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Hidden Order and Dynamics of SUSY VBS models

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Refs: D. Arovas, KH, X-L.Qi and S-C. Zhang, Phys.Rev.B 79, 224404 (2009)
KH and KT, preprint (2010) and in preparation

Outline

- Introduction
- Valence-Bond Solid (VBS) model(s)
  - definition, matrix product & analogies
  - Hidden (string) order
- Supersymmetric VBS state
  - Holes in VBS states
  - Analogies (FQHE, BCS) & hidden order
  - Low-lying excitation (SMA)
  - Some generalizations
- Summary

Introduction

- **VBS model**
  ... a paradigmatic model for spin-gap ststems in 1D
- generalization to other groups: SU(n), SO(n), ...
  Tu-Zhang-Xiang '08, etc.
- What about adding "holes" to VBS model ?
  - ✓ a single spin-0 hole ... an exact eigenstate for a t-J-like model (S=1 VBS background) Zhang-Arovas '89
  - ✓ a single spin-1/2 hole ... a model for Y_2BaNiO_5 Penc-Shiba '95
- Model with many holes??
- Hidden order (robust against hole) ? Excitation?
- Possible future generalizations??

Valence-bond solid (VBS) states

Valence-bond solid models

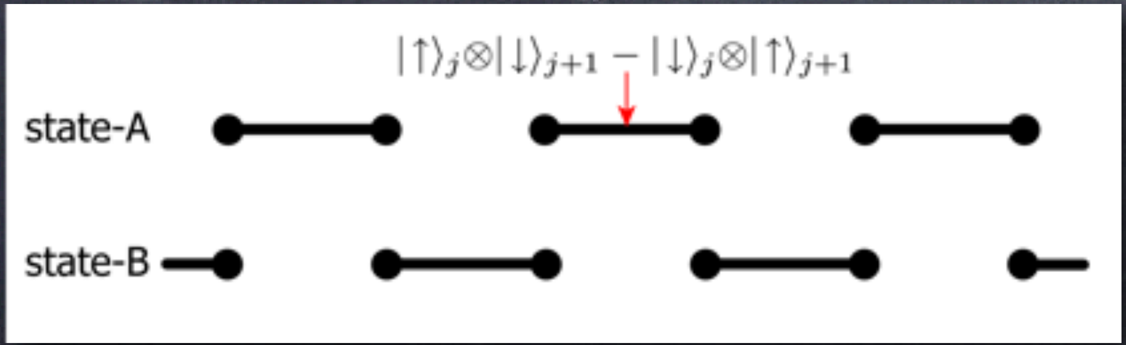
- Class of models for which short-range valence-bond crystals are (rigorous) ground states
  - Majumdar-Ghosh model (S=1/2 J1-J2 chain with J2/J1=1/2)
  - 2D Shastry-Sutherland model, and many more...
- Soluble models which realize Haldane's scenario:

Affleck et al '88, Arovas et al. '88

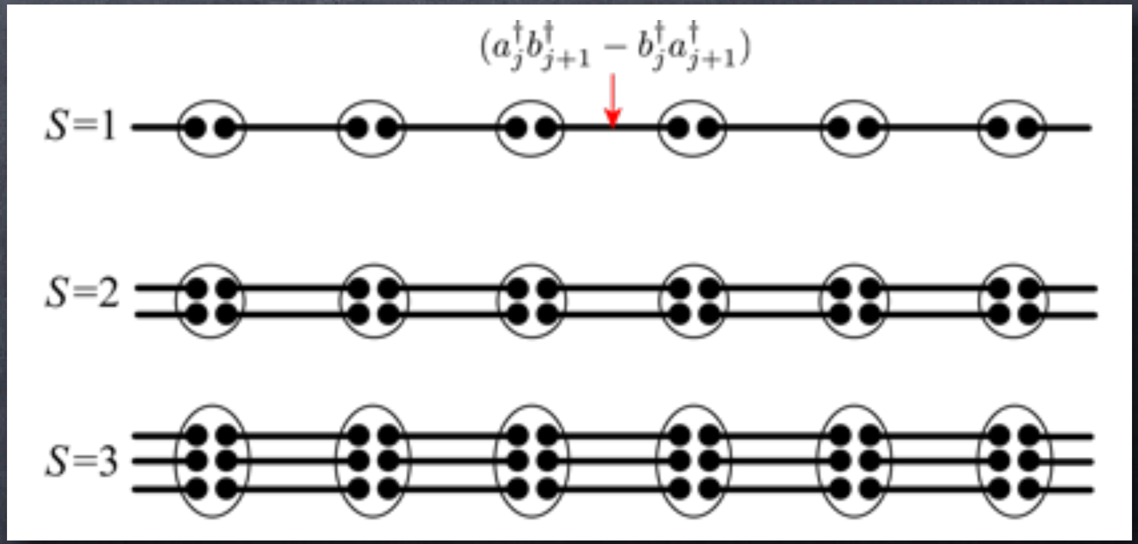
$$\mathcal{H}_{S=1} = \sum_i \left\{ \boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1} + \frac{1}{3}\left(\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1}\right)^2 + \frac{2}{3} \right\}$$

$$\mathcal{H}_{S=2} = \sum_i \left\{ \boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1} + \frac{2}{9}\left(\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1}\right)^2 + \frac{1}{63}\left(\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1}\right)^3 + \frac{10}{7} \right\}$$

$$\mathcal{H}_{S=3} = \sum_i \left\{ \boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1} + \frac{59}{358}\left(\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1}\right)^2 + \frac{19}{1611}\left(\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1}\right)^3 + \frac{1}{3222}\left(\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1}\right)^4 + \frac{396}{179} \right\}$$



Majumdar-Ghosh state: a "Kekule" structure

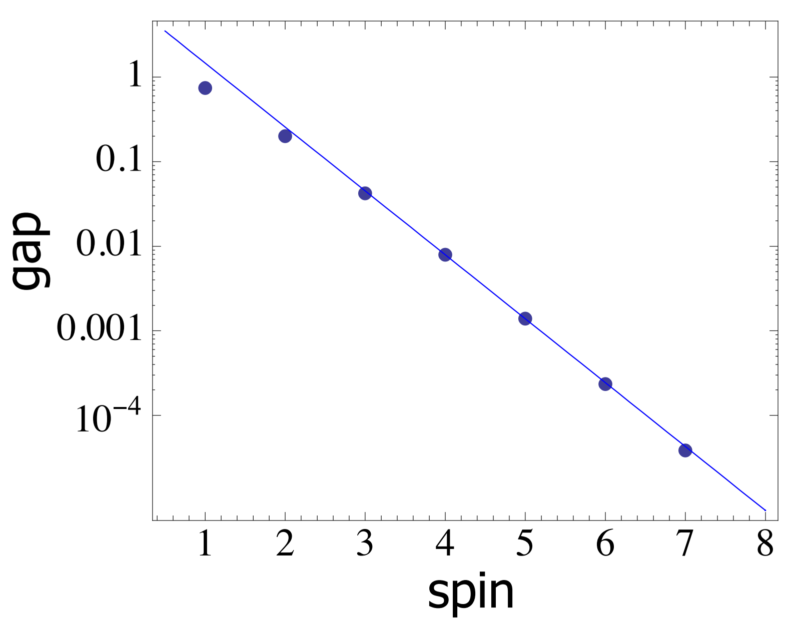


Valence-bond solid models

- (Many) important features:
  - ✓ short-range spin correlation: Arovas et al. '88, Freitag-Müller-Hartmann '91

$$\langle S_i^z S_j^z \rangle = \begin{cases} \frac{(S+1)^2}{3}\left(-\frac{S}{S+2}\right)^{|j-i|} & \text{for } i \neq j \\ \frac{S(S+1)}{3} & \text{for } i = j \end{cases}$$

  - ✓ "Haldane gap" to triplon excitation
  - ✓ edge states & (exact) degeneracy in a finite open system
  - ✓ Hidden order



Spin	Gap (SMA)	Approximation
1	$\frac{20}{27}$	$\to 7.41 \times 10^{-1}$
2	$\frac{1}{5}$	$\to 2.00 \times 10^{-1}$
3	$\frac{264}{6265}$	$\to 4.21 \times 10^{-2}$
4	$\frac{715}{90318}$	$\to 7.92 \times 10^{-3}$
5	$\frac{884}{634711}$	$\to 1.39 \times 10^{-3}$
6	$\frac{282625}{1203274644}$	$\to 2.35 \times 10^{-4}$

KT-Suzuki '95

Supersymmetric VBS states

A quick look at supersymmetry

... Schwinger boson construction of SU(2)

