional to the area of the (d-1) dim. boundary ∂A

$$S_A \sim \frac{\mathrm{Area}(\partial A)}{a^{d-1}} + (\text{subleading terms}),$$

[Bombelli-Koul-Lee-Sorkin 86', Srednicki 93']

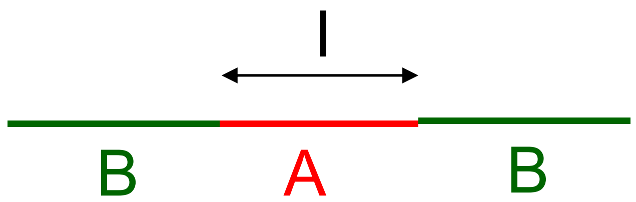
where a is a UV cutoff (i.e. lattice spacing).

Very similar to the Bekenstein-Hawking formula of black hole entropy

$$S_{BH} = \frac{\mathrm{Area}(\mathrm{horizon})}{4G_N}.$$

2D CFT (an exception of area law)

The entanglement entropy
in 2D CFT looks like



[Holzhey-Larsen-Wilczek 94' ; Calabrese-Cardy 04',
Recent review: Calabrese-Cardy arXiv:0905.4013]

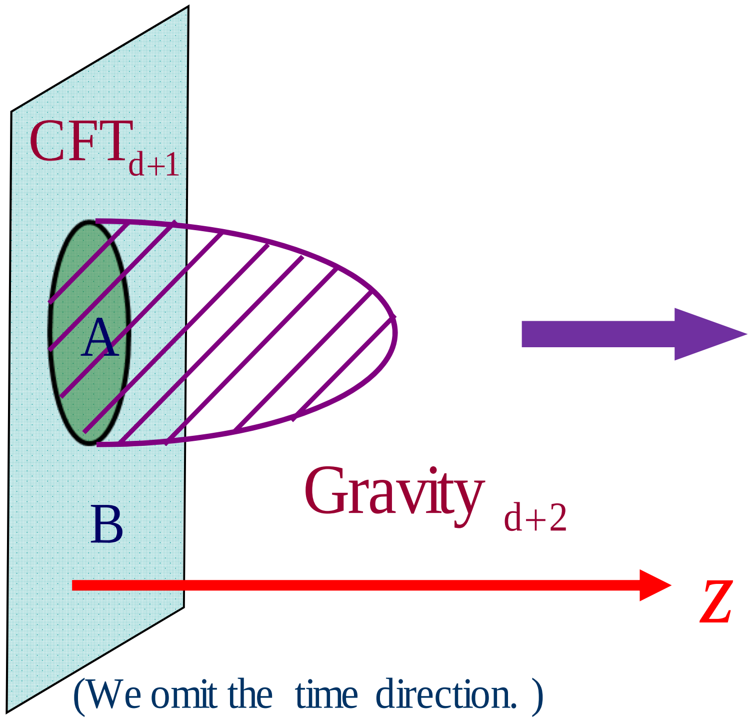
$$S_A = \frac{c}{3}\log\left(\frac{l}{a}\right) ,$$

where c is the central charge and a is UV cut off (lattice spacing).

(3-2) Holographic Entanglement Entropy

The holographic entanglement entropy S_A is given by the area of minimal surface whose boundary coincides with ∂A.

