One-dimensional Fermi gases: Density-matrix renormalization group study of ground state properties and dynamics

30 August 2011
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Plan of the talk

• Introduction
  – Low-dimensional cold atom systems
  – Density-matrix renormalization group
• Harmonically trapped imbalanced system
  – Larkin-Ovchinnikov state
• Optical lattice with disorder
  – Proving superfluidity by dynamics
• Collision dynamics
  – Non-classical reflectance
• Summary
Collaborators

• Masahito Ueda (University of Tokyo)
  – Harmonically trapped imbalanced system
    PRL 100, 110403 (2008)

• Antonio M. García-García (Cambridge University)
  – Optical lattice with disorder
    PRA 82, 043613 (2010)

• Jun’ichi Ozaki (Kyoto University)
• Norio Kawakami (Kyoto University)
  – Collision dynamics
  arXiv: 1107.0774
Low-dimensional cold atom systems

2D Bose-Einstein condensates (BEC) released and collide

⇒ Berezinskii-Kosterlitz-Thouless (BKT) transition is observed

Hadzibabic et al.: Nature 441, 1118 (2006)

Low-dimensional cold atom systems

Quasi-1D system in optical lattices

2D lattice of 1D systems

1D Bosons: absence of thermalization

Speckle potential for bosons

“1D” : Strong radial confinement
Radial level separation >> energy scale of axial motion
(Fermi energy in fermionic system)

- Low-dimensional but not necessarily solvable
- How to simulate numerically without uncontrollable approximations?
Density-Matrix Renormalization Group (DMRG)

System \( S = B + s + s' + B' \)

\[
\left| \Psi_{ijj'i'} \right\rangle = c_{ijj'i'} \sum_{ijj'i'} \phi_i \chi_j \chi_{j'} \phi_{i'}
\]

Partial density matrix for the left block \( L \) (\( k \equiv ij \))

\[
\rho_{kk'} = \sum_{i' j'} \left| \Psi_{kj'i'} \right\rangle \left\langle \Psi_{kj'i'} \right| = \sum_{i' j'} c_{kj'i'} c_{k'j'i'}^*
\]

Has all information on \( L = B + s \) when \( S \) is in the target state \( |\Psi\rangle \)

Diagonalize \( \rho \): states with larger eigenvalues are more important

cf. In NRG (numerical renormalization group) the lowest energy states are kept

Reduce the dimension to \( m \)
of the Hilbert space for \( L \)

= “renormalization”

Reviews:
Infinite system DMRG

Calculate the DM and reduce the dimension

Add two sites at the center

Iterate until the desired system size is reached
Finite system DMRG

One of the blocks can be made longer iteratively.

reduce the dimension to $m$

prepared in previous steps

we know the transfer matrix

wavefunction in new basis can be predicted

One of the blocks can be made longer iteratively
Finite system DMRG

Iterate until physical quantities (e.g. energy) converge

predicted wavefunction iteratively updated

m can be increased during the process for higher precision
Application of DMRG: Low-dimensional quantum systems

DMRG: variational method

Error in ground state energy is positive, and decrease as # of states \( m \) is increased

1D Heisenberg model (White: PRB 48, 10345 (1993))

\[
H = \sum_i S_i \cdot S_{i+1}
\]

Conjecture: gapped for spin \( N \in \mathbb{N} \), gapless for spin \( N+1/2 \) (Haldane PRL 1983)

\( S=1 \) Haldane gap detected

An improved method gives 0.4104792485(4) (Ueda and Kusakabe: PRB 84, 054446 (2011))

Eric Jeckelmann, April 16, 2010

http://www.itp.uni-hannover.de/~jeckelm/dmrg/paper_stat5.pdf
Application of DMRG: Low-dimensional quantum systems

Fractional Quantum Hall systems
(Landau levels: effectively 1D)

Shibata and Yoshioka: PRL 86, 5755 (2001)

