Non-perturbative beta functions and scaling laws in QCD with many quark flavors

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Yuki Kusafuka, Eri Ueno and H. T.
(Nara Women’s Univ. Japan)
Introduction

Conformal window of the many flavor QCD

- SU($N_c$) gauge theory with $N_f$ massless flavors
  - 2-loop gauge beta function \( \alpha_g = \frac{g^2}{(4\pi)^2} \)
    \[
    \beta_g^{[2]} \equiv \mu \frac{d\alpha_g}{d\mu} = -2b_0\alpha_g^2 - 2b_1\alpha_g^3
    \]
    \[
    b_0 = \frac{11}{3} N_c - \frac{2}{3} N_f
    \]
    \[
    b_1 = \frac{34}{3} N_c^2 - N_f \left( \frac{N_c^2 - 1}{N_c} + \frac{10}{3} N_c \right)
    \]

- IR (Banks–Zaks) fixed point
  - goes towards strong coupling region as $N_f$ decreases.

- Spontaneous breaking of the chiral symmetry for $N_f < N_{cr}$
  - Scale invariance is lost there. \( \Rightarrow \) Fixed point cannot exist!
  - Conformal window: \( N_{f,cr} < N_f < \frac{11}{2} N_c \)
Introduction

Boundary of the conformal window $N_{f \text{ cr}}$

Analysis of the Dyson–Schwinger equations


- DS equations with the fixed point gauge coupling are examined.
- The ladder approximation is used mostly.
- Motivation of study: application to New Technicolor models

Lattice simulations of the effective gauge coupling

Y. Iwasaki et al. PRL 69 (1992)
T. Appelquist et al. PRL 100 (2008); PRL 1 02 (2009); PRD 79 (2009)
A. Hasenfratz, PRD 80 (2009); PRD 82 (2010)
T. DeGrand, et al. PRD 80 (2009); PRD 82 (2010); PRD 83 (2011)
L. Del Debbio et al. PRD 81 (2010); PRD 82 (2010)
Z. Fodor et al. PL B 681 (2009); arXiv:0904.4662
M. Hyakawa et al. PRD 83 (2011)

$\Rightarrow \quad 8 < N_{\text{cr}} < 12$
Question: How does the beta function change vs. $N_f$?

- A fixed point is not allowed at the strong coupling region. Therefore the perturbative beta functions cannot transform into the asymptotically free one smoothly, then what happens?

お話しおり　Decoupling of fermions?

- Chiral symmetry breaking reduces the flavor number effectively. So the beta function may be modified at the strong coupling region.
- But a UV fixed point seems to appear in the strong coupling region for $N_f \geq N_{cr}$, which contradicts with chiral symmetry breaking.

No fixed point
We need some non-perturbative analyses of the beta functions in the conformal window.

Use of the Wilson (Exact) Renormalization Group

- **Scale invariance:**
  Dyson–Schwinger equations treating the chiral order parameters are useless in the conformal window. Also it would be difficult to study almost scale invariant theories by the Lattice MC simulation.

- **Non-perturbative calculation:**
  In the Wilson RG, renormalized theories can be defined by the renormalized trajectories (RTs), which are given as the continuum limit of the Wilson RG flows.

- **Beta function:**
  Then non-perturbative beta functions can be given by scale transformation on the RTs.

⇒ So the ERG is a quite suitable framework.
Non-perturbative beta function

Wilson RG

- **Wilsonian effective action**

  Integrating out higher momentum modes
  
  \[
  Z = \int_{|p| < \Lambda_0} D\phi(p) \exp(-S_0[\phi; \Lambda_0])
  = \int_{|p| < \Lambda} D\phi(p) \exp(-S_{\text{eff}}[\phi; \Lambda])
  \]

  \[
  S_{\text{eff}}[\phi; \Lambda] = \int d^Dx \sum_i \frac{g_i}{\Lambda^{d_i}} O_i[\phi]
  \]

  Wilsonian effective action contains infinitely many operators

- **Wilson RG**

  \[
  \Lambda \frac{dS_{\text{eff}}}{d\Lambda} = \mathcal{F}[S_{\text{eff}}], \quad \text{or} \quad \Lambda \frac{dg_i}{d\Lambda} = \beta_i \{g\}
  \]