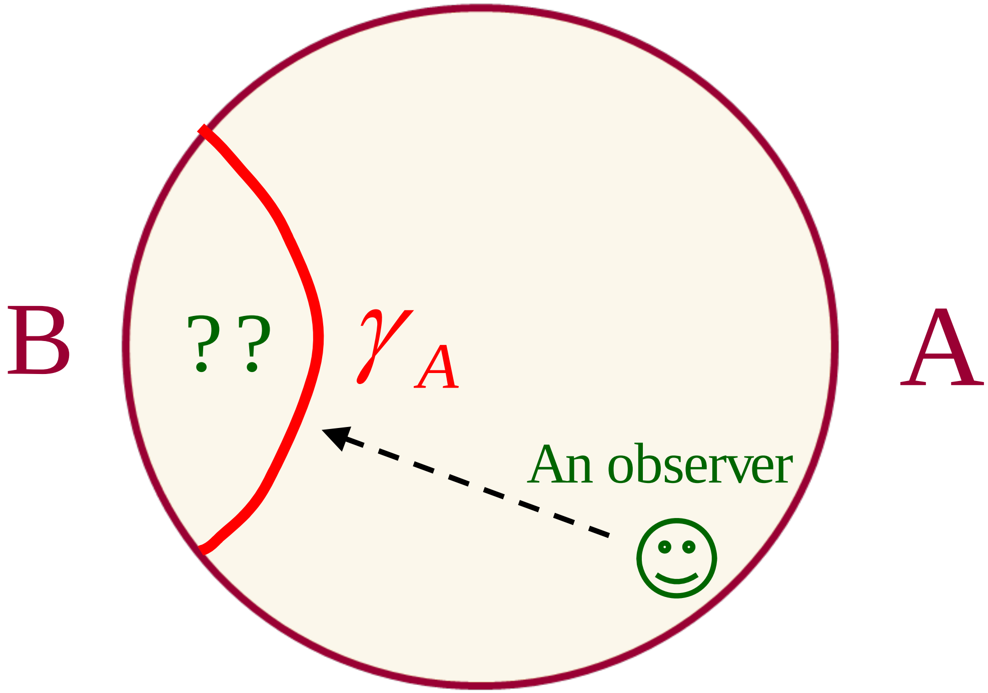
[Ryu-TT, hep-th/0603001, 0605073]



$$S_A = \frac{\mathrm{Area}(\gamma_A)}{4G_N}$$

(`Bekenstein-Hawking formula' if γ_A were the horizon.)

Heuristic Interpretation

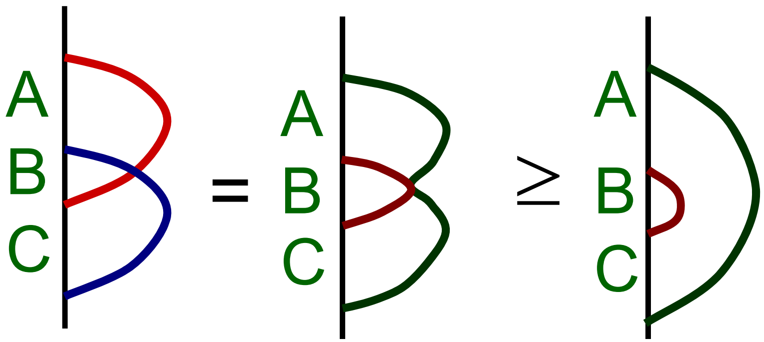


$\gamma_A \approx$ a sort of black hole for the observer

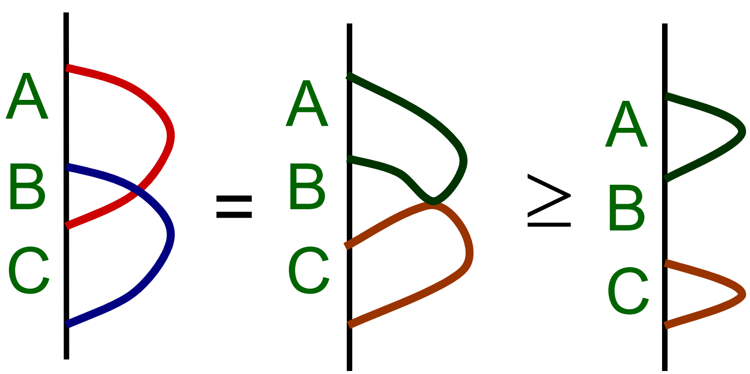
One Minute Proof of Strong Subadditivity [06' Hirata-TT, 07' Headrick-TT]

The strong subadditivity is known as the most important inequality satisfied by EE. [Lieb-Ruskai 73']

