Comments on lattice chiral symmetry and minimal doubling

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Outline

- Chiral symmetry in mass parameter space
- The anomaly and lattice complications
- Topology and doublers
- Minimal doubling
- Point split operators
- Counterterms
Frame the discussion in the path integral formulation

\[ Z = \int (dA)(d\psi)(d\bar{\psi}) \exp(-S_g(A) + \bar{\psi}D\psi) = \int (dA)|D(A)|e^{-S_g(A)} \]

- Concentrate on the determinant of the Dirac matrix \(|D(A)|\)

“Continuum” picture

\[ D = K + M \]
\[ K = \gamma_\mu(\partial_\mu + igA_\mu) = -K^\dagger = -\gamma_5K\gamma_5 \]
\[ M^\dagger = M = \gamma_5M\gamma_5 \]

- introduce \(\Theta\) later
Chiral symmetry as a symmetry in parameter space

- \([K, \gamma_5]_+ = 0\) implies
  \[K = e^{i\omega_i \tau_i \gamma_5} Ke^{i\omega_i \tau_i \gamma_5}\]

- since \((\text{Tr})\tau_i = 0\)
  \[|e^{i\omega_i \tau_i \gamma_5}| = \exp(i\omega_i \text{Tr} (\tau_i \gamma_5)) = 1\]

- therefore
  \[|K + M| = |K + e^{i\omega_i \tau_i \gamma_5} Me^{i\omega_i \tau_i \gamma_5}|\]

Physics unchanged with a modified mass matrix

\[M \rightarrow e^{i\omega_i \tau_i \gamma_5} Me^{i\omega_i \tau_i \gamma_5}\]
Combined with the vector counterpart

\[ M \rightarrow e^{-i\omega_i \tau_i} M e^{i\omega_i \tau_i}, \]

\[ SU(N_f) \otimes SU(N_f) \] symmetry

- symmetry of the massive theory in parameter space.

Specific example

- for two degenerate flavors the mass terms

\[ m\psi\psi \]
\[ -\bar{m}\bar{\psi}\psi \]
\[ im\bar{\psi}\tau_3\gamma_5\psi \]

- all give equivalent theories
Wine bottle analogy

\[ V \sim (\sigma^2 + \bar{\pi}^2 - v^2)^2 \]

- All tilt directions are equivalent
- Basis for twisted mass

Lattice artifacts improved
The anomaly

\[ M \rightarrow e^{i\theta \gamma_5} M \]

not a valid symmetry

- mixes \( \sigma \) with \( \eta' \)
- \( \eta' \) gets extra mass from topology
- details depend on specific cutoff
- involve complications in defining \( \gamma_5 \)

Fujikawa relates fermion measure and the index theorem

- in gauge field with topology, \( K(A) \) has zero modes
- eigenstates of \( \gamma_5 \)
- winding number \( \nu = n_+ - n_- \)
Use $K$ to cutoff high modes of the Dirac operator

- zero modes contribute to $\text{Tr} \gamma_5$

$$\text{Tr} \gamma_5 \rightarrow \text{Tr}(\gamma_5 e^{-K^\dagger K/\Lambda^2})$$

$$= \sum_i \langle \psi_i | \gamma_5 e^{-K^\dagger K/\Lambda^2} | \psi_i \rangle$$

$$= \nu \sim \int \tilde{F} \tilde{F}$$

- expand in powers of $K$

  at $K^4$ ultraviolet divergence cancels $1/\Lambda^4$

In the path integral, configurations weighted by $| e^{i\theta \text{Tr} \gamma_5} | = e^{i\theta N_f \nu}$

Leaving the usual CP violating parameter $\Theta$ of QCD
$m \leftrightarrow -m$ theories not equivalent for $N_f$ odd

- $-1$ is not in $SU(2N + 1)$
- $-m\bar{\psi}\psi$ gives the $\Theta = \pi$ theory
- spontaneous CP violation expected ($N_f > 1$)
  purely non-perturbative

