Auxiliary field Monte-Carlo study of the QCD phase diagram at strong coupling

A. Ohnishi (YITP)
in collaboration with
T.Z.Nakano (YITP/Kyoto U.), T. Ichihara (Kyoto U.)

- Introduction
- Auxiliary Field Monte-Carlo treatment of SC-LQCD
- Monte-Carlo estimate of the phase boundary
- Summary

Work in progress
QCD Phase Diagram

RHIC/LHC/Early Universe

$T$

$\mu_B$

QGP
AGS/SPS/FAIR/J-PARC

Hadron Matter

Quarkyonic

Neutron Star

CSC

Nucleon Gas

Nuclear Matter

Nuclear Matter

Quarkyonic
**QCD phase diagram (Exp. & Theor. Studies)**

*QCD phase transition is not only an academic problem, but also a subject which would be measured in HIC or Compact Stars*
Lattice QCD at finite density has the sign problem. → Approx. methods and/or Effective model studies are necessary.

- Approximate methods: Taylor exp. (LT04), Imag. μ, Canonical (LC04, 08), Reweighting (LR02, 04), Fugacity exp. (Nagata / Adams), Strong Coupling Lattice QCD

- Effective models: NJL, PNJL, PQM, ..

Direct Sampling in LQCD at finite μ → Histogram method (Ejiri) or SC-LQCD
Strong Coupling Lattice QCD

Pure YM

YM+Quarks (MF)

Wilson ('74), Creutz ('80), Munster ('80, '81), Lottini, Philipsen, Langelage's ('11)

Kawamoto ('80), Kawamoto, Smit ('81), Damgaard, Hochberg, Kawamoto ('85), Bilic, Karsch, Redlich ('92), Fukushima ('03); Nishida ('03), Kawamoto, Miura, AO, Ohnuma ('07).

Miura, Nakano, AO, Kawamoto ('09)
Nakano, Miura, AO ('10)
SC-LQCD with Polyakov Loop Effects at $\mu=0$


- P-SC-LQCD reproduces $T_c(\mu=0)$ in the strong coupling region ($\beta = 2N_c/g^2 \leq 4$)

MC data: SCL (Karsch et al. (MDP), de Forcrand, Fromm (MDP)), $N_\tau=2$ (de Forcrand, private), $N_\tau=4$ (Gottlieb et al.('87), Fodor-Katz ('02)), $N_\tau=8$ (Gavai et al.('90))
**Strong Coupling Lattice QCD**

**Pure YM**

- SC-LQCD
- MC

**YM+Quarks (MF)**

- Wilson ('74), Creutz ('80), Munster ('80, '81), Lottini, Philipsen, Langelage's ('11)
- Kawamoto ('80), Kawamoto, Smit ('81), Damagaard, Hochberg, Kawamoto ('85), Bilic, Karsch, Redlich ('92), Fukushima ('03); Nishida ('03), Kawamoto, Miura, AO, Ohnuma ('07), Miura, Nakano, AO, Kawamoto ('09), Nakano, Miura, AO ('10)

**YM+Q+Fluc. (MDP)**

- Mutter, Karsch ('89), de Forcrand, Fromm ('10), de Forcrand, Unger ('11)

**Challenge: YM+Q+Fluc.+Finite Coupling Effects**

- de Forcrand, Fromm, Langelage, Miura, Philipsen, Unger ('11), AO, Nakano, Ichihara (in prep.)

Ohnishi @ NTFL (Feb. 21, 2012)
Auxiliary field effective action in the Strong Coupling Limit
**Strong Coupling Expansion**

- **Lattice QCD action (aniso. lattice, unrooted staggered Fermion)**

\[
S_{LQCD} = S_F + S_G \\
S_G = -\frac{1}{g^2} \sum_{\text{plaq.}} \text{tr} [U_P + U_P^+] f_P \\
S_F = \frac{1}{2} \sum_x \left[V^+(x) - V^-(x)\right] + \frac{1}{2\gamma} \sum_{x,j} \eta_j(x) \left[\bar{\chi}_x U_{x,j} \chi_{x+j} - \bar{\chi}_{x+j} U_{x,j}^+ \chi_x\right] + \sum_x \frac{m_0}{\gamma} M_x \\
V^+(x) = e^{\mu/\gamma^2} \bar{\chi}_x U_{x,0} \chi_{x+0}, \quad V^-(x) = e^{-\mu/\gamma^2} \bar{\chi}_{x+0} U_{x,0}^+ \chi_x, \quad M_x = \bar{\chi}_x \chi_x, \quad a_\gamma = a/\gamma, \quad f_P = 1 \text{ or } 1/\gamma
\]

