Sneaking up on dense QCD using large N methods

Aleksey Cherman

Based on work with M. Hanada, D. Robles-Llana, B. Tiburzi...

UNIVERSITY OF CAMBRIDGE

at YITP, 16 February 2012
Dense matter is fascinating!

\[ n_B \gtrsim \Lambda_{QCD}^{-3} \]

Intrinsically interesting probe of QCD

Very important for neutron star physics

Finite density driven by a chemical potential for quark (~baryon) number

\[ \mathcal{L}_{QCD} \rightarrow \mathcal{L}_{QCD} + \mu_B \bar{\psi} \gamma^0 \psi \]

Many spectacular phenomena seen using weak-coupling methods, which apply for \( \mu_B / \Lambda_{QCD} \rightarrow \infty \)

For \( \mu_B / \Lambda_{QCD} \sim 1 \), not much is known reliably from first principles. Normally, this is where one would turn to lattice Monte Carlo methods.
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Lattice does not work at finite \( \mu_B \)!
Monte Carlo method: generate random $A_\mu$ configurations using $\det(D) e^{-S[A_\mu]}$ as a probability distribution, then evaluate the integral.

Works fine as long as distribution is $> 0$!
What makes Monte Carlo methods tick

\[
\langle O \rangle = \frac{\int dA_\mu d\psi d\bar{\psi} e^{-S[A_\mu, \psi, \bar{\psi}]} O[A_\mu, \psi, \bar{\psi}]}{\int dA_\mu d\psi d\bar{\psi} e^{-S[A_\mu, \psi, \bar{\psi}]} } = \frac{1}{Z} \int dA_\mu \det(D) e^{-S[A_\mu]} O[A_\mu]
\]

Monte Carlo method: generate random \( A_\mu \) configurations using
\[
\det(D) e^{-S[A_\mu]}
\]
as a probability distribution, then evaluate the integral.
Works fine as long as distribution is \( > 0! \)

QCD at \( \mu_B=0 \): \( \gamma_5 \gamma_5 = \gamma_5 \)

So then \( \det(D) = \prod^\lambda_i > 0 \)

Eigenvalues of \( D \) come in \( \lambda, \lambda^* \) pairs
Once $\mu_B > 0$, $\gamma^5$ symmetry breaks, and $\det(D)$ becomes complex, with a rapidly fluctuating phase.

Can’t use importance sampling anymore!

No known way to generically dodge this kind of problem.

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But maybe one just needs a clever algorithm to sum up the fluctuating phases?

Well...
Computational Complexity and Fundamental Limitations to Fermionic Quantum Monte Carlo Simulations

Matthias Troyer\textsuperscript{1} and Uwe-Jens Wiese\textsuperscript{2}

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(Received 11 August 2004; published 4 May 2005)

Quantum Monte Carlo simulations, while being efficient for bosons, suffer from the “negative sign problem” when applied to fermions—causing an exponential increase of the computing time with the number of particles. A polynomial time solution to the sign problem is highly desired since it would provide an unbiased and numerically exact method to simulate correlated quantum systems. Here we show that such a solution is almost certainly unattainable by proving that the sign problem is nondeterministic polynomial (NP) hard, implying that a generic solution of the sign problem would also solve all problems in the complexity class NP in polynomial time.

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P = NP

Clay Institute Prize
(1) Do not look for general solutions: exploit specifics of theory.

(2) Our approach: Exploit QCD details, but not in $N_c = 3$ world - too hard!

Go to the large $N$ limit!

Good (10-30%) approx. to real world for many observables at $\mu_B = 0$.

Probably much less close to our world for $\mu_B > 0$, but such is life.
So how to make progress?

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**The idea:** find sign-problem-free theory which is `orbifold-equivalent' to large N QCD at $\mu_B > 0$. 
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First glance:

\[ \text{QCD} \quad \equiv \quad \text{easier theory} + \frac{1}{N} \]

easier to \textquoteleft squeeze\textquoteright...

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Better picture

$$\text{QCD} \quad \equiv \quad \text{easier theory} \quad + \quad 1/N$$

Easier to `squeeze’...
First: Do sign-problem-free theories exist?