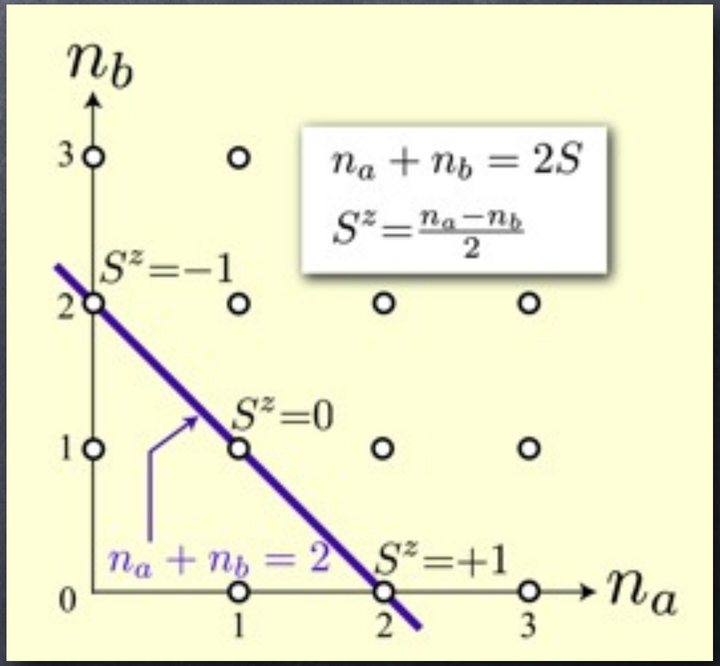
- Generators ("spin"): $\{\hat{S}^x \, , \, \hat{S}^y \, , \, \hat{S}^z\}$
- commutation relations:

$$[\hat{S}^a \, , \hat{S}^b] = i \, \epsilon_{abc} \, \hat{S}^c$$

- Schwinger-boson representation:

$$\hat{S}^+ = a^\dagger b \, , \; \hat{S}^- = b^\dagger a \, , \; \hat{S}^z = (a^\dagger a - b^\dagger b)/2 \;\; (a^\dagger a + b^\dagger b = 2S)$$

$$|S; m\rangle = \frac{1}{\sqrt{(S+m)!(S-m)!}} (a^\dagger)^{S+m} (b^\dagger)^{S-m} |0\rangle$$



- Label for irreps.
  = "boson number" = spin "S"

$$a^\dagger a + b^\dagger b = 2S$$

A quick look at supersymmetry

...Schwinger operator construction of UOSp(1|2)

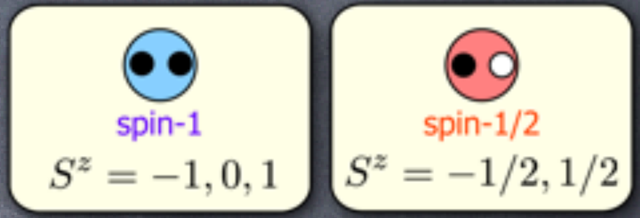
- 5 Generators & 3 Schwinger op. $(a^\dagger\,,\ b^\dagger\,,\ f^\dagger)$
  - ✓ Generators-(i) ("bosonic" = spin): $\{\hat{S}^x\,,\ \hat{S}^y\,,\ \hat{S}^z\}$
  - ✓ Generators-(ii) ("fermionic"):

$$K_1 {=} \frac{1}{2}(x^{-1} f a^\dagger + x f^\dagger b)\,,\ K_2 {=} \frac{1}{2}(x^{-1} f b^\dagger - x f^\dagger a)$$

- commutation relations:

$$[S^a\,,\,S^b] {=} i\,\epsilon^{abc} S^c\,,\ [S^a\,,\,K_\mu] = \frac{1}{2}(\sigma^a)_{\nu\mu} K_\nu$$

$$\{K_\mu\,,\,K_\nu\} = \frac{1}{2}(i\,\sigma^y \sigma^a)_{\mu\nu} S^a$$



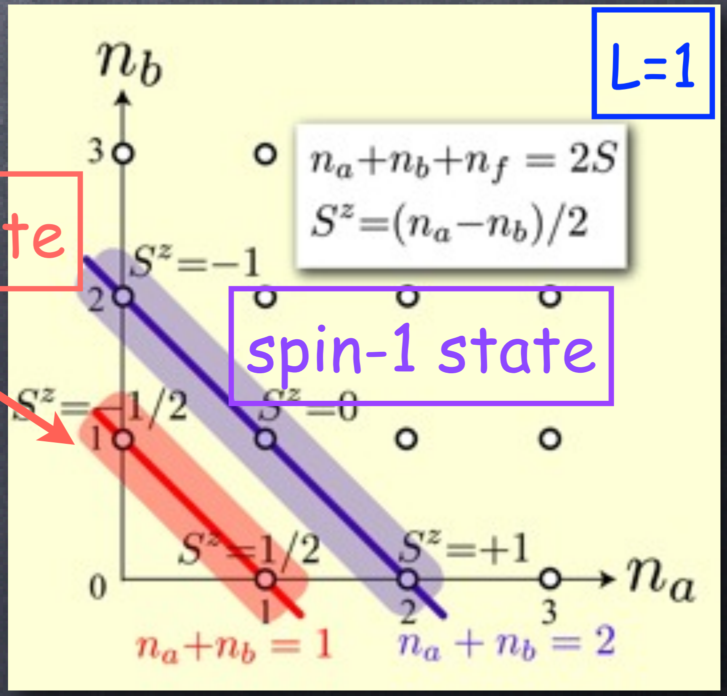
- Label of irreps.

$$n_a + n_b + n_f = 2L\ (\in \mathbb{Z})$$

  - two SU(2) irreps:

$$S = L\ (n_f = 0)\,,\quad S = L{-}1/2\ (n_f = 1)$$

1-hole (S=1/2) state



SUSY VBS state

... construction

- prepare two (super)spin-1/2s on each site:

(spin-↑ , spin-↓ , hole) ⇔ $(a^\dagger \, , \, b^\dagger \, , \, f^\dagger)$

☞ super-qubit... L=1/2

- "Physical" state @site = superspin-1
  - Schwinger-op rep. :

    spin-↑ spin-↓ hole
    $$a_j^\dagger a_j + b_j^\dagger b_j + f_j^\dagger f_j = 2L = 2$$
  - "spin-1" or "1-hole(S=1/2)"
- Form "entangled pairs":

$$\boxed{(a_j^\dagger b_{j+1}^\dagger - b_j^\dagger a_{j+1}^\dagger - r \, f_j^\dagger f_{j+1}^\dagger)}$$

- SUSY-VBS state:

Def $$|\mathrm{sVBS}\rangle = \prod_j (a_j^\dagger b_{j+1}^\dagger - b_j^\dagger a_{j+1}^\dagger - r \, f_j^\dagger f_{j+1}^\dagger)|0\rangle \quad (r \equiv x^2 \in \mathbb{R})$$

SUSY VBS state

... Physical picture

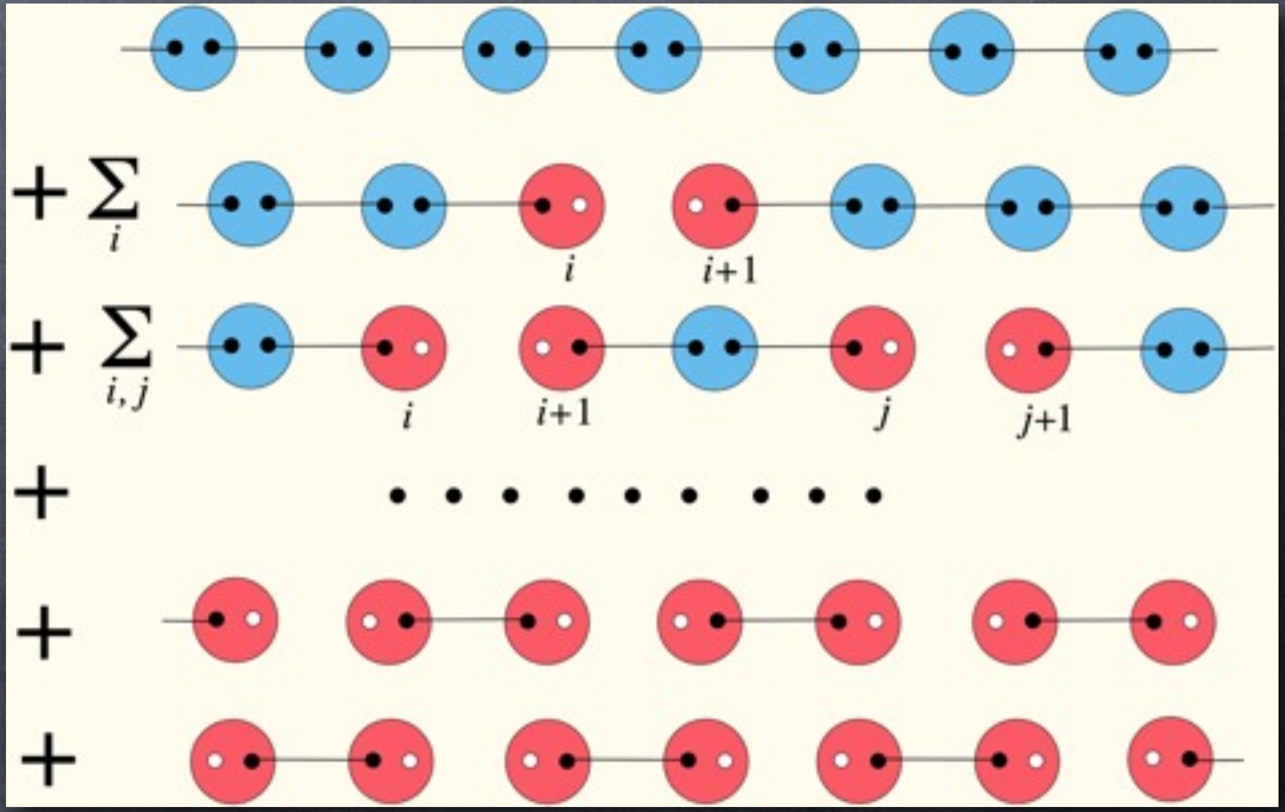
- Schwinger-op. rep. :