- **Strong coupling expansion**
  - (Strong coupling limit)

  - Ignore plaquette action \((1/g^2)\)
  - Integrate out spatial link variables of min. quark number diagrams \((1/d\) expansion)

\[
S_{\text{eff}} = \frac{1}{2} \sum_x \left[V^+(x) - V^-(x)\right] - \frac{1}{4 N_c \gamma^2} \sum_{x,j} M_x M_{x+j} + \frac{m_0}{\gamma} \sum_x M_x
\]

\[\int dU \ U_{ab} U_{cd}^{+} = \delta_{ad} \delta_{bc} / N_c\]
Introduction of Auxiliary Fields

- Bosonization of $MM$ term (Four Fermi (two-body) interaction)

$$S_F^{(s)} = -\alpha \sum_{j, x} M_x M_{x+j} = -\alpha \sum_{x, y} M_x V_{x, y} M_y \quad [V_{x, y} = \frac{1}{2} \sum_j (\delta_{x+j, y} + \delta_{x-j, y})]$$

- Meson matrix ($V$) has positive and negative eigen values

$$f_M(k) = \sum_j \cos k_j \quad , \quad f_M(\bar{k}) = -f_M(k) \quad [\bar{k} = k + (\pi, \pi, \pi)]$$

- Negative mode = “High” momentum mode
  → Involves a factor $\exp(i \pi (x_1 + x_2 + x_3)) = (-1)^*(x_1 + x_2 + x_3)$
  in coordinate representation

- Bosonization of Negative mode: Extended HS transf.
  → Introducing “$i$” gives rise to the sign problem.
  
  $Miura, Nakano, AO (09), Miura, Nakano, AO, Kawamoto (09)$

$$\exp(\alpha AB) = \int d\varphi d\phi \exp[-\alpha (\varphi^2 - (A+B)\varphi + \varphi^2 - i(A-B)\phi)]$$

$$\approx \exp[-\alpha (\bar{\psi}\psi - A\psi - \bar{\psi}B)]_{\text{stationary}}$$
Phase cancellation mechanism in $\sigma$MC

**Bosonized effective action**

$$S_{\text{eff}} = \frac{1}{2} \sum_{x,y} \bar{\chi}_x D_{x,y} \chi_y + \frac{\Omega}{4 N_c \gamma^2} \sum_{k, f_M(k) > 0} f_M(k) \left[ \sigma_k^* \sigma_k + \pi_k^* \pi_k \right]$$

$$D_{x,y} = \delta_{x+\hat{0}, y} \delta_{x,y} e^{\mu/\gamma^2} U_{x,0} - \delta_{x,y+\hat{0}} \delta_{x,y} e^{-\mu/\gamma^2} U_{y,0}^+ + 2 \left[ \sum_x + \frac{m_0}{\gamma} \right] \delta_{x,y}$$

$$\sigma(x) = \sum_{k, f_M(k) > 0} f_M(k) e^{i k x} \sigma_k \quad \pi(x) = \sum_{k, f_M(k) > 0} f_M(k) e^{i k x} \pi_k$$

**Fermion matrix is spatially separated**

→ Fermion det at each point

**Imaginary part ($\pi$) involves**

$$\varepsilon = (-1)^{x_0 + x_1 + x_2 + x_3} = \exp(i \pi(x_0 + x_1 + x_2 + x_3))$$

→ Phase cancellation of nearest neighbor spatial site det for $\pi$ field having low $k$
Auxiliary Field Monte-Carlo Integral