Yes!

1. QCD with N=2 colors, and
2. QCD with adjoint representation quarks.

\[ \gamma_5 \mathcal{D} \gamma_5 = \mathcal{D}^\dagger \] still broken when \( \mu_B > 0 \)

But now fermion representation is (pseudo)-real...

additional symmetry: \( C \gamma_5 \mathcal{D} (C\gamma_5)^{-1} = \mathcal{D}^* \)

No sign problem!

\[ \text{even when } \mu_B > 0! \]
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No sign problem!

But 1 & 2 have a number of major differences from \( N=3 \) QCD...

Goal is to use large \( N \) to get something equivalent to QCD.
Second: lightning review of large $N$

‘t Hooft large $N$ limit: $N \to \infty$, keeping $g^2 N$ fixed, $N_f$ fixed

Non-planar diagrams and quark loops suppressed

$\sim 1/N^{1/2}$ $\sim 1/N$ $\sim \frac{1}{N} \frac{N_f}{N} \sim \frac{1}{N^2}$

Mesons are stable, weakly-interacting; meson loops suppressed.
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Mesons are stable, weakly-interacting; meson loops suppressed.

Sign problem still present at large N.

Folklore says large N means we can set \( \det(\mathcal{D}) = 1 \) OK at \( \mu_B=0 \).

But at finite \( \mu_B \) this is known to give wrong answers: spurious phase transitions!

Setting \( \det(\mathcal{D}) = 1 \) by hand is a mutilation of the theory...

Expect \( \det(\mathcal{D}) \) to continue to have a fluctuating phase even at large N, so sign problem is still there...

\textit{e.g.} Barbour et al, 1986, Stephanov 1996
The proposal

1. \( \text{SU}(N) \) gauge theory + \( N_f \) fundamental fermions \( \Rightarrow \) \( \text{SO}(2N) \) gauge theory + \( N_f \) fundamental fermions

   - QCD
   - Orbifold equivalence
   - Easier theory

2. Equivalence can be made to hold even when \( \mu_B > 0 \).

   Use deformation approach due to Unsal+Yaffe

3. The \( \text{SO}(2N) \) theory does not have a sign problem at finite \( \mu_B \).

   Make sure \( D \) has enough symmetry, e.g. \( C \gamma_5 \mathcal{D} (C \gamma_5)^{-1} = \mathcal{D}^* \)
A quick look at SO gauge theories

$$\mathcal{L} = \frac{1}{4g^2} \text{tr} F_{\mu\nu} F^{\mu\nu} + \sum_{a=1}^{N_f} \bar{\psi}_a (\not{D} + m + \mu_B \gamma^4) \psi_a$$

Looks a lot like QCD: has both mesons and baryons

Still have $SU(N_f)_L \times SU(N_f)_R \times U(1)_B$ symmetry.

But SO is real, so all fermion reps are real

Flavor symmetry enhanced to $SU(2N_f)$

$\langle \bar{\psi} \psi \rangle \neq 0$

$$SU(2N_f) \rightarrow SO(2N_f)$$

$$N_f^2 - 1 + N_f(N_f - 1)$$

NG bosons

Witten & Coleman, Peskin, 1980
Two ways to make color singlets in SO(2N)

QCD: $\bar{\psi}_a \gamma^5 \psi_b \quad N_f^2 - 1$ pions, $P=-1$

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SO(2N): all of above, + \( \psi_a^T C \gamma^5 \psi_b \ N_f (N_f - 1) \) Baryonic pions, \( P=+1 \)
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Will refer to these NGBs with \( U(1)_B \) charge as `bpions`. 
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+ theory also bmeson relatives of the other usual mesons

Ex.: \( b\rho \) mesons
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Ex.: \( b \rho \) mesons

In what sense can such a weird theory be `equivalent' to QCD?
Orbifold Equivalence

Pick “mother” theory with a global symmetry $G$.

Pick a discrete cyclic subgroup $\mathbb{Z}_\Gamma \subset G$

Set to zero all degrees of freedom in the mother not invariant under $\mathbb{Z}_\Gamma \subset G$

The orbifold projection:

$\mathbb{Z}_\Gamma$ orbifold “daughter theory”

If $\mathbb{Z}_\Gamma$ symmetry is not spontaneously broken

Correlation functions of `neutral’ operators in mother and daughter theories will coincide in the large N limit.