  Legendre flow equation (Wetterich eq.) for the cutoff effective action

  \[
  \frac{\partial \Gamma_\Lambda}{\partial \Lambda} = \frac{1}{2} \int_p \text{tr} \left[ \frac{\partial R}{\partial \Lambda} \cdot \left( \frac{\delta^2 \Gamma_\Lambda}{\delta \phi_p \delta \phi_{-p}} \right)^{-1} \right]
  \]

K.G.Wilson, I.G.Kogut (1974)
Non-perturbative beta function

Scalar field theory as a toy model

- RG flows in \((\lambda_4, \lambda_6)\) space

  Operator truncation

  \[
  \mathcal{L}_{\text{eff}} = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{\lambda_4}{2!} \left(\frac{\phi^2}{2}\right)^2 - \frac{\lambda_6}{3! \Lambda^2} \left(\frac{\phi^2}{2}\right)^3
  \]

- Wetterich eqn (sharp cutoff limit)

  \[
  \Lambda \frac{d\lambda_4}{d\Lambda} = a \lambda_4^2 - b \lambda_6 \\
  \Lambda \frac{d\lambda_6}{d\Lambda} = 2 \lambda_6 - c \lambda_4^3 + 2d \lambda_4 \lambda_6
  \]

  \[a = 9A, \quad b = 10A\]  
  \[c = 27A, \quad d = (45/2)A\]  
  \[(A = 2/(4\pi)^2)\]

- Renormalized trajectory

  \(\Leftrightarrow\) renormalized theory
Non-perturbative beta function

- Renormalized trajectory
  - Perturbative analysis
    \[ \lambda_6^* = \frac{c}{2} \lambda_4^3 + \frac{c}{4} (3a - 2d) \lambda_4^3 - \frac{c}{8} (-12a^2 + 3bc + 14ad - 4d^2) \lambda_4^5 + \cdots. \]

- Non-perturbative beta function
  - Renormalized trajectory
    We can find the RT numerically without perturbative expansion.
    \[ \lambda_6 = \lambda_6^*(\lambda_4) \]
  - “Non-perturbative” beta function
    The beta function of a renormalized parameter is given by the scale transformation on the RT.
    \[ \beta_4(\lambda_4) = a \lambda_4^2 - b \lambda_6^*(\lambda_4) \]
    \[ = a \lambda_4^2 - \frac{bc}{2} \lambda_4^3 - \frac{bc}{4} (3a - 2d) \lambda_4^4 + \cdots \]
RG flow equations for SU(N) gauge theories

Wilsonian effective action

- **Four–fermi operators**
  - Important to describe the chiral symmetry breaking
  - Symmetries
    - Gauge symmetry : \( SU(N_c) \)
    - Chiral flavor symmetry : \( SU(N_f)_L \times SU(N_f)_R \)
    - Parity
  - 4 invariant four–fermi operators

\[
\mathcal{L}_{4f} = \frac{G_S}{\Lambda^2} \mathcal{O}_S + \frac{G_V}{\Lambda^2} \mathcal{O}_V + \frac{G_{V1}}{\Lambda^2} \mathcal{O}_{V1} + \frac{G_{V2}}{\Lambda^2} \mathcal{O}_{V2}
\]

\[
\mathcal{O}_S = 2 \bar{L}_i R^j \bar{R}_j L^i = \frac{1}{2} [\bar{\psi}_i \gamma^j \psi_j \bar{\psi}_i \gamma^i - \bar{\psi}_i \gamma^5 \psi_j \bar{\psi}_j \gamma^5 \psi_i]
\]

\[
\mathcal{O}_V = \bar{L}_i \gamma^\mu L^j \bar{L}_j \gamma_\mu L^i + (L \leftrightarrow R)
\]

\[
= \frac{1}{2} [\bar{\psi}_i \gamma^\mu \psi_j \bar{\psi}_j \gamma_\mu \psi_i + \bar{\psi}_i \gamma^\mu \gamma_5 \psi_j \bar{\psi}_j \gamma_\mu \gamma_5 \psi_i]
\]