spin-↑ spin-↓ hole

$$a_j^\dagger a_j + b_j^\dagger b_j + f_j^\dagger f_j = 2L = 2$$

Def $$|\mathrm{sVBS}\rangle = \prod_j (a_j^\dagger b_{j+1}^\dagger - b_j^\dagger a_{j+1}^\dagger - r\, f_j^\dagger f_{j+1}^\dagger)|0\rangle \quad (r \equiv x^2 \in \mathbb{R})$$

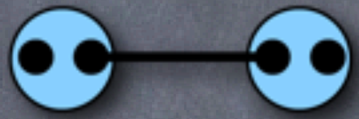
- expansion w.r.t. “r”:
  - lowest-order in “r”...
    S=1 VBS state
  - highest-order in “r”...
    S=1/2 Majumdar-Ghosh
  - r→0: VBS, r→∞: MG



SUSY VBS state

... Parent Hamiltonian

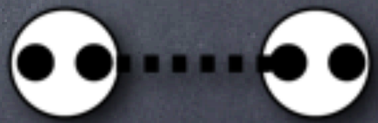
- usual VBS (S=1): cf) Klein model '82
  - valence-bond = "SU(2) singlet": $(a_j^{\dagger} b_{j+1}^{\dagger} - b_j^{\dagger} a_{j+1}^{\dagger})$
  - addition of angular-momenta:



$$\mathbf{1} \otimes \mathbf{1} \simeq \mathbf{0} \oplus \mathbf{1} \oplus \cancel{\mathbf{2}}$$

  - Projection operator: $h_{\text{local}} = V_2 \, \hat{P}_2 = \frac{V_2}{2} \left\{ \boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1} + \frac{1}{3} (\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1})^2 + \frac{2}{3} \right\}$

- SUSY VBS:
  - valence-bond = "UOSp(1|2) singlet": $(a_j^{\dagger} b_{j+1}^{\dagger} - b_j^{\dagger} a_{j+1}^{\dagger} - r \, f_j^{\dagger} f_{j+1}^{\dagger})$
  - addition of angular momenta:

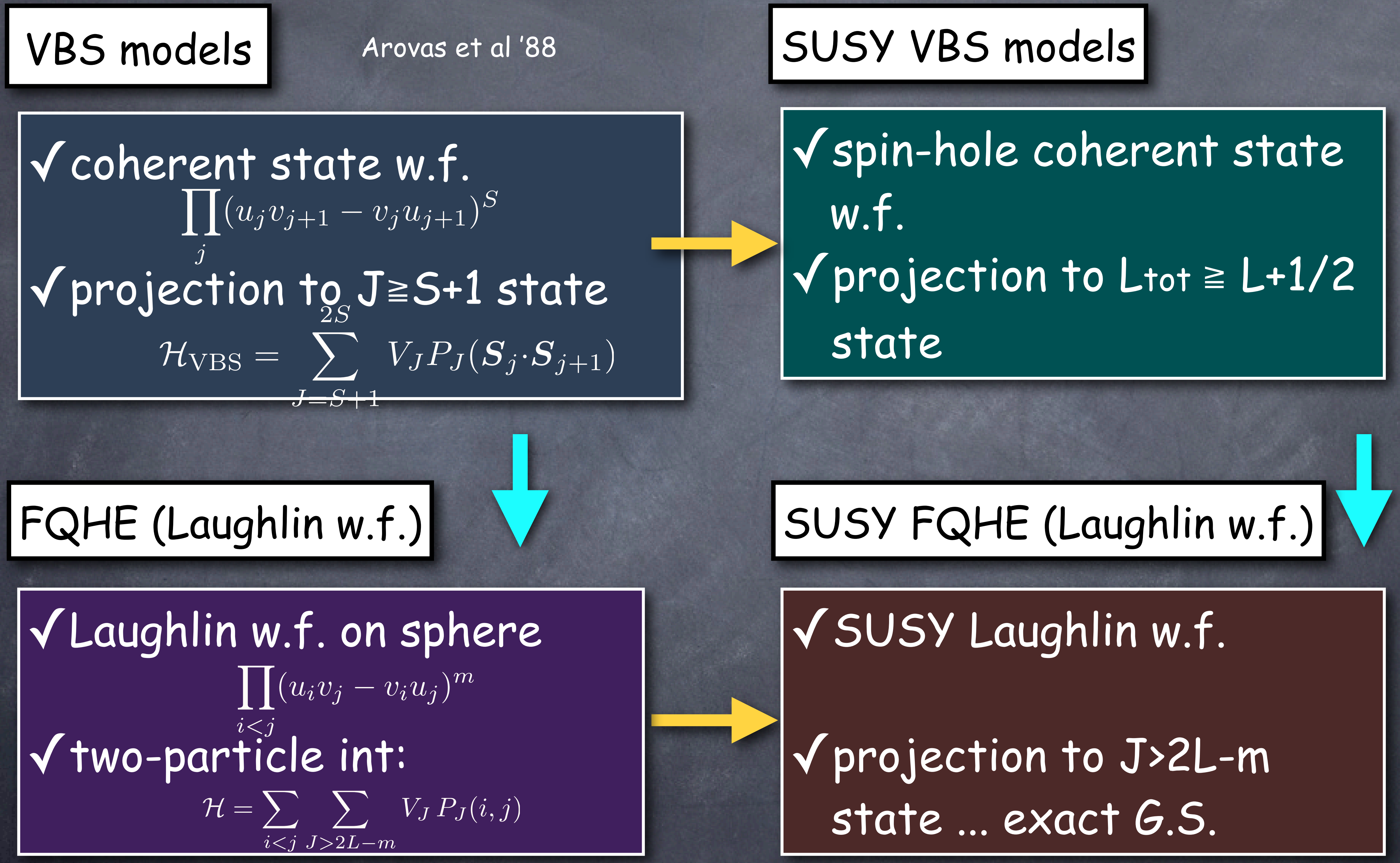


$$(L=1) \otimes (L=1) \simeq (0) \oplus (1/2) \oplus (1) \oplus (\cancel{3/2}) \oplus (\cancel{2})$$

  - Projection operator:

$$V_{3/2} (\hat{P}_{3/2}^{\dagger} \hat{P}_{3/2}) + V_2 (\hat{P}_2^{\dagger} \hat{P}_2) \quad (V_{3/2}, V_2 > 0)$$

Analogy to FQHE



VBS models

Arovas et al '88

- ✓coherent state w.f.

$$\prod_j (u_j v_{j+1} - v_j u_{j+1})^S$$

- ✓projection to J≧S+1 state

$$\mathcal{H}_{\text{VBS}} = \sum_{J=S+1}^{2S} V_J P_J(\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1})$$

SUSY VBS models

- ✓spin-hole coherent state w.f.
- ✓projection to L_{tot} ≧ L+1/2 state

FQHE (Laughlin w.f.)

- ✓Laughlin w.f. on sphere

$$\prod_{i<j} (u_i v_j - v_i u_j)^m$$

- ✓two-particle int:

$$\mathcal{H} = \sum_{i<j} \sum_{J>2L-m} V_J P_J(i,j)$$

Haldane '83

SUSY FQHE (Laughlin w.f.)

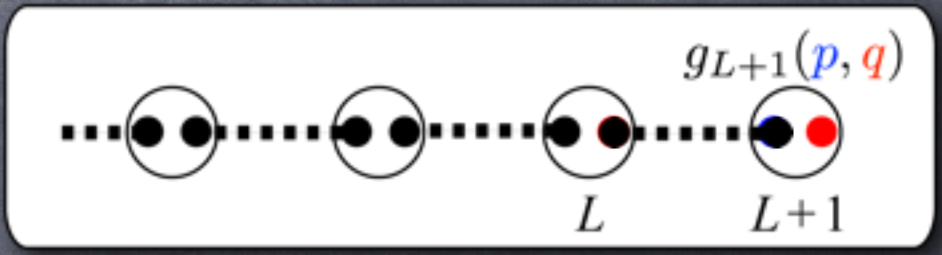
- ✓SUSY Laughlin w.f.
- ✓projection to J>2L-m state ... exact G.S.