Two light flavors with non-degenerate mass

$$m_{\pi_0}^2 \sim \frac{m_u + m_d}{2} - c (m_u - m_d)^2 + O(m_q^3)$$

- $c$ is a “low energy constant”
- associated with isospin breaking and $\pi_0 \eta \eta'$ mixing
  lowers $m_{\pi_0}$
Negative $m_{\pi_0}^2$ can give pion condensation

Mass gap does not vanish when only one quark is massless

- no singularity along $m_u = 0$ axis
- scale dependence in defining a massless up quark
Isospin breaking

$$\frac{m^2_{\pi^0}}{m^2_{\pi^+}} = 1 - 2C \frac{(m_d - m_u)^2}{m_d + m_u}$$

- in continuum limit either
  
  $C$ diverges

  quark mass ratios not constant

Mass independent regularization tricky

- not natural with lattice regulator

- perturbative matching requires caution
Lattice complications

Lattice at finite volume has no infinities

• how can an anomaly appear?

Consider any lattice Dirac operator $D$

• assume gamma five hermiticity $\gamma_5 D \gamma_5 = D^\dagger$
  
  all operators in practice satisfy this (except twisted mass)

Divide $D$ into hermitean and antihermitean parts

$$K = (D - D^\dagger)/2$$

$$M = (D + D^\dagger)/2$$
Then

\[ [K, \gamma_5]^+ = 0 \]
\[ [M, \gamma_5]^-= 0 \]

\( M \rightarrow e^{i\theta \gamma_5} M \) an exact symmetry of the determinant

- Where is the anomaly?

Naive fermions solve this with doublers

- half use \( \gamma_5 \) and half \(-\gamma_5\)
- the naive chiral symmetry is actually flavored

This carries through to staggered and minimally doubled actions
How about Wilson fermions?

- doublers given masses of order the cutoff
- the rotation $M \rightarrow e^{i\omega_i \gamma_5} M$ also rotates their phases

Physical $\Theta$ is a relative angle

- independently rotate the fermion mass and the Wilson term

  Seiler and Stamatescu

The overlap operator

- eigenvalues on a circle
- each zero eigenmode has counterpart on the opposite side
- rotation of Hermitean part rotates heavy mode as well
- anomaly brings in $\hat{\gamma}_5$

$$\nu = \text{Tr}(\gamma_5 + \hat{\gamma}_5)/2$$
Message for continuum QCD:

- physical $\Theta$ can be moved around
- placed on any one flavor at will

$\Theta$ can be entirely moved into the top quark phase

- some aspects of the top quark are relevant to low energy physics!
- decoupling theorems don’t apply non-perturbatively
Topology and doublers

Above effectively a restatement of Nielsen-Ninomiya

Consider the quark propagator at large momentum

- \( D = i\slashed{p} + O(m, gA_\mu) \)

  a gauge choice needed to keep things smooth

Study the leading part \( D = i\slashed{p} \)

- maintain \([i\slashed{p}, \gamma_5]_+ = 0\)
Convenient gamma matrix convention

\[
\gamma_5 = \sigma_3 \otimes 1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

\[
\gamma_4 = \sigma_2 \otimes 1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}
\]

\[
\vec{\gamma} = \sigma_1 \otimes \vec{\sigma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}
\]

Then \( i\vec{p} = \begin{pmatrix} 0 & z(p) \\ -z^*(p) & 0 \end{pmatrix} \)

with \( z \) a quaternion

\[
z(p) = p_4 + i\vec{p} \cdot \vec{\sigma} = |p| g(p) \quad g(p) \in SU(2) \sim S_3
\]
For the path integral we want

$$|D| = z^* z = |z|^2 = p^2_4 + \vec{p}^2$$

Consider a surface of constant $|D|$

- a sphere in four dimensions of constant radius $|p|

Over this sphere, $z$ wraps non-trivially

- around a similar sphere in quaternionic space
On the lattice

Momentum bounded to the Brilloin zone

\[ -\pi/a < p_\mu \leq \pi/a \]

- the propagator must be periodic in momentum space.