\[
\mathcal{O}_{V1} = 2 \bar{L}_i \gamma^\mu L^i \bar{R}_j \gamma_\mu R^j = \frac{1}{2} [(\bar{\psi}_i \gamma^\mu \psi_i)^2 - (\bar{\psi}_i \gamma^\mu \gamma_5 \psi_i)^2]
\]

\[
\mathcal{O}_{V2} = (\bar{L}_i \gamma^\mu L^i)^2 + (L \leftrightarrow R) = \frac{1}{2} [(\bar{\psi}_i \gamma^\mu \psi_i)^2 + (\bar{\psi}_i \gamma^\mu \gamma_5 \psi_i)^2]
\]
RG flow equations for SU(N) gauge theories

Invariant four–fermi operators

**Apparent invariants**

\[ \psi^a_L = L^{ai}, \psi^a_R = R^{ai} \quad (a = 1, \cdots, N_c) \]
\[ (i = 1, \cdots, N_f) \]

\[ 2\bar{L}_{ai} \gamma^\mu L^{ai} \bar{R}_{bj} \gamma_\mu R^{bj} = \frac{1}{2} \left[ (\bar{\psi}_{ai} \gamma^\mu \psi^{ai})^2 - (\bar{\psi}_{ai} \gamma^\mu \gamma_5 \psi^{ai})^2 \right] \]

\[ 2\bar{L}_{ai} \gamma^\mu L^{bi} \bar{R}_{bj} \gamma_\mu R^{aj} = \frac{1}{2} \left[ \bar{\psi}_{ai} \gamma^\mu \psi^{bi} \bar{\psi}_{bj} \gamma_\mu \psi^{aj} - \bar{\psi}_{ai} \gamma^\mu \gamma_5 \psi^{bi} \bar{\psi}_{bj} \gamma_\mu \gamma_5 \psi^{aj} \right] \]

\[ \bar{L}_{ai} \gamma^\mu L^{ai} \bar{L}_{bj} \gamma_\mu L^{bj} + (L \leftrightarrow R) = \frac{1}{2} \left[ (\bar{\psi}_{ai} \gamma^\mu \psi^{ai})^2 + (\bar{\psi}_{ai} \gamma^\mu \gamma_5 \psi^{ai})^2 \right] \]

\[ \bar{L}_{ai} \gamma^\mu L^{bi} \bar{L}_{bj} \gamma_\mu L^{aj} + (L \leftrightarrow R) \]
\[ = \frac{1}{2} \left[ \bar{\psi}_{ai} \gamma^\mu \psi^{bi} \bar{\psi}_{bj} \gamma_\mu \psi^{aj} + \bar{\psi}_{ai} \gamma^\mu \gamma_5 \psi^{bi} \bar{\psi}_{bj} \gamma_\mu \gamma_5 \psi^{aj} \right] \]

\[ \bar{L}_{ai} \gamma^\mu L^{aj} \bar{L}_{bj} \gamma_\mu L^{bi} + (L \leftrightarrow R) \]
\[ = \frac{1}{2} \left[ \bar{\psi}_{ai} \gamma^\mu \psi^{aj} \bar{\psi}_{bj} \gamma_\mu \psi^{ai} + \bar{\psi}_{ai} \gamma^\mu \gamma_5 \psi^{aj} \bar{\psi}_{bj} \gamma_\mu \gamma_5 \psi^{bi} \right] \]

\[ \bar{L}_{ai} \gamma^\mu L^{bj} \bar{L}_{bj} \gamma_\mu L^{ai} + (L \leftrightarrow R) \]
\[ = \frac{1}{2} \left[ \bar{\psi}_{ai} \gamma^\mu \psi^{bj} \bar{\psi}_{bj} \gamma_\mu \psi^{ai} + \bar{\psi}_{ai} \gamma^\mu \gamma_5 \psi^{bj} \bar{\psi}_{bj} \gamma_\mu \gamma_5 \psi^{ai} \right] \]
RG flow equations for SU(N) gauge theories