Matrix-product representation

… SUSY VBS states

- Schwinger-op. rep. : $|\mathrm{sVBS}\rangle = \prod_j (a_j^\dagger b_{j+1}^\dagger - b_j^\dagger a_{j+1}^\dagger - r\, f_j^\dagger f_{j+1}^\dagger)|0\rangle$
- Matrix-product representation: Fannes et al. '92,93
  - size = number of (L/R) edge states (up, down & 1-hole...3)
  - KEY: "valence-bond insertion" = "matrix multiplication"

$$g_1 \otimes g_2 \otimes \cdots \otimes g_L \otimes g_{L+1}$$



$$g_j = \begin{array}{cccc} & \downarrow & \uparrow & \text{hole} \quad \Leftarrow\text{Right} \\ \uparrow & -a_j^\dagger b_j^\dagger & -(a_j^\dagger)^2 & -\sqrt{r} a_j^\dagger f_j^\dagger \\ \downarrow & (b_j^\dagger)^2 & a_j^\dagger b_j^\dagger & \sqrt{r} b_j^\dagger f_j^\dagger \\ \text{hole} & -\sqrt{r} f_j^\dagger b_j^\dagger & -\sqrt{r} f_j^\dagger a_j^\dagger & 0 \\ \Uparrow\text{Left} & & & \end{array} \qquad g_j^\dagger = \begin{pmatrix} -a_j b_j & (b_j)^2 & -\sqrt{r} b_j f_j \\ -(a_j)^2 & a_j b_j & -\sqrt{r} a_j f_j \\ -\sqrt{r} f_j a_j & \sqrt{r} f_j b_j & 0 \end{pmatrix}$$

$$|\mathrm{sVBS}\rangle = \begin{cases} g_1 \otimes \cdots \otimes g_j \otimes \cdots g_L & \text{. . . for open chain} \\ \mathrm{STr}\,[g_1 \otimes \cdots \otimes g_j \otimes \cdots g_L] & \text{. . . for periodic chain} \end{cases}$$

Correlation functions (i)

…spin-spin correlation functions

- "transfer-matrix" formalism: Klümper et al '92, KT-Suzuki '94-95, etc..

$$\langle \mathrm{sVBS}|\mathrm{sVBS}\rangle = G^L\,,$$

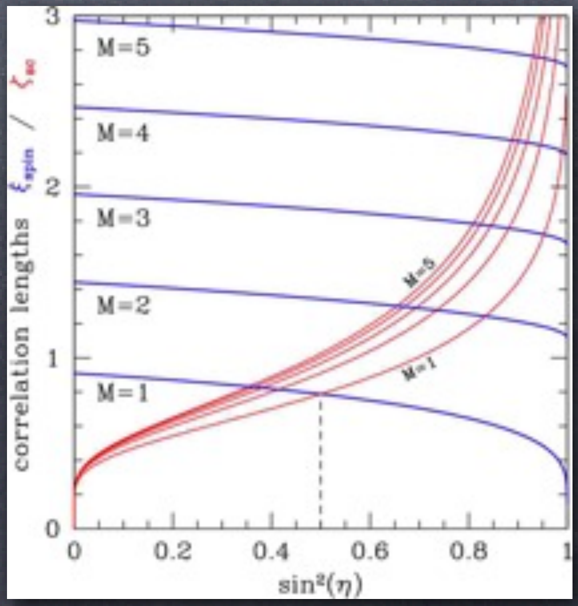
$$\langle \mathrm{sVBS}|\hat{\mathcal{O}}^A_x\hat{\mathcal{O}}^B_y|\mathrm{sVBS}\rangle = G^{x-1}G(\mathcal{O}^A)G^{y-x-1}G(\mathcal{O}^B)G^{L-y}$$

$$G_{\bar{a},a;\bar{b},b} \equiv g^*(\bar{a},\bar{b})g(a,b)\,,\ G(\mathcal{O}^A)_{\bar{a},a;\bar{b},b} \equiv g^*(\bar{a},\bar{b})\hat{\mathcal{O}}^A\,g(a,b)$$

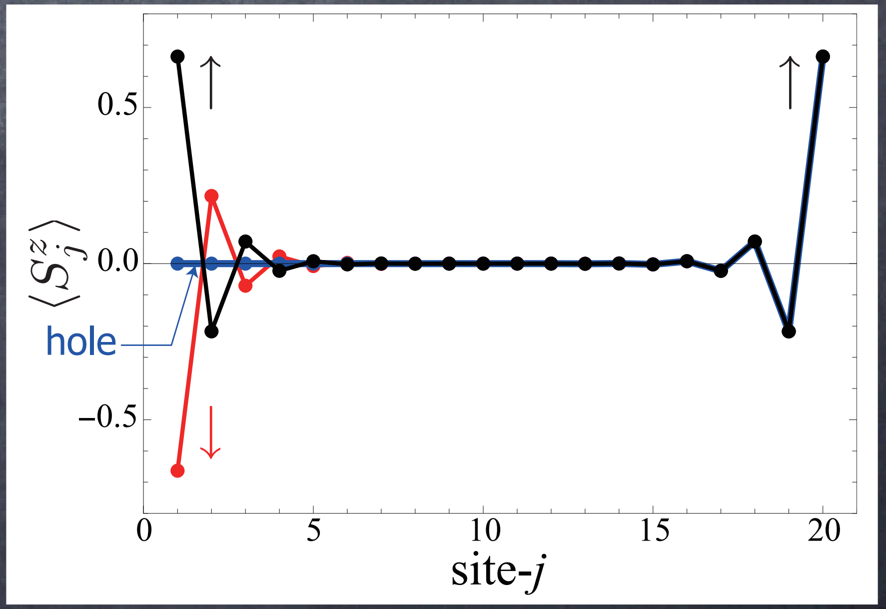
- spin-spin correlation function: Arovas et al. '09

$$\langle S_i^\alpha S_j^\alpha\rangle = \begin{cases} \frac{2\left(r^2+\sqrt{8r^2+9}+3\right)}{\sqrt{8r^2+9}\left(\sqrt{8r^2+9}+3\right)} & \text{for } i=j \\ \frac{13r^2+24+\left(r^2+8\right)\sqrt{8r^2+9}}{2\sqrt{8r^2+9}\left(\sqrt{8r^2+9}+3\right)}\left\{-\frac{2}{\sqrt{8r^2+9}+3}\right\}^{|j-i|} & \text{for } i\neq j \end{cases}$$

  - Correlation length:
  - 9 edge states ((2L+1)^2, in general)



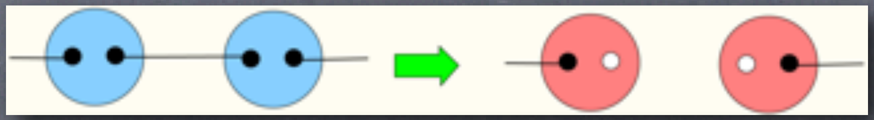
"edge states"



Correlation functions (ii)

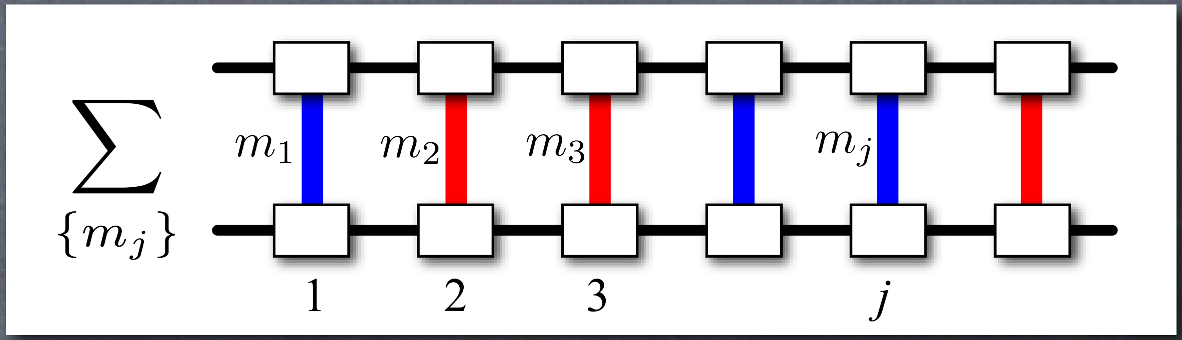
…"superconducting" correlation

- Possible "hole pairing"…



$$(a_j b_{j+1} - b_j a_{j+1}) f_j^\dagger f_{j+1}^\dagger$$

- Need to modify "transfer-matrix" formalism:



$$\langle \mathrm{sVBS} | \mathrm{sVBS} \rangle = G^L \,,$$

$$\langle \mathrm{sVBS} | \hat{\mathcal{O}}_x^A \hat{\mathcal{O}}_y^B | \mathrm{sVBS} \rangle = G^{x-1} G(\mathcal{O}^A) \widetilde{G}^{y-x-1} G(\mathcal{O}^B) G^{L-y}$$