Effective action of Auxiliary Fields

\[
S_{\text{eff}} = \frac{\Omega}{4 N_c \gamma^2} \sum_{k, f_M(k) > 0} f_M(k) \left[ \sigma_k^* \sigma_k + \pi_k^* \pi_k \right] \\
- \sum_x \log \left[ X_N(x)^3 - 2 X_N(x) + 2 \cosh (3 N \tau \mu) \right]
\]

\[
X_N(x) = X_N[\sigma(x, \tau), \pi(x, \tau)]
\]

- $\mu$ dependence appears only in the log.
- $\sigma_k, \pi_k$ have to be generated in momentum space, while $X_N$ requires $\sigma(x)$ and $\pi(x) \rightarrow$ Fourier transf. in each step.
- $X_N$ is complex, and this action has the sign problem. But the sign problem is milder because of the phase cancellation and is less severe at larger $\mu$.

Let's try at finite $\mu$!
Auxiliary Field Monte-Carlo (σMC) estimate of the phase boundary
Numerical Calculation

- $4^4$ asymmetric lattice + Metropolis sampling of $\sigma_k$ and $\pi_k$.

- Metropolis sampling of full configuration ($\sigma_k$ and $\pi_k$) at a time. (efficient for small lattice)

- Initial cond. = const. $\sigma$

- Chiral limit ($m=0$) simulation $\rightarrow$ Symmetry in $\sigma \leftrightarrow -\sigma$

- Sign problem is not severe ($<\cos \theta> \sim (0.9-1.0)$) in a $4^4$ lattice.

- Computer: My PC (Core i7)
Results (1): $\sigma$ distribution

- Fixed $\mu/T$ simulation: $\mu/T = 0 \sim 2.4$
- Low $\mu$ region: Second order
  (Single peak: finite $\sigma \rightarrow$ zero)
- High $\mu$ region: First order
  (Dist. func. has two peaks)
Results (2): Susceptibility and Quark density

- Weight factor $<\cos \theta>$
  \[
  \langle \cos \theta \rangle = \frac{Z}{Z_{\text{abs}}}
  \]
  \[
  Z = \int D\sigma_k D\pi_k \exp(-S_{\text{eff}})
  \]
  \[
  = \int D\sigma_k D\pi_k \exp(-\text{Re } S_{\text{eff}}) e^{i\varphi}
  \]
  \[
  Z_{\text{abs}} = \int D\sigma_k D\pi_k \exp(-\text{Re } S_{\text{eff}})
  \]

- Chiral susceptibility
  \[
  \chi = -\frac{1}{L^3} \frac{\partial^2 \log Z}{\partial m_0^2}
  \]

- Quark number density
  \[
  \rho_q = -\frac{1}{L^3} \frac{\gamma^2}{N_\tau} \frac{\partial \log Z}{\partial \mu}
  \]
Results (3): Phase diagram

By taking $T = \gamma^2 / N_\tau$, 
$\gamma$ dep. of the phase boundary becomes small. *Bilic et al.* ('92)

Definitions of phase boundary

- $\phi^2 = \sigma^2 + \pi^2$ dist. peak: finite or zero (red curve)
- Chiral susceptibility peak (blue)

Fluctuation effect

- Reduction of $T_c$ at $\mu=0$
- Enlarged hadron phase at medium $T$

→ Consistent with MDP

*de Forcrand, Fromm* ('09); *de Forcrand, Unger* ('11)
Results (4): Larger Lattice

Can we go to a larger lattice?

d\(\mu/dT\) > 0 at low T.
→ How about in a larger lattice?

Suggestion by de Forcrand
→ low T behavior is sensitive to N.
→ 4\(^3\) x 8
Summary

- We have proposed an auxiliary field MC method (σMC) in SC-LQCD.
  - To simulate the SCL quark-$U_0$ action (LO in strong coupling ($1/g^0$) and $1/d$ ($1/d^0$) expansion) without further approximation.
    _c.f. Determinantal MC by Abe, Seki_
  - Sign problem is mild in small lattice ($<\cos \theta> \sim (0.9-1)$ for $4^4$), because of the phase cancellation coming from nearest neighbor interaction.
  - Extension to NLO SC-LQCD is straightforward.