**Fiertz identities**

\[
\bar{\psi}_1 \gamma^\mu \psi_2 \ \bar{\psi}_3 \gamma_\mu \psi_4 + \bar{\psi}_1 \gamma^\mu \gamma_5 \psi_2 \ \bar{\psi}_3 \gamma_\mu \gamma_5 \psi_4 \\
= \bar{\psi}_1 \gamma^\mu \psi_4 \ \bar{\psi}_3 \gamma_\mu \psi_2 + \bar{\psi}_1 \gamma^\mu \gamma_5 \psi_4 \ \bar{\psi}_3 \gamma_\mu \gamma_5 \psi_2 \\
= -\frac{1}{2} \left[ \bar{\psi}_1 \gamma^\mu \psi_4 \ \bar{\psi}_3 \gamma_\mu \psi_2 - \bar{\psi}_1 \gamma^\mu \gamma_5 \psi_4 \ \bar{\psi}_3 \gamma_\mu \gamma_5 \psi_2 \right]
\]

**Current–current interactions**

\[
2 \sum_{A=1}^{\text{dim}G} (T^A)^a_d \ (T^A)^c_b = \delta^a_b \ \delta^c_d - \frac{1}{N_c} \delta^a_d \ \delta^c_b
\]

\[
2 \sum_A \bar{L}_i T^A \gamma^\mu L^i \ \bar{R}_j T^A \gamma_\mu R^j = -\mathcal{O}_S - \frac{1}{2N_c} \mathcal{O}_{V1}
\]

\[
\sum_A \bar{L}_i T^A \gamma^\mu L^i \ \bar{L}_j T^A \gamma_\mu L^j + (L \leftrightarrow R) = \frac{1}{2} \mathcal{O}_V - \frac{1}{2N_c} \mathcal{O}_{V2}
\]

\[
\sum_A \bar{L}_i T^A \gamma^\mu L^j \ \bar{L}_j T^A \gamma_\mu L^i + (L \leftrightarrow R) = \frac{1}{2} \mathcal{O}_{V2} - \frac{1}{2N_c} \mathcal{O}_V
\]
Spontaneous breaking of the chiral symmetry

K.−I.Aoki, K.Morikawa, W.Souma, J.−I.Sumi, H.T.,M.Tomoyose,

\[ G_S \rightarrow \infty \quad \text{: Chiral symmetry breaking} \]

\[ \langle \bar{\psi}^i \psi^j \rangle = M^3 \delta^j_i \Rightarrow \quad SU(N_f)_L \times SU(N_f)_R \rightarrow SU(N_f)_V \]

Approximation scheme

Operator truncation

\[ \mathcal{L}_{\text{eff}} = -\frac{1}{4g^2} F^A_{\mu\nu} F^{A\mu\nu} + \bar{\psi} f i \slashed{D} \psi^f + \mathcal{L}_{4f} \]

We discard all gauge non–invariant corrections.

Note: Cutoff breaks gauge invariance. Gauge non–invariant
    corrections may be controlled by the modified WT identities.
RG flow equations (sharp cutoff limit)

**Four–fermi couplings**

\( g_i = G_i / 4\pi^2, \quad \alpha_g = g^2 / (4\pi)^2 \)


H.Gies, J.Jackel, EPJC 46 (2006)

\[
\Lambda \frac{dg_S}{d\Lambda} = 2g_S - 2N_c g_S^2 + 2N_f g_S g_V + 6g_S g_{V_1} + 2g_S g_{V_2} - 12C_2(F) g_S \alpha_g + 12g_{V_1} \alpha_g - \frac{3}{2} \left( 3N_c - \frac{4}{N_c} - \frac{1}{N_c^2} \right) \alpha_g^2
\]

\[
\Lambda \frac{dg_V}{d\Lambda} = 2g_V + (N_f / 4) g_S^2 + (N_c + N_f) g_V^2 - 6g_V g_{V_2} - \frac{6}{N_c} (g_V + g_{V_2}) \alpha_g - \frac{3}{4} \left( N_c - \frac{8}{N_c} + \frac{3}{N_c^2} \right) \alpha_g^2
\]

\[
\Lambda \frac{dg_{V_1}}{d\Lambda} = 2g_{V_1} - (1/4) g_S^2 - g_S g_V - 3g_{V_1}^2 - N_f g_S g_{V_2} + 2(N_c + N_f) g_V g_{V_1} + 2(N_c N_f + 1) g_{V_1} g_{V_2} + \frac{6}{N_c} g_{V_1} \alpha_g + \frac{3}{4} \left( 1 + \frac{3}{N_c^2} \right) \alpha_g^2
\]