Mixtures of bosons and fermions
- Spin-polarized half-filled fermions + bosons

Mering and Fleischhauser: PRA 81, 011603 (R) (2010)

Phase separation between CDW and Mott insulator

- Correlated electrons + phonons

Phase diagram of Hubbard-Holstein model
Tezuka, Arita and Aoki: PRB 76, 155114 (2007)

Also, quantum chemistry, 2D & 3D classical systems, ...
Two-component Fermi gas

Neutral atoms: bosons or fermions depending on parity of $A + Z$ (nucleon number + electron number)

Atom: fine structure, hyper fine structure electron spin $S$, orbital degrees of freedom, nuclear spin $I$

Fermions in two hyperfine states:
Loss due to three-body collisions is rare \(\leftrightarrow\) Pauli principle (pseudo-) spin population preserved

\(\Rightarrow\) Fermi gas with fixed spin imbalance can be realized

Feshbach resonance

A (highly excited) molecular state close to $E_B=0$ can modify the scattering length by controlling the internuclear distance.

Finite overlap of wavefunctions

Closed channel: Quantum numbers differ from initial state of

When the open and close channels differ in magnetic moment, we can utilize Rapid change of scattering length as function of magnetic field $B$

Feshbach resonance:

One bound (=molecular) state has binding energy = 0

$1/(k_Fa)>0$

$1/(k_Fa)=0$

$1/(k_Fa)<0$

$\Rightarrow$ interaction (including sign) can be controlled
For equal number of atoms,

**BEC-BCS crossover**

BEC: Bosons might break into fermions at energy $\Delta$, but $\Delta$ is not correlated with $T_c$.

BCS-type condensate: Pairing gap $\Delta$ is in proportion to $T_c$ (density determines $T_c$).

BEC of diatomic molecules: smoothly connected with BCS condensate?
Theory: Eagles (1969), Leggett (1980), Nozières and Schmidt-Rink (1985), ...

Experiment for the same number of Fermi atoms in two of the hyperfine states:

What happens when the numbers of two spins are not equal?

1) Harmonically trapped imbalanced system

Motivation: condensation of population imbalanced fermions in elongated traps

Partridge et al. (Rice): Science 311, 503 (2006)

\( \Rightarrow \) Discrepancy in cloud shape and maximum imbalance \( P \) for condensate

\[ P = \frac{N_{\uparrow} - N_{\downarrow}}{N_{\uparrow} + N_{\downarrow}} \]

Q. What happens when the atom trap is essentially 1D?

s-wave scatt. length \( (\propto g^{-1}) \)

\[ a_{1D} = -\frac{a_{\perp}^2}{2a} \left( 1 - C \frac{a}{a_{\perp}} \right) \]

Kinetic energy \( \ll \) level separation of the radial trap

Olshanii: PRL 1998

2D optical lattice \( \Rightarrow \) array of 1D traps realized
(Esslinger group, ETH Zürich; Hulet group, Rice)

### 3D: Controversy over experiments

<table>
<thead>
<tr>
<th></th>
<th>MIT</th>
<th>Rice</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>Trap aspect ratio $\lambda$</td>
<td>~10$^7$</td>
</tr>
<tr>
<td>$\sim 10^7$</td>
<td>Number of $^6\text{Li}$ atoms $N$</td>
<td>$\sim 10^5$</td>
</tr>
<tr>
<td>$\sim$equipotential surface</td>
<td>Density distribution</td>
<td>significant deformation</td>
</tr>
</tbody>
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$P_{CC}=75-80\%$

Upper limit for imbalance $P$ for condensation

$P_{CC}>95\%$

$P = (N_\uparrow - N_\downarrow)/(N_\uparrow + N_\downarrow)$

$|\uparrow\rangle$: majority

$|\downarrow\rangle$: minority

(two hyperfine states of $^6\text{Li}$)

What happens in 1D?