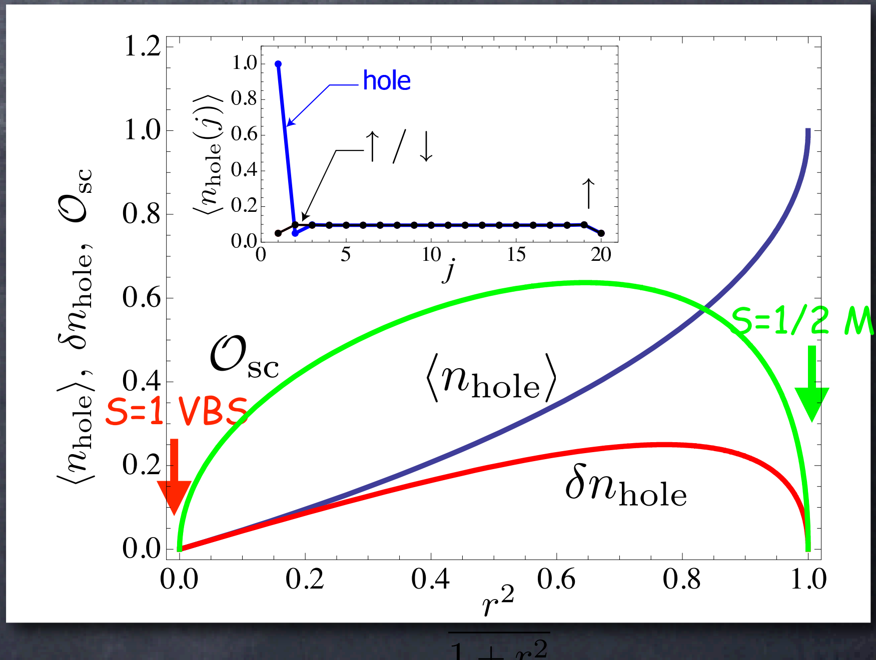
$$\widetilde{G}_{\bar{a},a;\bar{b},b} \equiv g^*(\bar{a},\bar{b})(-1)^F g(a,b) \,, \; G(\mathcal{O}^A)_{\bar{a},a;\bar{b},b} \equiv g^*(\bar{a},\bar{b}) \hat{\mathcal{O}}^A g(a,b)$$

- Order parameter for "hole pairing":

Arovas et al. '09

$$\langle \Delta_j \rangle = \langle (a_j b_{j+1} - b_j a_{j+1}) f_j^\dagger f_{j+1}^\dagger \rangle$$

$$= \frac{144 r \left(r^2 + 3 + \sqrt{8r^2+9} \right)}{\sqrt{8r^2+9} \left(3 + \sqrt{8r^2+9} \right)^2 \left(5 + \sqrt{8r^2+9} \right)}$$



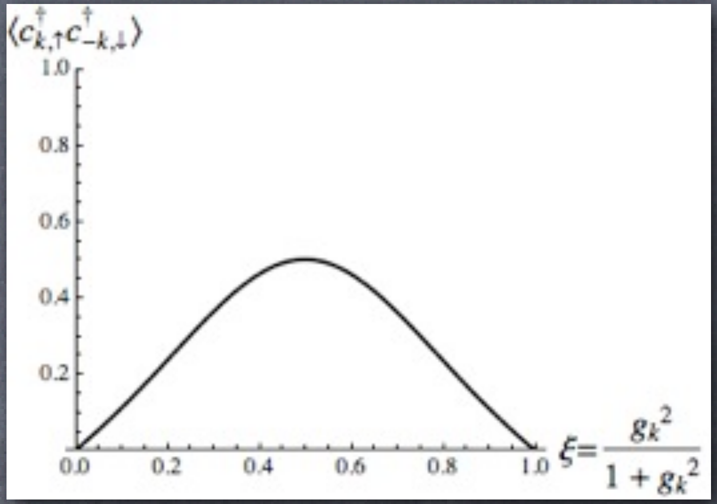
Analogy ... BCS wavefunction

- BCS wavefunction:

$$|\Psi_{\mathrm{BCS}}\rangle = \prod_{\boldsymbol{k}} (1 + g_{\boldsymbol{k}} c^{\dagger}_{\boldsymbol{k},\uparrow} c^{\dagger}_{-\boldsymbol{k},\downarrow})|0\rangle$$

electron pair

  - g_k=0: (trivial) vacuum
  - 0< g_k <∞: superconducting
  - g_k=∞: filled Fermi sphere (normal state)

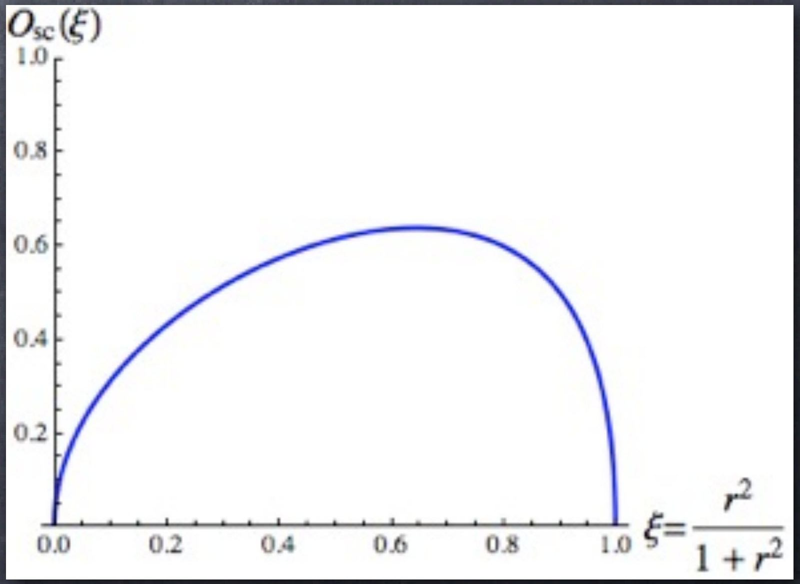


- SUSY VBS state:

$$|\mathrm{sVBS}\rangle = \prod_{j} (a^{\dagger}_{j} b^{\dagger}_{j+1} - b^{\dagger}_{j} a^{\dagger}_{j+1} - r\, f^{\dagger}_{j} f^{\dagger}_{j+1})|0\rangle \quad (r \equiv x^2 \in \mathbb{R})$$

hole pair

  - "vacuum" = spin-1 VBS state
  - r=0: pure VBS state
  - 0<r<∞: "hole-pair condensate"
  - r=∞: Majumdar-Ghosh state