Problem

- at the edge of the Brilloin zone
  periodicity forbids wrapping

The propagator must unwrap somewhere

- if \( z \) remains finite, there must be another zero.

Not a bad thing

- anomaly says one flavor QCD has no chiral symmetry
- with \( N_f \) flavors chiral symmetry is \( SU(N_f) \otimes SU(N_f) \otimes U(1)_B \)
  not \( U(N_f) \otimes U(N_f) \).
Note: zeros at non-zero momentum not a problem

- redefining $\psi(x) \rightarrow e^{i\alpha \cdot x} \psi$

- moves a zero at $p_\mu = 0$ to $p_\mu = \alpha_\mu$

  can always translate our zeros anywhere in momentum space.

Nielsen-Ninomiya conclusion:

- local lattice action that anticommutes with $\gamma_5$

- must have an even number of fermion species

  Note: 3 is also bad
Wilson: unwrapping occurs at the doublers
  • doublers given mass by Wilson term

Overlap: unwrapping occurs at the opposite side of the overlap circle
  • each zero mode has a corresponding large real eigenvalue

Staggered: four tastes unwrap each other
  • rooting: unwrapping forces unphysical singularities
    at $m_q = 0$ even with non-degenerate quarks

SLAC fermions: unphysical singularities in momentum space

Minimal doubling
  • 2 species unwrap each other
Minimal doubling examples

(units with $a = 1$)

Karsten (1981); Wilczek (1987)

\[ K = i \sum_{i=1}^{3} \gamma_i \sin(p_i) + \frac{i\gamma_4}{\sin(\alpha)} \left( \sum_{\mu=1}^{4} \cos(p_\mu) - \cos(\alpha) - 3 \right) \]

- propagator poles along $p_4$ axis at $\vec{p} = 0$, $p_4 = \pm \alpha$

- strictly antihermitean kinetic term, exactly anticommutes with $\gamma_5$.
- similar to a Wilson term for space momenta with an $i\gamma_4$ factor
Tatsu Misumi

- “twisted ordering”

\[ K = i\gamma_1 (\sin(p_1) + \cos(p_2) - 1) \]
\[ i\gamma_2 (\sin(p_2) + \cos(p_3) - 1) \]
\[ i\gamma_3 (\sin(p_3) + \cos(p_4) - 1) \]
\[ i\gamma_4 (\sin(p_4) + \cos(p_1) - 1) \]

- propagator poles at \( p = (0, 0, 0, 0) \) and \( p = (\pi/2, \pi/2, \pi/2, \pi/2) \)

- \( 1 \rightarrow C \) shifts distance between poles
Graphene motivated: generalize $z = e^{ip_1} + e^{ip_2} + 1$ to

$$z = e^{+i\sigma_1 p_1} + e^{+i\sigma_1 p_2} + e^{-i\sigma_1 p_3} + e^{-i\sigma_1 p_4}$$
$$+ e^{+i\sigma_2 p_1} + e^{-i\sigma_2 p_2} + e^{-i\sigma_2 p_3} + e^{+i\sigma_2 p_4}$$
$$+ e^{+i\sigma_3 p_1} + e^{-i\sigma_3 p_2} + e^{+i\sigma_3 p_3} + e^{-i\sigma_3 p_4}$$
$$- 12C$$

- minimal doubling for $\frac{1}{2} < C < 1$.
- poles along major diagonal $p_\mu = p_\nu = \arccos(C)$

$C = 1/\sqrt{2}$ gives orthogonal lattice

$C = \cos(\pi/5)$ gives 4d analogue of graphene/diamond
Point splitting

One field gives two particles

- separate them by combining fields at nearby points

For the Karsten Wilczek form above

- start with free fields momentum space

\[
    u(q) = \frac{1}{2} \left( 1 + \frac{\sin(q_4 + \alpha)}{\sin(\alpha)} \right) \psi(q + \alpha e_4)
\]

\[
    d(q) = \frac{1}{2} \Gamma \left( 1 - \frac{\sin(q_4 - \alpha)}{\sin(\alpha)} \right) \psi(q - \alpha e_4)
\]