Kachru, Silverstein 1998

Kovtun, Unsal, Yaffe, 2003-4
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Existing proofs of large N equivalence require some generalizations for this application: no general proof yet that necessary conditions above are also sufficient for fund. fermion case.

Truth in advertising:

Kachru, Silverstein 1998

Kovtun, Unsal, Yaffe, 2003-4
How does one connect an SO(2N) gauge theory to an SU(N) theory?

1. Change the gauge group: project onto SU(N) subgroup

\[ J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SO(2N) \quad A_\mu \rightarrow JA_\mu J^T = A_\mu \]

2. The bmesons better get killed by projection...

\[ \omega = e^{i\pi/2} \in U(1)_B \quad \psi \rightarrow \omega J\psi = \psi \]

**Result of orbifold:**

\[ \mathcal{L}^{SO} \rightarrow \mathcal{L}^{SU} \]
Survivors of projection

All gauge-invariant operators in pure-glue sector of SO theory

All meson operators

neutral sector in SO

Victims of projection

All bmeson operators

non-neutral sector

Operators of the form $\psi^T \psi$ have $\mathbb{Z}_2$ charge -1

Projection sets to zero all degrees of freedom not invariant under $\mathbb{Z}_2$

Baryons: orbifold prescription still needs to be worked out! AC, Mike Blake, 1203.XXXX
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Baryons: orbifold prescription still needs to be worked out! AC, Mike Blake, 1203.XXXX
Cartoon picture of orbifold equivalence

Mother:

\[ \frac{1}{\sqrt{N}} + \frac{1}{\sqrt{N}} = \frac{1}{\sqrt{N}} + \frac{1}{\sqrt{N}} \]

Daughter:

\[ \frac{1}{\sqrt{N}} + \frac{1}{\sqrt{N}} = \frac{1}{\sqrt{N}} + \frac{1}{\sqrt{N}} \]

\[ m = \text{meson} \]
\[ b = \text{bmeson} \]

Discard bmesons
Cartoon picture of orbifold equivalence

\[ \frac{1}{N} \]

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\[ \frac{1}{N} \]

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Processes in Mother not possible in Daughter:

\[ \frac{1}{N} \]

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Not allowed if \( U(1)_B \) unbroken
Cartoon picture of orbifold equivalence

Mother:

Daughter:

Processes in Mother not possible in Daughter:

Not allowed if $U(1)_B$ unbroken

Allowed but suppressed
The good news

No bmeson condensation at $\mu_B=0$.  

SO theory should be large N equivalent to QCD at $\mu_B=0$  

In fact, can show that there is no bmeson condensation at least for $\mu_B < m_\pi / 2$.  

So at least up to $\mu_B < m_\pi / 2$, expect equivalence to hold.
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Vafa-Witten theorem

SO theory should be large N equivalent to QCD at $\mu_B=0$

In fact, can show that there is no bmeson condensation at least for $\mu_B < m_\pi / 2$.

Using XPT analysis

So at least up to $\mu_B < m_\pi / 2$, expect equivalence to hold.

But (in principle) large N QCD has a sign problem for any $\mu_B > 0$!

So orbifold equivalence gives a way to dodge the sign problem at least for $\mu_B < m_\pi / 2$.

Already enough to think about physics at small $\mu_B/T$ - see Hanada-Yamamoto 2011

But we need to go past $\mu_B < m_\pi / 2$ to study nuclear matter...
The bad news

Once $\mu_B > m_\pi / 2$ bpions condense: $\langle \psi^T C \gamma^5 \psi \rangle \neq 0$

Equivalence is lost for $\mu_B > m_\pi / 2$!
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?
The proposal

1. SO(2N) gauge theory with \(N_f\) flavors of fundamental Dirac fermions

\[\cong\]

SU(N) gauge theory with \(N_f\) flavors of fundamental Dirac fermions

Orbifold equivalence

Large N QCD

2. Equivalence can be made to hold even when \(\mu_B > m_\pi/2\).

Use deformation approach due to Unsal+Yaffe

3. The SO(2N) theory does not have a sign problem at finite \(\mu_B\).

Make sure D has enough symmetry, e.g.

\[C\gamma_5 \mathcal{D} (C\gamma_5)^{-1} = \mathcal{D}^*\]
Protecting $U(1)_B$

We deform the SO(2N) theory so that

1. the modified theory still maps to QCD, and
2. prevent bpion condensation.