\[
\Lambda \frac{dg_{V_2}}{d\Lambda} = 2g_{V_2} - 3g_V^2 - N_c N_f g_{V_1}^2 + (N_c N_f - 2) g_{V_2}^2 - N_f g_S g_{V_1} + 2(N_c N_f + 1) g_V g_{V_2} + 6(g_V + g_{V_2}) \alpha_g - \frac{3}{4} \left( 3 + \frac{1}{N_c^2} \right) \alpha_g^2
\]
RG flow equations for SU(N) gauge theories

Loop corrections for the four–fermi operators

\[ \begin{align*}
&\begin{array}{c}
\text{Note: Four–fermi couplings } g_{V1}, g_{V2} \text{ do not involve in the large } N_c \\
\text{and } N_f \text{ limit.}
\end{array} \\
&\begin{array}{c}
\text{Note: The large } N_c \text{ corrections contain only the ladder diagrams.}
\end{array} \\
&\begin{array}{c}
\text{But the non–ladder ones come through the large } N_f \text{ part.}
\end{array}
\end{align*} \]
RG flow equations for SU(N) gauge theories

**Gauge coupling**

- We use the perturbative beta functions in the large $N_c, N_f$ limit and **add a part of higher order corrections via the four-fermi effective couplings.**

1. **Vertex correction:**
   - We discard all vertex corrections with the four-fermi couplings, since the gauge symmetry should forbid them.

2. **Vacuum polarization:**
   - The higher order corrections via four-fermi effective operators should be incorporated into the vacuum polarization.

\[ \beta_g^{[2]} = -\frac{2}{3} (11 - 2r) \alpha_g^2 - \frac{2}{3} (34 - 13r) \alpha_g^3 \]

\[ \Lambda \frac{d\alpha_g}{d\Lambda} = \beta_g^{[2]} + 2rgV \alpha_g^2 \]
Aspect of RG flows

Numerical analysis of the flow equations

- RG flows in large $N_c$ and $N_f$
  - RG flows are given in 3 dimensional coupling space of $(\alpha_g, g_V, g_S)$.

- Fixed points in the conformal window
  - A UV fixed point exists as well as the IR fixed point.
  - The UV fixed point and the IR fixed point merge with each other at $r = 4.05$.

- RG flows in $(g_S, g_V)$ space
  - One linear combination of $g_S$ and $g_V$ gives the relevant operator, which induces the chiral phase transition.
Aspect of RG flows

- **RG flows in the 3D space**
  - There is the phase boundary of chiral symmetry and the UV fixed point lies on the boundary.
  - Flows in the unbroken phase approach towards the IR fixed point.
  - The phase boundary disappears for $r < 4.05$ and the entire region becomes the broken phase.

\[
N_f = 13
\]

\[
N_f = 12
\]
“Non-perturbative” gauge beta functions

RT in the conformal window

- Perturbative RT
  - We may extract the RT by solving the truncated RG flow equations in perturbative expansion.

\[ g_s^* = \frac{9}{4} \alpha_g^2 - \frac{9}{4} (-3 + 2b_0) \alpha_g^3 + \frac{9}{32} (90 - 120b_0 + 48b_0^2 - 16b_1 - 3r) \alpha_g^4 + \cdots, \]

\[ g_V^* = \frac{3}{8} \alpha_g^2 - \frac{3}{4} b_0 \alpha_g^3 + \frac{3}{128} (-3 + 96b_0^2 - 32b_1 - 30r) \alpha_g^4 + \cdots. \]

Note: These equations give continuum limit of the truncated ERG equation, not the full QCD.

- This “RT” does not seem to give a continuum limit.

- However this “RT” seems to survive for \( N_f > (11/2)N_c \).