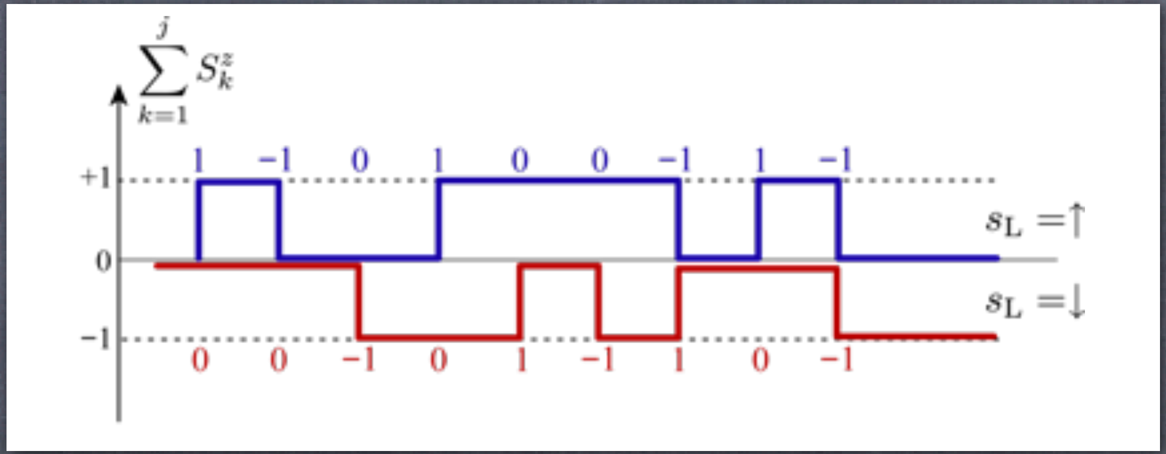


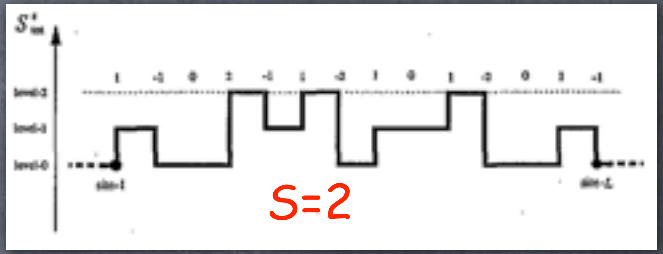
Hidden Order

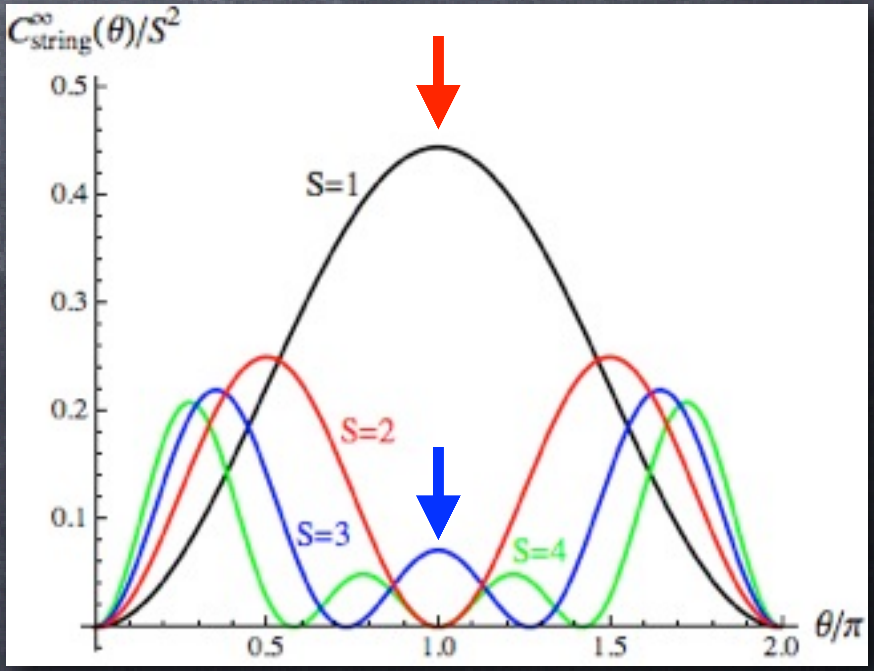
... (usual) VBS state

- Order in VBS state:
  - No ordinary (local) magnetic LRO, no dimerization...
  - But there is "hidden" order ("string order")

den Nijs-Rommelse '89, Tasaki '91
Pérez-García et al '08







Oshikawa '92, KT-Suzuki '95

- How to detect ?

cf) Gu-Wen '09

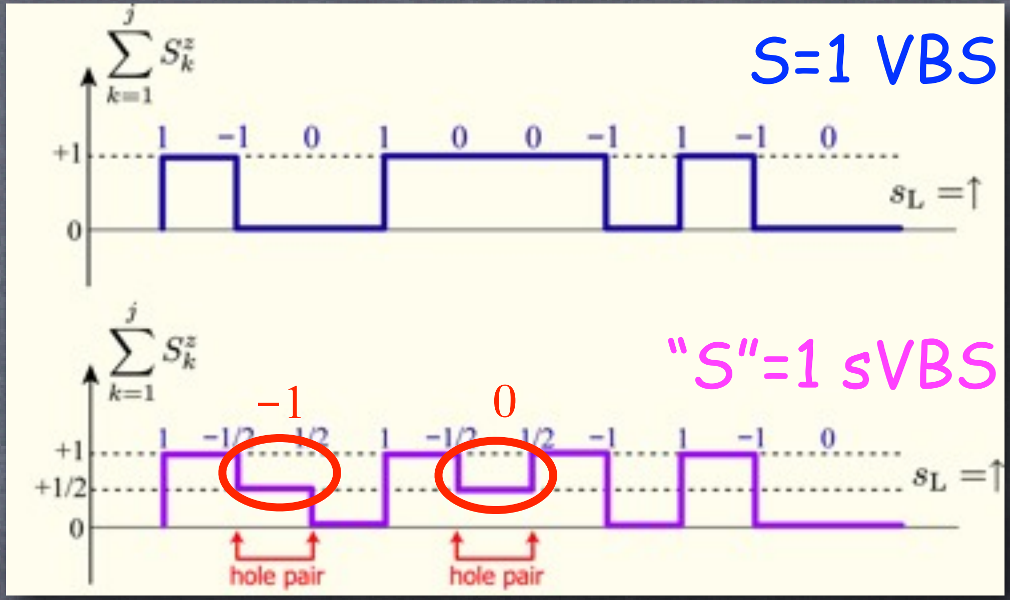
  - Néel order: $S^z_j (-1)^n S^z_{j+n}$ ⇒ ferro
  - VBS: Girvin-Arovas '90

$$\mathcal{O}^z_{\rm string} = \lim_{n\nearrow\infty} \langle \mathrm{sVBS}| S_j \exp\left[i\pi \sum_{k=j}^{j+n-1} S^z_k \right] S^z_{j+n} |\mathrm{sVBS}\rangle$$

Hidden Order

... SUSY VBS state

- "string order" in SUSY VBS ?
  - matrix-product rep. ⇒ "almost" flat surface



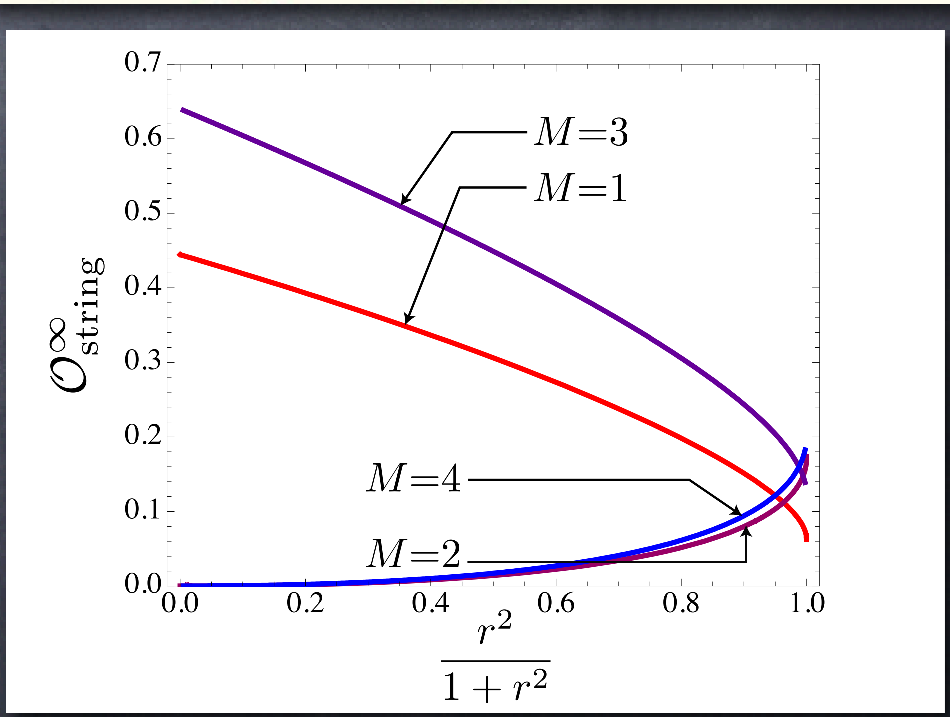
$$g_j = \begin{pmatrix} -a_j^\dagger b_j^\dagger & -(a_j^\dagger)^2 & -\sqrt{r} a_j^\dagger f_j^\dagger \\ (b_j^\dagger)^2 & a_j^\dagger b_j^\dagger & \sqrt{r} b_j^\dagger f_j^\dagger \\ -\sqrt{r} f_j^\dagger b_j^\dagger & -\sqrt{r} f_j^\dagger a_j^\dagger & 0 \end{pmatrix}$$

$$\bigotimes_{j=1}^{k} g_j = \begin{pmatrix} |S_{\text{tot}}^z=0\rangle & |S_{\text{tot}}^z=+1\rangle & |S_{\text{tot}}^z=+1/2\rangle \\ |S_{\text{tot}}^z=-1\rangle & |S_{\text{tot}}^z=0\rangle & |S_{\text{tot}}^z=-1/2\rangle \\ |S_{\text{tot}}^z=-1/2\rangle & |S_{\text{tot}}^z=+1/2\rangle & |S_{\text{tot}}^z=0\rangle \end{pmatrix}$$

$$\left(S_{\text{tot}}^z \equiv \sum_{j=1}^{k} S_j^z \right)$$

- string order parameter captures hidden order...

$$\boxed{\mathcal{O}_{\text{string}}^z = \langle \text{sVBS} | S_j \exp\left[i\pi \sum_{k=j}^{j+n-1} S_k^z \right] S_{j+n}^z | \text{sVBS} \rangle}$$