Note: deformation term orbifolds to zero.

Cartoon picture: should act like a mass term for bpions.

So system pays extra cost for condensing when $C > 0$...

So use deformations to discourage bpion condensation.

Next step: make sure this is more than a cartoon.
Sometimes irrelevant operators are quite relevant

Original theory: YM on lattice + naive fermions

Natural scale for physical $m_q$ on lattice: $m \sim 1/a$

Symmetries: Chiral sym, doubler sym

Consequences: Doubler sym locks $m_d$ of $2^D-1$ tastes to $m_{phys}$, $\chi$-sym keeps $m_{phys} = m_{bare}$

Deformed theory: YM + naive fermions + Wilson term

$$\mathcal{L}_{naive} \rightarrow \mathcal{L}_{naive} + ra \bar{\psi} D^2 \psi$$

Deformation breaks doubler symmetry

Doubler masses zoom off to the natural scale $m \sim 1/a$ when $r \sim 1$

($\chi$-sym broken too, but by tuning $m_{bare}$ can tune $m_{phys}$ to anything.)
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Natural scale for meson masses: $m_{hadron} \sim \Lambda_{QCD}$

In SO theory, deformation breaks $SU(2N_f)$ symmetry keeping $m_{bpion}$ locked to $m_{\pi}$

For $\mathcal{C} \sim 1$ expect $m_{bpion}$ to zoom off to $m_{bpion} \sim \Lambda_{QCD}$

Of course, lattice simulations critical to better understand deformed theory
Deformations and Effective Field Theory

Hard to understand deformed theory analytically in general.

But if $m_q, \mu_B \ll \Lambda_{QCD}$ and $\mathcal{C} \ll 1$, low-energy physics can be systematically describable using effective field theory.

Here EFT is just chiral perturbation theory adapted for SO gauge theory.

In XPT it is easiest to work with the deformations

$$V_{\pm} = \frac{\mathcal{C}^2 a^2}{N} \sum_{a, b=1}^{N_f} \left( S_{ab}^\dagger S_{ab} \pm P_{ab}^\dagger P_{ab} \right)$$

$$P_{ab} = \psi_a^T C \psi_b$$

$$S_{ab} = \psi_a^T C \gamma^5 \psi_b$$

Without deformations, the EFT has the Lagrangian

$$\mathcal{L} = \frac{F_{\Pi}^2}{4} \text{tr} \left[ D_\mu \Sigma D_\mu \Sigma^\dagger \right] - \frac{\lambda F_{\Pi}^2}{4} \text{tr} \left[ \Sigma \mathcal{M} + \Sigma^\dagger \mathcal{M}^\dagger \right]$$

Deformations induce new terms in the low-energy action...

Just have to work them out...
Two deformations

To capture effects of deformations, use spurion analysis.

Deformation is 4-quark operator, so can borrow standard techniques used in XPT to understand e.g. finite lattice-spacing effects

\[ V_+ \text{ produces just one new term in the EFT} \]

\[ c_+ F_{\Pi}^2 \sum_{a,b=1}^{N_f} \left( \text{tr} \left[ \Sigma L^{(ab)} \right] \text{tr} \left[ \Sigma^\dagger L^{(ab)\dagger} \right] + \text{tr} \left[ \Sigma R^{(ab)} \right] \text{tr} \left[ \Sigma^\dagger R^{(ab)\dagger} \right] \right) \]

\[ V_- \text{ produces two new terms in the EFT} \]

\[ c_- F_{\Pi}^2 \sum_{a,b=1}^{N_f} \left( \text{tr} \left[ \Sigma L^{(ab)} \right] \text{tr} \left[ \Sigma R^{(ab)} \right] + \text{tr} \left[ \Sigma^\dagger L^{(ab)\dagger} \right] \text{tr} \left[ \Sigma^\dagger R^{(ab)\dagger} \right] \right) + d_- F_{\Pi}^2 \sum_{a,b=1}^{N_f} \left( \text{tr} \left[ \Sigma L^{(ab)} \Sigma R^{(ab)} \right] + \text{tr} \left[ \Sigma^\dagger L^{(ab)\dagger} \Sigma^\dagger R^{(ab)\dagger} \right] \right) \]