In AdS/CFT, this can be proved geometrically as follows



$$\Rightarrow S_{A+B}+S_{B+C}\geq S_{A+B+C}+S_{B}$$



$$\Rightarrow S_{A+B}+S_{B+C}\geq S_{A}+S_{C}$$

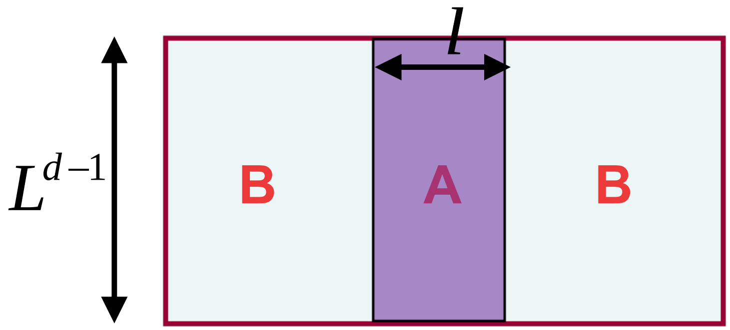
(3-3) Calculations of Holographic Entanglement Entropy

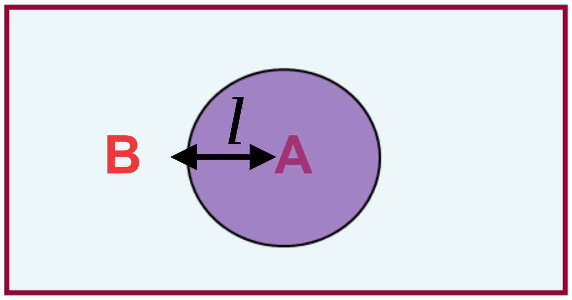
Since it is very complicated to compute EE in higher dim., the AdS/CFT provides a powerful analytical method for this purpose.

Two examples of the subsystem A:

(a) Infinite strip

(b) Circular disk





Entanglement Entropy for (b) Circular Disk from AdS

$$S_A = \frac{\pi^{d/2} R^d}{2G_N^{(d+2)}\Gamma(d/2)} \left[p_1 \left(\frac{l}{a}\right)^{d-1} + p_3 \left(\frac{l}{a}\right)^{d-3} + \cdots \right.$$

$$\left. \cdots + \begin{cases} p_{d-1}\left(\frac{l}{a}\right) + p_d & (\text{if } d = \text{even}) \\ p_{d-2}\left(\frac{l}{a}\right)^2 + q \log\left(\frac{l}{a}\right) & (\text{if } d = \text{odd}) \end{cases} \right],$$

Area law divergence

where $p_1 = (d-1)^{-1},\ p_3 = -(d-2)/[2(d-3)], \ldots.$

$\ldots.. \quad q = (-1)^{(d-1)/2}(d-2)!!/(d-1)!! \quad .$

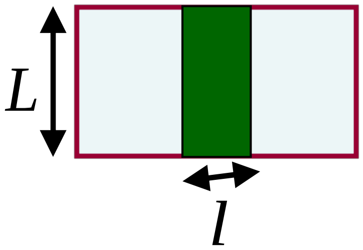
A universal quantity which characterizes odd dimensional CFT

⇒ Satisfy 'C-theorem' [Myers-Sinha 10']

Conformal Anomaly (~central charges)
In 2D, c/3 agrees with the CFT result.

Explicit example of AdS5/CFT4

$$\mathrm{AdS}_5 \times \mathrm{S}^5 \Leftrightarrow N = 4\ \mathrm{SU(N)\ SYM}$$



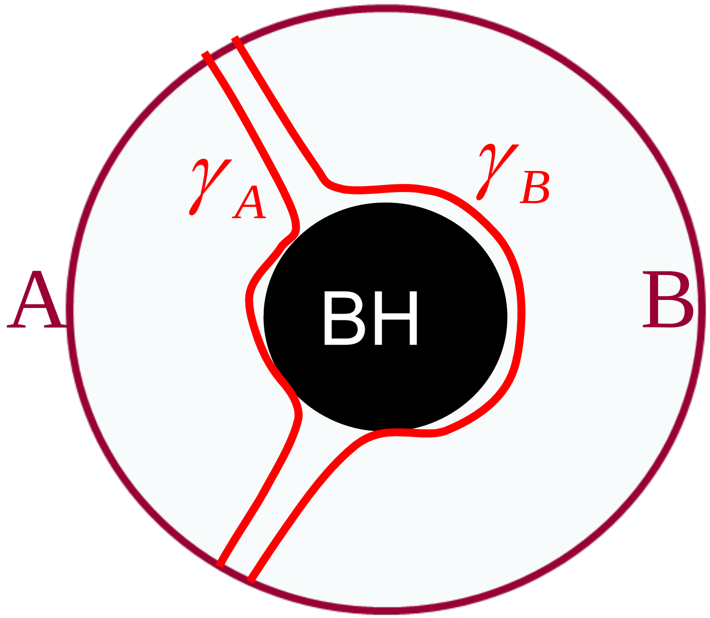
$$\text{Free CFT}: \quad S_A^{freeCFT} = K \cdot \frac{N^2 L^2}{a^2} - 0.087 \cdot \frac{N^2 L^2}{l^2}.$$