Note: How about the RT of QED?
“Non-perturbative” gauge beta functions

RTs near boundary of the conformal window

- The perturbative continuum limit lines approaches towards the non-perturbative RT as the flavor number is lowered.
- Out of the window the fixed points disappears. However the non-perturbative RT survives as the RT of the asymptotically free QCD.
- The perturbative continuum limit seems to converge towards the RT.

\[ N_f = 13 \]

\[ N_f = 12 \]
Non-perturbative gauge beta functions

- We define the non-perturbative gauge beta function by scale transformation of the gauge coupling on the RTs.
- A UV fixed point appears in the gauge beta function due to higher order corrections generated through the four-fermi operators.
- This behavior is not due to chiral symmetry breaking.
- The UV fixed point does not appear in the strong coupling region.

“Conformality lost”

- The IR fixed point merges with the UV fixed point at the edge of conformal window.
Anomalous dimensions in many flavor QCD

**Critical flavor number**

- Analysis for the many flavor QCD for comparison with the lattice MC.
  - Solve the RG equations with a finite $N_f$, ($N_c=3$).
  - Use 2-, 3-, and 4-loop perturbative beta functions

- Non-perturbative gauge beta functions

- Critical flavor numbers
  
  $N_{f_{cr}} \simeq 12.78$ (2-loop)  
  $N_{f_{cr}} \simeq 11.24$ (3-loop)  
  $N_{f_{cr}} \simeq 11.58$ (4-loop)

Note: Lattice analyses indicate that QCD with 12 flavors is conformal.

*E.g.* E.Itou et.al. arXiv:10110516
Anomalous dimensions in many flavor QCD

Anomalous dimensions of fermion mass

- Anomalous dimension of $\bar{\psi}\psi$ in the ERG approach

$$\gamma_{\bar{\psi}\psi} = \begin{array}{c}
\hline
\hline

\end{array} + \begin{array}{c}
\hline
\hline

\end{array} = -6C_2(F')\alpha_g - 2N_c g_S + 4gV_1$$

- RG scheme and gauge independent at the fixed points

- Results by the RG equations
  
  Note: the anomalous dimension is fairly suppressed compared with the conventional value in the large N and ladder approx.

- Lattice MC results
  
  T.Appelquist et.al. (2011)

$$\gamma_{m^*} \simeq 0.386 \pm 0.010 \ (N_f = 12)$$

- The 3– and 4–loop results are close to the Lattice estimations.
Scaling laws in nearly conformal theories

Scaling of the dynamical scale in the broken phase

“Conformality lost” and the Miransky scaling


- Suppose that a UV fixed point and an IR fixed point merge.
  The beta function with an external parameter $c$ is given as
  \[ \beta(g; c) = \Lambda \frac{d g}{d \Lambda} = (c - c_{cr}) - (g - g_*)^2 \]
  Then the fixed point couplings are $g_\pm = g_* \pm \sqrt{c - c_{cr}}$

- BKT type phase transition for $\Lambda_{IR}/\Lambda_{UV} = \exp \left( \int_{g_{UV}}^{g_{IR}} \frac{d g}{\beta(g; c)} \right)$
  \[ \approx \exp \left( - \frac{\pi}{\sqrt{c_{cr} - c}} \right) \]

- Fermion mass generation $N_f \leq N_{cr}$
  Miransky scaling $m_f \sim M e^{-\frac{c}{\sqrt{N_{cr} - N_f}}}$
Scaling laws in nearly conformal theories

Approximation by a parabolic function

- We may approximate the RT as a parabolic function as follows;

1. Expand the RG flow equations around the critical fixed point.

\[ x^i = x_*^i + \tilde{x}^i : \text{effective couplings near a fixed point } x_*^i \]

\[ \Lambda \frac{d\tilde{x}^k}{d\Lambda} = M[x_*]^k_i \tilde{x}^i + \frac{1}{2} \frac{\partial^2 \beta^k}{\partial x^i \partial x^j} [x_*] \tilde{x}^i \tilde{x}^j + \cdots \]

\[ M[x_*]^k_i = \frac{\partial \beta^k}{\partial x^i} [x_*] \]

Note: An exactly marginal operator appears at fixed point merger.