New low-energy constants \( c_+, c_-, d_- \)
## Spectrum of the deformed theory

### Without symmetry breaking:

<table>
<thead>
<tr>
<th>Mode</th>
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<tr>
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Matching to microscopic theory gives $N_c$ scaling of the new LECs

$c_-, c_+ \sim N_c^0$, $d_- \sim N_c^{-1}$

Can also show that the sign of $C$ in microscopic theory controls the signs of the LECs in the EFT.

So deformations work by raising the bpion masses, while leaving neutral-sector stuff alone.

To nail down symmetry realization pattern, minimize effective potential in deformed theory.
The proposal

1. SO(2N) gauge theory with $N_f$ flavors of fundamental Dirac fermions $\cong$ SU(N) gauge theory with $N_f$ flavors of fundamental Dirac fermions
   - Orbifold equivalence
   - Large N QCD

2. Equivalence can be made to hold even when $\mu_B > m_\pi/2$.
   - Use deformation approach due to Unsal+Yaffe

3. The SO(2N) theory does not have a sign problem at finite $\mu_B$.
   - Make sure D has enough symmetry, e.g.
     $C \gamma_5 \mathcal{D} (C \gamma_5)^{-1} = \mathcal{D}^*$
Sign-free implementation of deformations

Deformations are four-quark operators, so must use auxiliary fields to put them on the lattice.

Sign problem reappears if aux field implementation breaks enough symmetries!

$$S_{ab} = \psi_a^T C \gamma_5 \psi_b$$ must be complex, sign problem

For $V$, we found a rather baroque way to implement auxiliary fields that avoids reintroducing the sign problem
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Fierz rearrangement: $S_{ab}^\dagger S_{ab} = \sum_\Gamma \overline{q}_a^i \Gamma q_j^a$.

Aux fields coupling to $S_{ab} = \psi_a^T C \gamma_5 \psi_b$ must be complex, sign problem.

Can couple real auxiliary fields $f_{ij}$ to $\overline{q}_a^i \Gamma q_j^a$.

But color group is real! $U(1)_B$ singlet, color tensor.
Sign-free implementation of deformations

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Aux fields coupling to \( S_{ab} = \psi_a^T C\gamma_5 \psi_b \) must be complex, sign problem

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Fierz rearrangement: \( S^\dagger_{ab} S_{ab} = \sum \frac{1}{2} f_{ij} f_{ij}^\dagger + ic \Gamma p\bar{q}_a \Gamma q_j^a \) + similar terms for \( P^\dagger_{ab} P_{ab} \)

Integration over \( f_{ij} \) gives original 4-quark terms
Sign-free implementation of $V$-deformations

Result of integrating in auxiliary fields in flavor-singlet channel:

\[
\frac{c^2}{\Lambda^2} (S^+_{\alpha\kappa} S_{\beta\kappa} - P^+_{\alpha\kappa} P_{\beta\kappa}) \rightarrow \frac{(f_{ij})^2}{2} + \frac{(g_{ij})^2}{2} + \frac{(h_{\mu\nu,ij})^2}{2} + ic_1 f_{ij} \bar{\psi}_i^a \psi_j^a + ic_2 g_{ij} \bar{\psi}_i^a \gamma^5 \psi_j^a + ic_3 h_{\mu\nu,ij} \bar{\psi}_i^a \gamma^{\mu\nu} \psi_j^a
\]

Factors of $i$ break $C\gamma^5$ symmetry.
Sign-free implementation of $V$-deformations