$$\text{Gravity}: \quad S_A^{AdS} = K' \cdot \frac{N^2 L^2}{a^2} - 0.051 \cdot \frac{N^2 L^2}{l^2}.$$

$$= \text{Strongly coupled CFT}$$

The order one deviation is expected since the AdS result corresponds to the strongly coupled Yang-Mills.

EE at Finite temp. from AdS BH



Minimal surfaces wrap the horizon.

$S_A \neq S_B$ *if* ρ_{tot} is not pure.

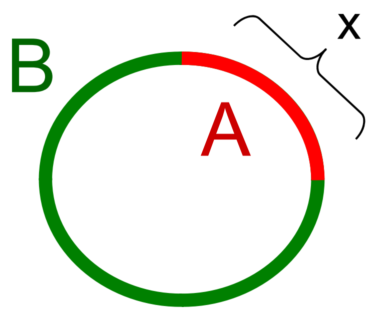
For example, the AdS3/CFT2 reproduces the known formula :

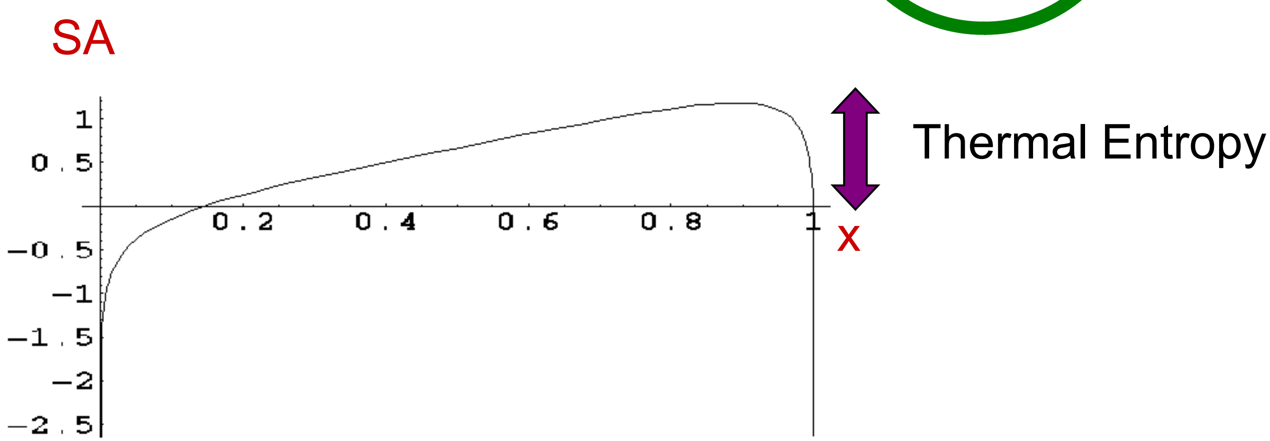
$$S_A = \frac{c}{3}\log\left(\frac{\beta}{a}\sinh\left(\frac{\pi l}{\beta}\right)\right).$$

cf. Finite Size System at Finite Temperature
(2D free fermion c=1) [07' Azeyanagi-Nishioka-TT]

$$S_A = \frac{1}{3}\log\left(\frac{\beta}{\pi a}\sinh\left(\frac{\pi x}{\beta}\right)\right) + \frac{1}{3}\sum_{i=1}^{\infty}\log\left[\frac{(1-e^{2\pi x/\beta}e^{-2\pi m/\beta})(1-e^{-2\pi x/\beta}e^{-2\pi m/\beta})}{(1-e^{-2\pi m/\beta})^2}\right]$$