- Phase boundary is obtained and found to be compatible with recent MDP results.
  - Phase boundary is moderately modified from MF results by fluctuations, if $T = \gamma^2/N_\tau$ and $\mu = \gamma^2 \mu_0$ scaling is adopted.
  - σMC results are compatible with MDP results, while the shift of $T_c$ at $\mu=0$ is around half (LO in $1/d$ expansion in σMC).
Future work

To do:

- Larger lattice ($8^4$, $16^3 \times 8$, ...)
- Finite coupling effects (NLO, NNLO, Polyakov loop, ...)
- Higher 1/d terms including baryonic action
- Polyakov coupling (back reaction)
- Unrooted staggered fermion corresponds to 4 flavors (tates) in continuum
  → Different Fermion (e.g. staggered-Wilson fermion).
Thank you
**Clausius-Clapeyron Relation**

- **First order phase boundary** → two phases coexist

\[
P_h = P_q \quad \rightarrow \quad dP_h = dP_q \quad \rightarrow \quad \frac{d\mu}{dT} = -\frac{s_q - s_h}{\rho_q - \rho_h}
\]

\[
dP_h = \rho_h d\mu + s_h dT, \quad dP_q = \rho_q d\mu + s_q dT
\]

- **Continuum theory**
  → Quark matter has larger entropy and density (d\(\mu\)/dT < 0)

- **Strong coupling lattice**
  - SCL: Quark density is larger than half-filling, and “Quark hole” carries entropy → d\(\mu\)/dT > 0
  - NLO, NNLO → d\(\mu\)/dT < 0

*AO, Miura, Nakano, Kawamoto ('09)*)
Introduction of Auxiliary Fields

\[ S^{(s)} = -\frac{1}{4N_c\gamma^2} \sum_{x,j} M_x M_{x+j} = -\frac{L^3}{4N_c\gamma^2} \sum_{\tau,k} f_M(k) \tilde{M}_k(\tau) \tilde{M}_{-k}(\tau) \]
\[ = \frac{L^3}{4N_c\gamma^2} \sum_{\tau,k,f_M(k)>0} f_M(k) \left[ \varphi_k(\tau)^2 + \phi_k(\tau)^2 + \varphi_k(\tilde{M}_k + \tilde{M}_{-k}) - i\phi_k(\tilde{M}_k - \tilde{M}_{-k}) \right. \]
\[ \left. + \varphi_k(\tau)^2 + \phi_k(\tau)^2 + i\varphi_k(\tilde{M}_k + \tilde{M}_{-k}) + \phi_k(\tilde{M}_k - \tilde{M}_{-k}) \right] \]
\[ = \frac{\Omega}{2N_c\gamma^2} \sum_{k,f_M(k)>0} f_M(k) \left[ \sigma_k^* \sigma_k + \pi_k^* \pi_k \right] + \frac{1}{2N_c\gamma^2} \sum_x M_x \left[ \sigma(x) + i\varepsilon(x)\pi(x) \right] \]

\[ \Omega = L^3 N_\tau \]

\[ \sigma(x) = \sum_{k,f_M(k)>0} f_M(k) e^{ikx} \sigma_k, \quad \pi(x) = \sum_{k,f_M(k)>0} f_M(k) e^{ikx} \pi_k \]
\[ \sigma_k = \varphi_k + i\phi_k, \quad \pi_k = \varphi_k + i\phi_k \]
\[ V_{x,y} = \frac{1}{2} \sum_j \left( \delta_{x+j,y} + \delta_{x-j,y} \right), \quad f_M(k) = \sum_j \cos k_j, \quad \tilde{k} = k + (\pi, \pi, \pi) \]
Fermion action is separated to each spatial point and bi-linear
→ Determinant of Nτ x Nc matrix

\[
\exp(-V_{\text{eff}}/T) = \int dU_0 \det \left[ X_N [\sigma] \otimes \mathbf{1}_c + e^{-\mu/T} U^+ + (-1)^{N_\tau} e^{\mu/T} U \right]
\]

\[
= X_N^3 - 2 X_N + 2 \cosh(3 N_\tau \mu)
\]

\[
I_\tau/2 = [\sigma(\chi) + i \varepsilon(\chi) \pi(\chi)] / 2 N_c \gamma^2 + m_0 / \gamma
\]

\[
X_N = B_N + B_{N-2} (2; N-1)
\]

\[
B_N = I_N B_{N-1} + B_{N-2}
\]