Result of integrating in auxiliary fields in flavor-singlet channel:

\[
\frac{c^2}{\Lambda^2} (S^{\dagger \, ab} S_{ab} - P^{\dagger \, ab} P_{ab}) \rightarrow \frac{(f_{ij})^2}{2} + \frac{(g_{ij})^2}{2} + \frac{(h_{\mu \nu, ij})^2}{2} + i c_1 f_{ij} \bar{\psi}_a^i \psi_a^j + i c_2 g_{ij} \bar{\psi}_a^i \gamma^5 \psi_a^j + i c_3 h_{\mu \nu, ij} \bar{\psi}_a^i \gamma^{\mu \nu} \psi_a^j
\]

Factors of $i$ break $C\gamma^5$ symmetry.

But for $m_q = 0$, aux fields preserve $CD(\mu_B, c)C^{-1} = -D(\mu_B, c)^*$
Sign-free implementation of $V_-$ deformations

Result of integrating in auxiliary fields in flavor-singlet channel:

$$\frac{c^2}{\Lambda^2} (S^\dagger_{ab} S_{ab} - P^\dagger_{ab} P_{ab}) \rightarrow \frac{(f_{ij})^2}{2} + \frac{(g_{ij})^2}{2} + \frac{(h_{\mu\nu,ij})^2}{2} + ic_1 f_{ij} \bar{\psi}_a^i \psi_a^j + ic_2 g_{ij} \bar{\psi}_a^i \gamma^5 \psi_a^j + ic_3 h_{\mu\nu,ij} \bar{\psi}_a^i \gamma^{\mu\nu} \psi_a^j$$

Factors of $i$ break $C\gamma^5$ symmetry.

But for $m_q = 0$, aux fields preserve $CD(\mu_B, c)C^{-1} = -D(\mu_B, c)^*$

Enough symmetry to ensure positivity as $m_q \rightarrow 0$, even when $c > 0$

Finally:

✓ No sign problem in the chiral limit.

✓ Large N equivalence to QCD kept past $\mu_B = m_\pi / 2$

The same trick does not work for $V_+$. Are there other tricks that do?
Summary and open questions

Using SO theory, we can dodge sign problem even past $m_\pi/2$.

Vanishing of sign problem as $m_q \rightarrow 0$  

Sign-quenching should be a good approximation for light quarks.
Summary and open questions

Using SO theory, we can dodge sign problem even past $m_\pi/2$.

Vanishing of sign problem as $m_q \to 0$ \quad Sign-quenching should be a good approximation for light quarks.

Does equivalence hold through nuclear matter transition?

- Do bmesons with charge/mass less than lightest baryons exist, even in deformed theory?
- If so, expect condensation for big enough $\mu_B$, killing equivalence.

We need non-perturbative tests!

Lattice, AdS/CFT, ...

To do:

Extend equivalence proofs, look for sign-free way to work with $V_+$, try to get away from chiral limit, try to dodge other sign problems, ...
Phase diagram of the $V_+\text{-deformed theory}$

\[ \frac{c_+}{(m_\pi/2)^2} \]

\[ \mu_B^2/(m_\pi/2)^2 \]

- $\langle b \rangle \neq 0$, $\langle \eta' \rangle = 0$
- $\langle b \rangle = 0$, $\langle \eta' \rangle = 0$
Phase diagram of the $V$-deformed theory

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Exotic metastable phase
Orbifold equivalence past $\mu_B = m_\pi/2$

With both deformations, the SO theory can be forced to stay in a $U(1)_B$-unbroken phase past $\mu_B = m_\pi/2$.

The correlation functions of neutral operators are identical with both deformations in the normal phase.

The $V$-deformed theory has an exotic phase with $\eta'$-condensation. This phase is always metastable in our analysis.
Orbifold equivalence past $\mu_B = m_\pi/2$

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At level of EFT, large N-equivalence is `obvious':

$$\frac{U(N_f)_L \times U(N_F)_R}{U(N_f)_V} \subset \frac{SU(2N_f)}{SO(2N_f)}$$

At large $N$, neutral correlators in $SU(2N_f)/SO(2N_f)$ EFT with given LECs trivially coincide with correlators computed in an $SU(N_f)$ EFT with the same LECs, so long as $U(1)_B$ is not broken.