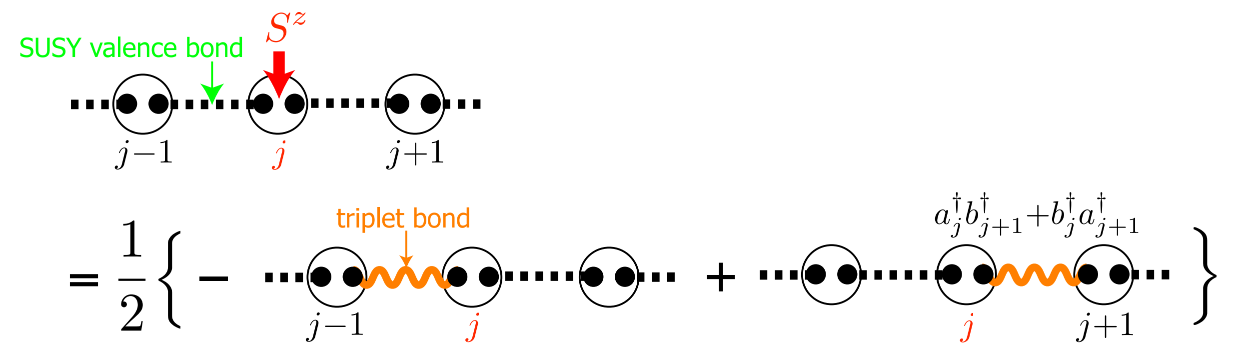
Low-lying excitations …"spin"

- Failure of "Lieb-Schultz-Mattis" like excitation
  = want for alternative way to physical excitations

- triplon = "crackion"

Fáth-Sólyom '93, KT-Suzuki '94-95, Rommer-Östlund '95,'97

$$S_j^z|\mathrm{sVBS}\rangle = \frac{1}{2}\left\{-|\psi_{j-1}^{(0)}\rangle + |\psi_j^{(0)}\rangle\right\}$$

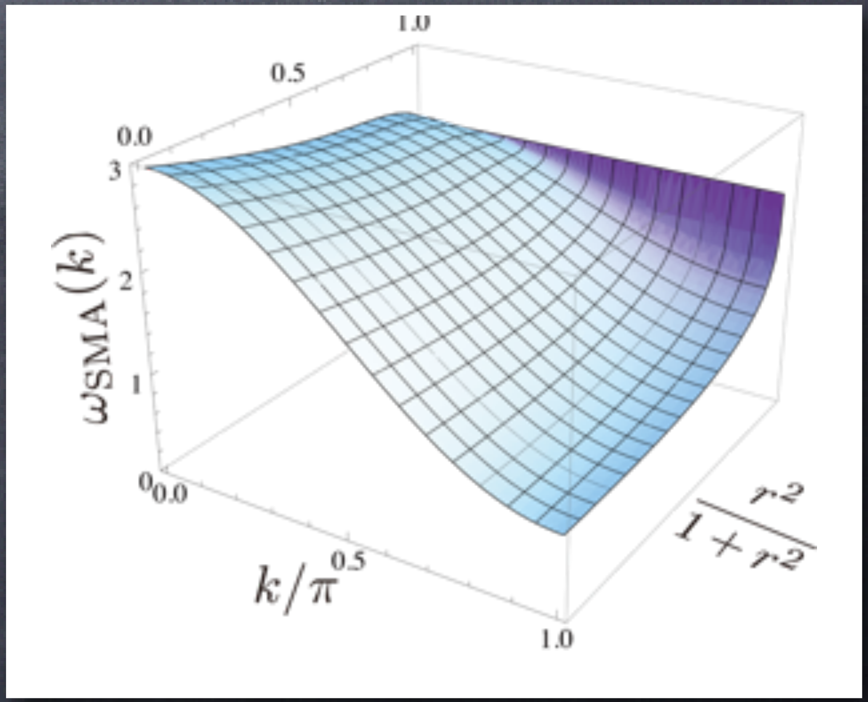


- single-mode approximation:

$$\omega_{\mathrm{SMA}}^z(k) = -\frac{1}{2}\frac{\langle\mathrm{sVBS}|\,[[\mathcal{H}, S^z(k)], S^z(-k)]\,|\mathrm{sVBS}\rangle}{\langle\mathrm{sVBS}|S^z(k)S^z(-k)|\mathrm{sVBS}\rangle}$$

$$= \frac{\langle\psi^{(0)}(k)|\mathcal{H}|\psi^{(0)}(k)\rangle}{\langle\psi^{(0)}(k)|\psi^{(0)}(k)\rangle}$$

$$= \frac{1}{2}\frac{(1-\cos k)}{S^{zz}(k)}\langle\psi_x^{(0)}|h_{x,x+1}|\psi_x^{(0)}\rangle$$



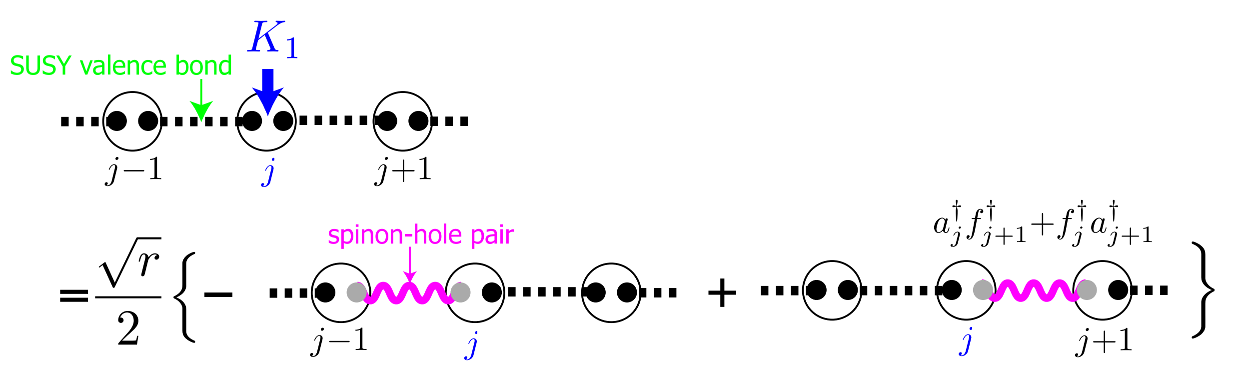
Low-lying excitations ..."hole"

- In SUSY, we have two more (fermionic) generators

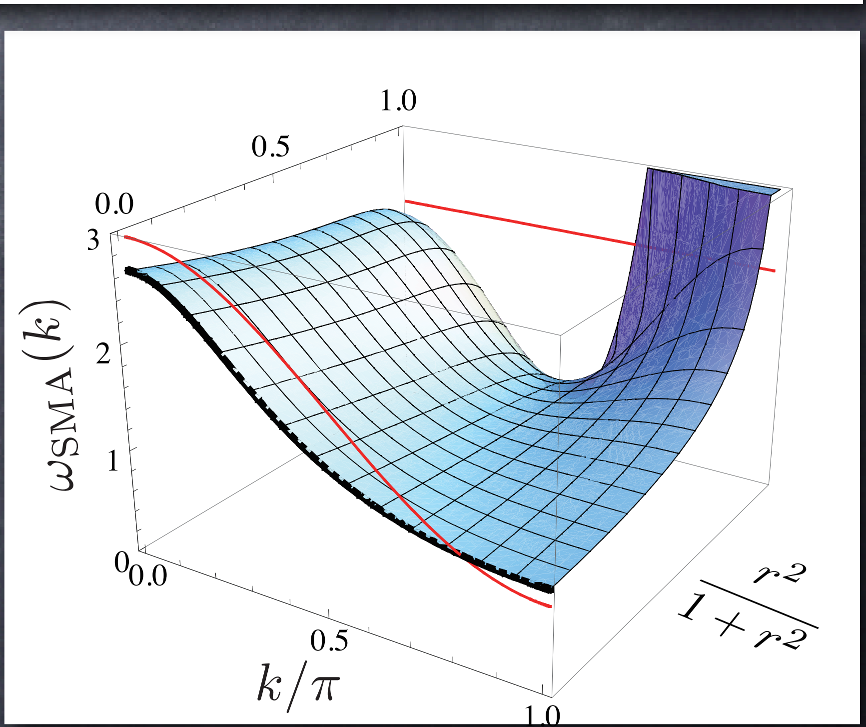
$$K_1(j) = \frac{1}{2}\left(\frac{1}{\sqrt{r}}a_j^\dagger f_j + \sqrt{r} f_j^\dagger b_j\right) , \; K_2(j) = \frac{1}{2}\left(\frac{1}{\sqrt{r}}b_j^\dagger f_j - \sqrt{r} f_j^\dagger a_j\right)$$

- "charege-crackion" = "spinon-hole pair"

$$K_{1,j}|\mathrm{sVBS}\rangle = \frac{1}{2}\sqrt{r}\left\{|\psi_{j-1}^{(1/2)}\rangle - |\psi_{j}^{(1/2)}\rangle\right\}$$



- can be treated by MPS, too
- lack of unitary operation connecting "spin" and "charge" ➡ different spectra ...