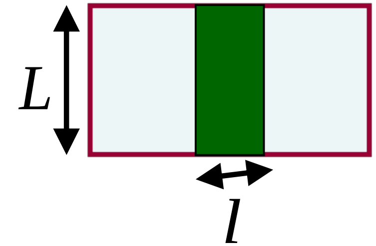
$$+2\sum_{m=1}^{\infty}\frac{(-1)^m}{m}\cdot\frac{\frac{\pi m x}{\beta}\cot\left(\frac{\pi m x}{\beta}\right)-1}{\sinh\left(\frac{\pi m}{\beta}\right)}\quad .$$



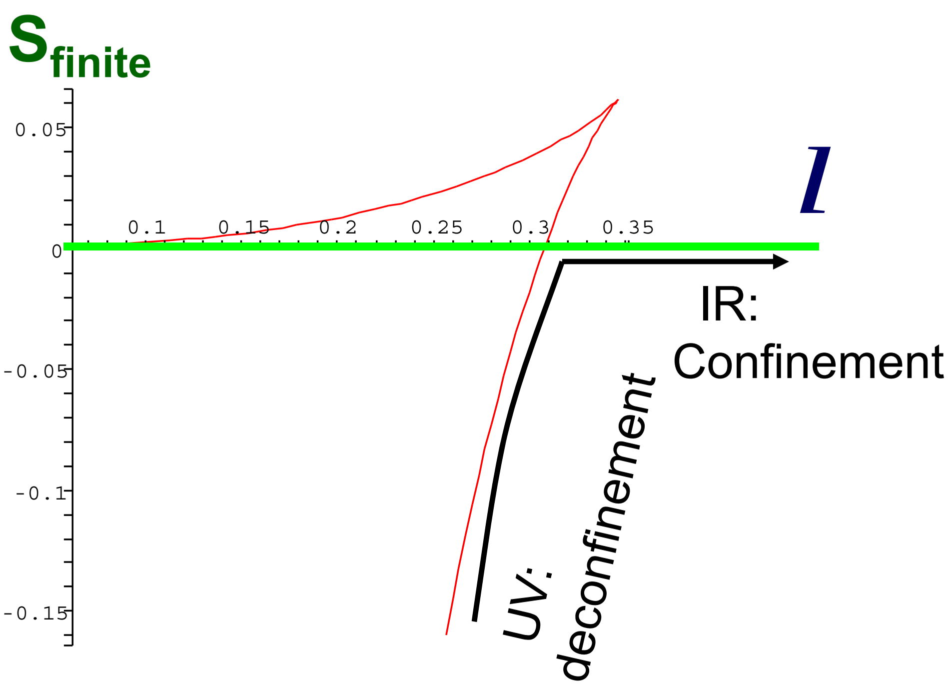


(3-4) EE and confinement/deconfinement phase transition

EE as an order parameter of phase transition.

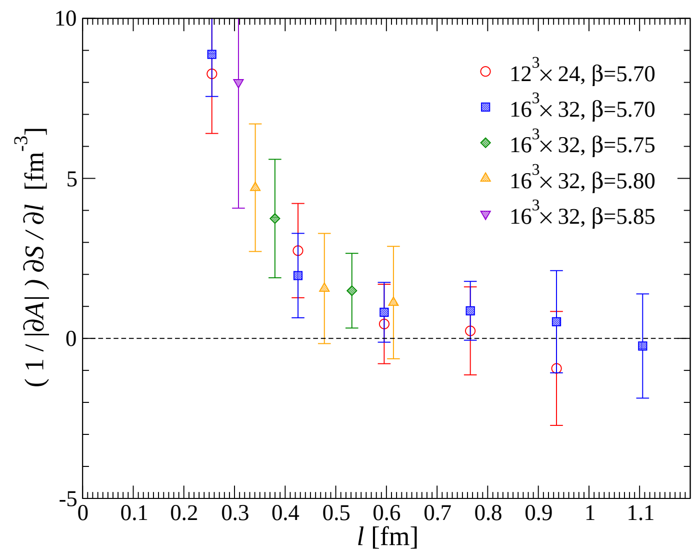


HEE of pure SU(N) YM



[Nishioka-TT 06', Klebanov-Kutasov-Murugan 07']