- factor of \( \Gamma = i\gamma_5\gamma_4 \) corrects for opposite chirality of one mode

Go to position space and insert gauge fields

\[
    u_x = \frac{e^{i\alpha x_4}}{2} \left( \psi_x + i \frac{e^{-i\alpha} U_{x,x-e_4} \psi_{x-e_4} - e^{i\alpha} U_{x,x+e_4} \psi_{x+e_4}}{2 \sin(\alpha)} \right)
\]

\[
    d_x = \frac{\Gamma e^{-i\alpha x_4}}{2} \left( \psi_x - i \frac{e^{-i\alpha} U_{x,x-e_4} \psi_{x-e_4} - e^{i\alpha} U_{x,x+e_4} \psi_{x+e_4}}{2 \sin(\alpha)} \right)
\]
Meson operators ($\alpha = \pi/2$):

$$
\pi_0(x) = \frac{i}{16} (4\bar{\psi}_x \gamma_5 \psi_x + \bar{\psi}_{x-e_4} \gamma_5 \psi_{x-e_4} + \bar{\psi}_{x+e_4} \gamma_5 \psi_{x+e_4}
- \bar{\psi}_{x+e_4} UU \gamma_5 \psi_{x-e_4} - \bar{\psi}_{x-e_4} UU \gamma_5 \psi_{x+e_4})
$$

$$
\eta'(x) = \frac{1}{8} (\bar{\psi}_{x-e_4} U \gamma_5 \psi_x - \bar{\psi}_x U \gamma_5 \psi_{x-e_4}
+ \bar{\psi}_{x+e_4} U \gamma_5 \psi_x - \bar{\psi}_x U \gamma_5 \psi_{x+e_4})
$$

- $\pi_0$ connects sites of the same parity
- $\eta'$ connects even and odd sites; no on-site contribution

Tatsu and Taro: use $\bar{u}u$ and $\bar{d}d$ fields to split masses

- variation on the Wilson term.
- use this form to study eigenvalue flow and the index theorem
Counterterms

All actions pick a special direction

- $s_\mu$ points between the two zeros
- hypercubic symmetry broken

Capitani, Weber, Wittig, MC

- introduces new perturbative counterterms
- the lattice gets distorted
  - the distance between the zeros
  - fermion speed of light
  - gluon speed of light

Three relevant operators

\[
\begin{align*}
\psi \gamma_\mu s_\mu \psi \\
\psi \gamma_\mu s_\mu s_\nu \partial_\nu \psi \\
F_{\mu\nu} F_{\mu\rho} s_\nu s_\rho
\end{align*}
\]

One operator of naive dimension 3, similar to Wilson

- two of dimension 4
Are the counterterms really necessary?
  - Need to remove the special direction

Not yet realized

Two possible approaches:

Zeros half way around Brilloin zone
  - $p_\mu = 0$ and $p_\mu = p_\nu = +\pi$
  - zeros in a body centered hyper-cubic arrangement

- adjusting $C$ above gives extra zeros before reaching this
Distort lattice to form a hyperdiamond

- make $0, 0, 0, 0$ to $p, p, p, p$ distance equal $p, p, p, p$ to $2\pi, 0, 0, 0$
- each site has five equidistant neighbors
- analog of graphene in 2-d

$C = \cos(\pi/5)$ reaches these points
- but the action as we travel along different bonds not equivalent
- brings in higher harmonics

Can non-nearest hoppings fix these examples?
- can we remain local over hypercubes?
Summary

Chiral symmetry is a symmetry in mass parameter space

Breaking of continuum chiral symmetry gives rich physics
  - spontaneous
  - anomaly
  - masses

Understanding this on the lattice is challenging but instructive

Minimal doubling maintains some chiral symmetry
  - fast for simulations
  - avoids rooting
  - counterterm questions remain