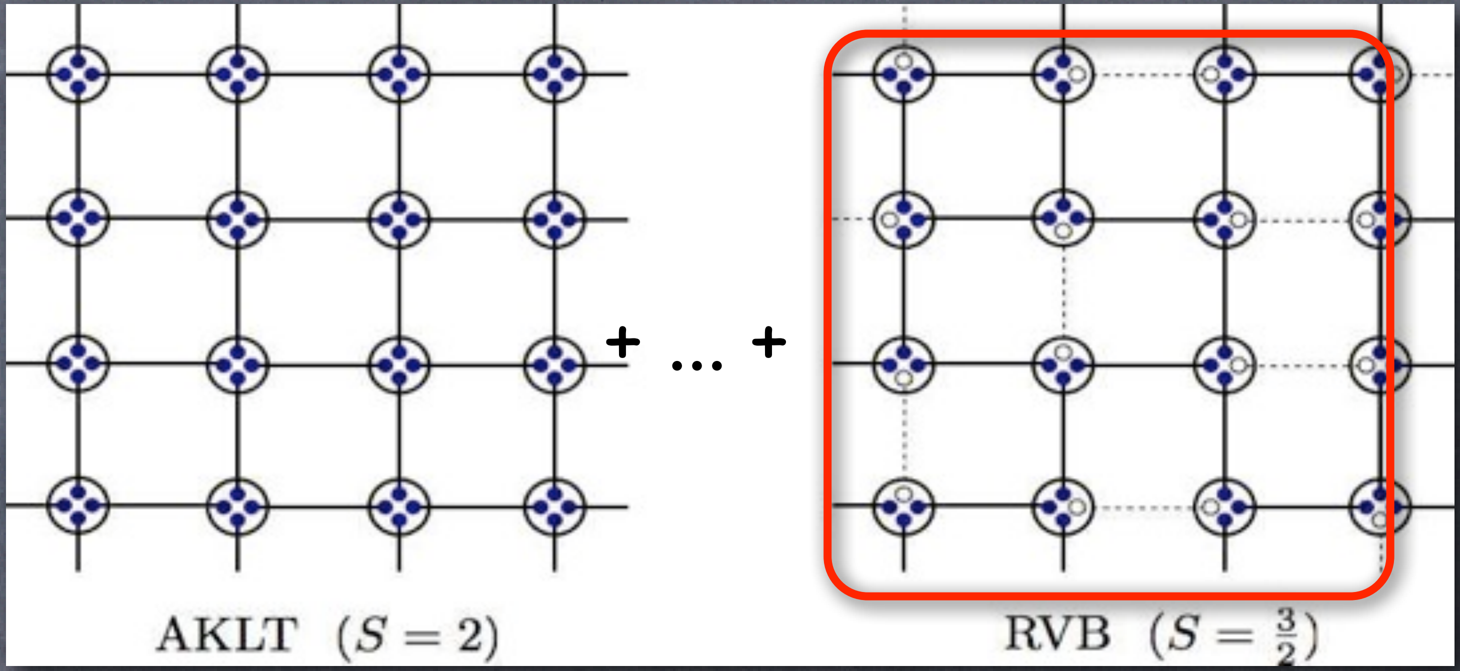
Generalizations

On Square Lattice...

- model-I: 1-species of fermion

$$|\Psi^{\text{2D-(I)}}_{\text{sVBS}}\rangle = \prod_{\langle i,j\rangle}(a_i^\dagger b_j^\dagger - b_i^\dagger a_j^\dagger - r\, f_i^\dagger f_j^\dagger)|0\rangle$$

$$=$$



- model-II: 3-species of fermion

$$|\Psi^{\text{2D-(II)}}_{\text{sVBS}}\rangle = \prod_{\langle i,j\rangle}(a_i^\dagger b_j^\dagger - b_i^\dagger a_j^\dagger - r_1\, f_i^\dagger f_j^\dagger - r_3\, g_i^\dagger g_j^\dagger - r_3\, h_i^\dagger h_j^\dagger)|0\rangle$$

$\xrightarrow{r_1,r_2,r_3\to\infty} \sum$ all possible dimer coverings

Rokhsar-Kivelson RVB liquid

Rokhsar-Kivelson '88

More fermions...

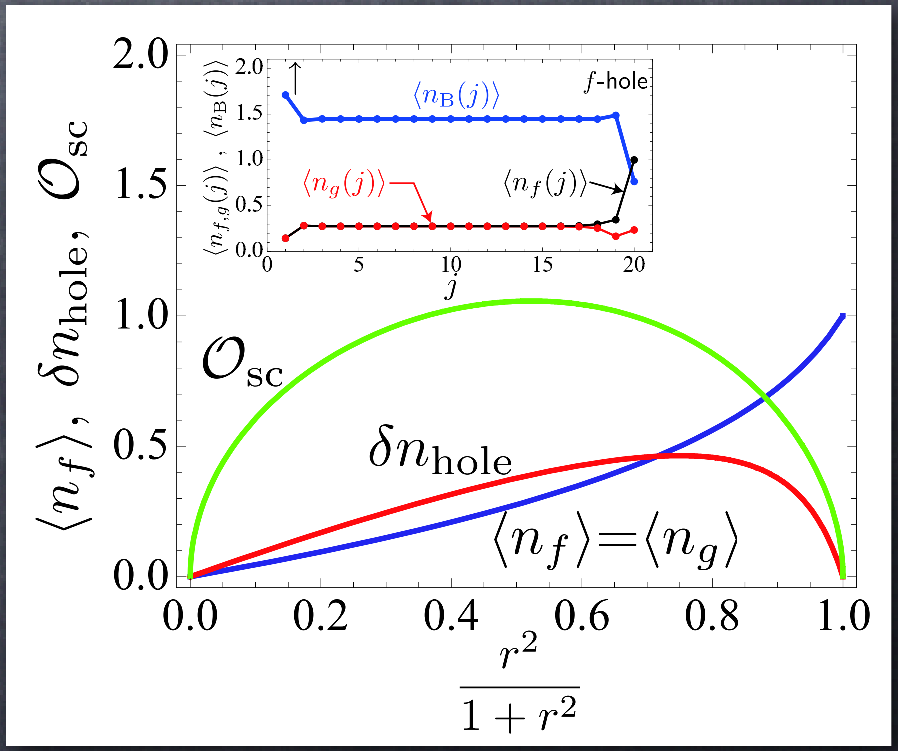
- Case with two species of holes (1D)...

$$|\mathrm{sVBS}\rangle = \prod_j \left\{ a_j^\dagger b_{j+1}^\dagger - b_j^\dagger a_{j+1}^\dagger - r(f_j^\dagger f_{j+1}^\dagger + h_j^\dagger h_{j+1}^\dagger) \right\} |0\rangle$$

- Matrix-product rep.:

$$g_j = \begin{pmatrix} -a_j^\dagger b_j^\dagger & -(a_j^\dagger)^2 & -\sqrt{r} a_j^\dagger f_j^\dagger & -\sqrt{r} a_j^\dagger h_j^\dagger \\ (b_j^\dagger)^2 & a_j^\dagger b_j^\dagger & \sqrt{r} b_j^\dagger f_j^\dagger & \sqrt{r} b_j^\dagger h_j^\dagger \\ -\sqrt{r} h_j^\dagger b_j^\dagger & -\sqrt{r} h_j^\dagger a_j^\dagger & -r h_j^\dagger f_j^\dagger & 0 \\ -\sqrt{r} f_j^\dagger b_j^\dagger & -\sqrt{r} f_j^\dagger a_j^\dagger & 0 & -r f_j^\dagger h_j^\dagger \end{pmatrix} |0\rangle_j$$

- Matrix-product formalism:
  - finite string order
  - symmetric behavior w.r.t. "r"



Summary

- SU(2) -> UOSp(1|2) generalization of VBS model(s)
  - ✓ "spin-L" or "spin-(L-1/2)+one (spin-1/2) hole"

$$(a_j^\dagger b_{j+1}^\dagger - b_j^\dagger a_{j+1}^\dagger) \;\; \Rightarrow \;\; (a_j^\dagger b_{j+1}^\dagger - b_j^\dagger a_{j+1}^\dagger - r f_j^\dagger f_{j+1}^\dagger)$$

- Matrix-product representation in terms of (2L+1)×(2L+1) matrix:
  - ✓ short-range spin-spin cor., finite hole-pairing correlation
  - ✓ hidden string order (symmetry-protected phase ?)
  - ✓ single-mode approx. ⇒ gapped magnetic excitations
- Generalizations:
  - ✓ 2D case ⇒ relation to Rokhsar-Kivelson point...
  - ✓ physics of QDM, topological order ??