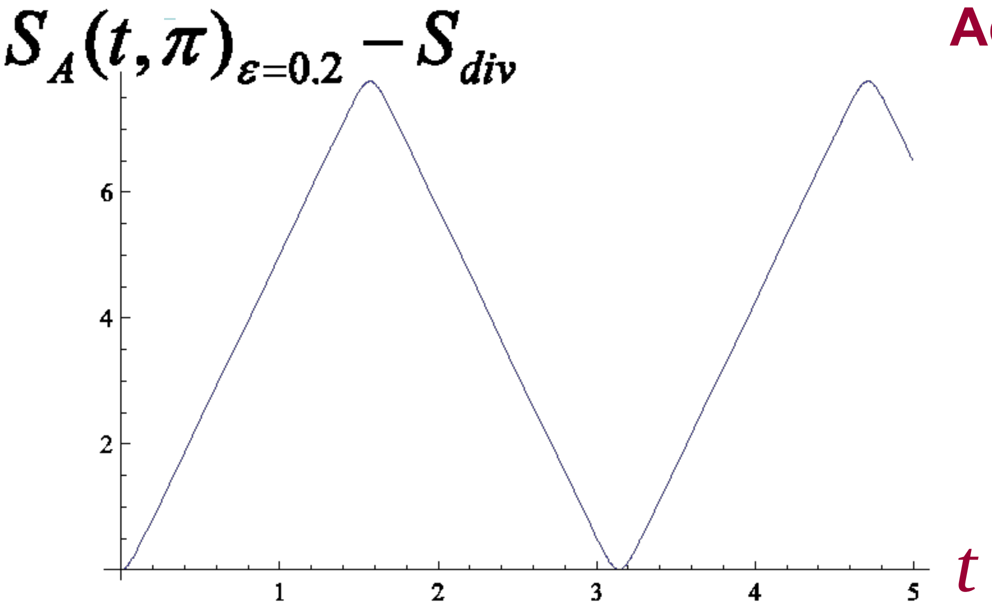
Lattice Result for pure YM



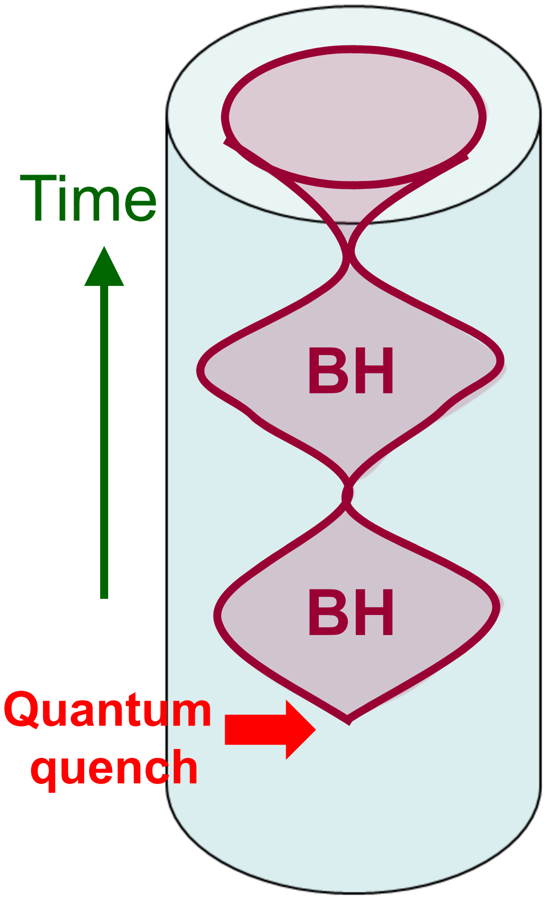
[4D SU(3), Nakagawa-Nakamura-Motoki-Zakharov 0911.2596]

(3-5) Relation to Black Hole Information Paradox [Ugajin-TT, 10']

Is a black hole a pure state
in spite of Hawking radiation ?



AdS/CFT ⟺



Evolution of EE in c=1 free fermion CFT under quantum quench ⇒ Pure state

BH formation and evaporation in extremely quantum gravity

No information paradox at all in either side !

④ Conclusions

- We gave an introductory review of the basics of AdS/CFT and recent progresses on condensed matter interpretation of AdS/CFT.

- Applications of AdS/CFT to condensed matter physics have been just initiated and we have to say that it is still in the stage of trial and error.

- However, the knowledge of condensed matter physics and quantum information theory seems to be very important for the deep understandings of quantum gravity.