The RT passes along the exactly marginal direction

\[ M[x_{cr}]^k_i u^i = 0 \]

2. Extract the beta function along the exactly marginal direction.

\[ \tilde{x}^i = \tilde{\alpha}_g u^i \Rightarrow \Lambda \frac{d\alpha_g}{d\Lambda} = -A(\alpha_g - \alpha_{gcr})^2 + \cdots \]

3. Find the (imaginary) fixed points \( \alpha_{g*1,2} \)

for a off–critical flavor number Nf.

\[ \beta_g = -A(\alpha_g - \alpha_{g*1})(\alpha_g - \alpha_{g*2}) + \cdots \]
Scaling laws in nearly conformal theories

- **Dynamical scale of the chiral symmetry breaking**
  - Running effect must be taken into account in the broken phase.
    
    J.Braun, C.S. Fischer, H. Gies arXiv: 10124279
  - The four-fermi coupling diverges at the dynamical scale $\Lambda_{SB}$.
    
    Note: Breakdown of the description in terms of the local fermi fields indicates the spontaneous chiral symmetry breaking.
  
  - Take difference with $\Lambda_{QCD}$ obtained by the 1-loop beta function.

- **Approximation for the scaling law**
  - Large deviation from the Miransky scaling.
  - Fit with perturbative beta functions + the parabolic function is good.
Scaling laws in nearly conformal theories

Scaling of the explicit fermion mass

- Recent lattice analyses
  - MC simulations of mass deformed QCD (adding a bare fermion mass)
    - L.Del Debbio, R.Zwicky, PRD82 014502 (2010); arXiv:1009.2894
    - Z.Foder et. al. arXiv:1104.3124

- Scaling law in the conformal window
  - The RG eqn for a fermion mass and IR enhancement
    \[ \Lambda \frac{dm}{d\Lambda} = -\gamma_{m^*}m \]
    \[ m(\Lambda) = \left( \frac{\Lambda_0}{\Lambda} \right)^{\gamma_{m^*}} m_0 \]
  - Decoupling at the scale of the fermion mass:
    \[ m(\Lambda = m_f) = m_f \]
  - The scaling law of the dimensionless mass parameter
    \[ \tilde{m}_f = m_f / \Lambda_0 \]
    \[ \tilde{m}_f = \tilde{m}_0 \frac{1}{1 + \gamma_{m^*}} \]
Scaling laws in nearly conformal theories

Scaling of the explicit fermion mass

Scaling laws in the slightly broken case

We may solve the RG equations for the effective couplings including the fermion mass $m$ on the RT numerically.

It seems to be difficult to distinguish whether the theory is conformal or chirally broken.

It is necessary to see dynamical mass generation to future work.

\[ SU(3) \; N_f = 12 \; (3\text{-loop}) \]

\[ SU(3) \; N_f = 11 \; (3\text{-loop}) \]
Scaling laws in nearly conformal theories

Scaling of the chiral condensate

- Hyperscaling relation in the conformal window

  We can also deduce the hyper scaling relation by the RG flow equations as

  \[
  \frac{\langle \bar{\psi} \psi \rangle}{\Lambda_0^3} = \frac{N_c N_f}{4\pi^2(1 - \gamma_{m^*})} \left( \tilde{m}_0 - \frac{2}{1 + \gamma_{m^*}} \tilde{m}_{0^*} \right) \quad (\tilde{m}_0 = m_0/\Lambda_0)
  \]

  \[
  \eta^*_* = \frac{3 - \gamma_{m^*}}{1 + \gamma_{m^*}}
  \]

  The linear term (contact term) is dominant for \( \gamma_{m^*} < 1 \).

  Therefore, it seems to be difficult to see the hyperscaling relation.
Summary and discussions

We extended the RG flow equations for the gauge couplings so as to include the “non-perturbative” corrections through the effective four-fermi operators.

We gave the non-perturbative gauge beta functions by scale transformation on the RT, which shows merge of the UV and the IR fixed points. ⇒ manifestation of the “Conformality Lost” picture.

The anomalous dimension of the fermion mass and the critical flavor number were evaluated for QCD by using the 3-, 4-loop beta functions.

Scaling of the dynamical scale was evaluated by using the beta functions.

The scaling relations of the fermion mass for the mass deformed QCD were also examined near the boundary of the conformal window.

Future issues

- Derivation of the RG flow equation for the gauge coupling including the four-fermi couplings by the ERG formalism.
- Evaluation of the chiral order parameters near the conformal boundary.
Many Thanks