- We presented a geometric calculation of entanglement entropy via AdS/CFT. This is especially powerful in higher dimensions, where field theory computations are very hard. This offers us a systematic viewpoint of entanglement entropy.

References(1): AdS/cond-mat

[AdS/CFT]

Reviews: Aharony-Gubser-Maldacena-Ooguri-Oz, hep-th/9905111
Klebanov, hep-th/9901018 (short review)

Original refs: Maldacena, hep-th/9711200; Gubser-Klebanov-Polyakov , hep-th/9802109; Witten, hep-th/9802150

[AdS/CMT]

Reviews: Hartnoll, 0903.3246; Sachdev, arXiv:1002.2947;
Herzog 0904.1975; Mcgreevy, 0909.0518

Quantum criticality and Nernst effect:
Herzog-Kovtun-Sachdev-Son, hep-th/0701036

Superconductor: Hartnoll-Herzog-Horowitz, 0803.3295,0810.1563,1002.1722

(Non) Fermi liquid: Liu-Mcgreevy-Vegh 0903.2477, 0907.2694,1003.1728

FQHE: Davis-Kraus-Shah 0809.1876, Fujita-Li-Ryu-TT 0901.0924

Viscosity in QGP: Policastro-Son-Starinets hep-th/0104066, 0704.0240

References(2): Entanglement Entropy

[Reviews]

EE in QFT: Calabrese-Cardy 0905.4013; Casini-Huerta 0905.2562

Holographic EE: Nishioka-Ryu-TT 0905.0932.

[Holographic Entanglement Entropy]

Original Ref. of HEE: Ryu-TT, hep-th/0603001, 0605073

Covariant Version: Hubeny-Rangamani-TT, 0705.0016

Strong Subadditivity: Headrick-TT, 0704.3719; Hirata-TT, hep-th/0608213

Log term: Lohmayer-Neuberger-Schwimmer-Theisen, 0911.4283
Caisni-Huerta, 1007.1813

Entropic C-theorem: Myers- Sinha, 1006.1263

Boundary Entropy: Azeyanagi-Karch-Thompson-TT, 0712.1850
Chiodaroli-Gutperle-Hung, 1005.4433

Massive Theory: Hertzberg-Wilczek, 1007.0993

Black holes: Ugajin-TT arXiv:1008.3439; Azeyanagi-Nishioka-TT arXiv:0710.2956

[EE in Confinement/deconfinement Phase Transition]

Original Ref: Nishioka-TT hep-th/0611035; Klebanov-Kutasov-Murugan, 0709.2140

Lattice Gauge Theory: Nakagawa-Nakamura-Motoki-Zakharov, 0911.2596

EE Gauge theory on compact spaces: Fujita-Nishioka-TT arXiv:0806.3118

Comment 1

To know the precise dual gauge theory, we need to consider *D-branes* in string theory.

However, gravity calculations in AdS/CFT predict that many quantities do not depend on details of strongly coupled CFTs.

→ To calculate physical quantities, we only need GR.

Comment 2

The SU(N) gauge sym. might be confusing from the viewpoint of strongly correlated electron systems.

We regard this SU(N) as an emergent gauge symmetry as is familiar in the slave particle method. We treat the U(1) ele-mag field as an external field independent from SU(N).

Boundary condition

EOM of a scalar field with mass m in AdS near the boundary z=0:

$$\partial_z^2\phi+\frac{1-d}{z}\partial_z\phi+\left(\partial_x^2-\frac{m^2}{z^2}\right)\phi=0,$$

$$\Rightarrow \phi(x,z)\approx\phi^{(0)}(x)\cdot z^{d-\Delta}+\phi^{(1)}(x)\cdot z^{\Delta}+\ldots\quad (z\to 0),$$

$$\Delta\equiv\frac{d}{2}\pm\sqrt{m^2+\frac{d^2}{4}}.$$

For each choice of $\phi^{(0)}(x)$, we can calculate $Z_{Gravity}(M)$ in a unique way assuming the smoothness of the solution.

$$\int dx^{d+1}\phi^{(0)}(x)O(x)\quad\Rightarrow\quad \Delta=\text{Conformal dim. of } O(x).$$

$$(\text{Remember } z=\text{length scale})$$