(Situation in 2006-2008)
Pair with finite momentum: FF and LO states

Different Fermi momentum for up and down spins
Pairs with non-zero momentum condense

LO: density difference oscillates with wave number $2q$

Fulde-Ferrell state
$\Delta \sim \exp(iqx)$

Larkin-Ovchinnikov state
$\Delta \sim \cos(qx)$

Density difference

Order parameter

Discretization of the space

Apply DMRG

Confining potential
\[ \phi(r) = A r^2 \]

Kinetic energy
\[ K(k) = \hbar^2 k^2 / 2m \]

Site potential
\[ \phi_i = A a^2 \left( i - \frac{L-1}{2} \right)^2 \]

K. E. dispersion
\[ K(k) = 2t (1 - \cos ka) \approx ta^2 k^2 \]

L-site Hubbard model
\[ H = -t \sum_{\langle i,j \rangle, \sigma = \uparrow, \downarrow} \left( \hat{c}_{i \sigma}^\dagger \hat{c}_{j \sigma} + \hat{c}_{j \sigma}^\dagger \hat{c}_{i \sigma} \right) + U \sum_i \hat{n}_{i \uparrow} \hat{n}_{i \downarrow} + \sum_{i, \sigma} \phi_i \hat{n}_{i \sigma} \]

cf. optical lattice systems

\[ \lambda = 2a \]

2V0
DMRG simulation of continuous system with the lattice introduced

Smaller lattice spacing $\rightarrow$ continuum limit approached

Density difference: shows oscillation incommensurate with lattice
Pair correlation and density distribution

Pair correlation

$$\langle \psi_0^{(N)} | \hat{c}_{i,\downarrow} \hat{c}_{i,\uparrow} \hat{c}_{j,\uparrow} \hat{c}_{j,\downarrow} | \psi_0^{(N)} \rangle \approx \Delta(z_i)^* \Delta(z_j)$$

imbalance parameter

$$P = \frac{N_{\uparrow} - N_{\downarrow}}{N_{\uparrow} + N_{\downarrow}}$$

M. Tezuka and M. Ueda, PRL 100, 110403 (2008)
Pair correlation and density distribution

Pair correlation

\[
\left\langle \psi_0^{(N)} \middle| \hat{c}_{i,\downarrow} \hat{c}_{i,\uparrow} \hat{c}_{j,\uparrow} \hat{c}_{j,\downarrow} \middle| \psi_0^{(N)} \right\rangle \approx \Delta(z_i)^* \Delta(z_j)
\]

imbalance parameter

\[
P = \frac{N_{\uparrow} - N_{\downarrow}}{N_{\uparrow} + N_{\downarrow}}
\]

\[P=0.05\]

\[0\]

\[-1\]

\[Z_i\]

\[Z_j\]

positive

negative

M. Tezuka and M. Ueda, PRL 100, 110403 (2008)
Pair correlation and density distribution

Pair correlation

\[
\langle \psi_0^{(N)} | \hat{c}_{i,\downarrow}^\dagger \hat{c}_{i,\uparrow}^\dagger \hat{c}_{j,\uparrow} \hat{c}_{j,\downarrow} | \psi_0^{(N)} \rangle \approx \Delta(z_i)^* \Delta(z_j)
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imbalance parameter

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Pair correlation

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\]

imbalance parameter

\[
P = \frac{N_{\uparrow} - N_{\downarrow}}{N_{\uparrow} + N_{\downarrow}}
\]

\(P=0.2\)

positive

negative

M. Tezuka and M. Ueda, PRL 100, 110403 (2008)
Pair correlation and density distribution

Pair correlation

\[
\langle \psi_0^{(N)} \mid \hat{c}_{i,\downarrow}^\dagger \hat{c}_{i,\uparrow}^\dagger \hat{c}_{j,\uparrow} \hat{c}_{j,\downarrow} \mid \psi_0^{(N)} \rangle \approx \Delta(z_i)^* \Delta(z_j)
\]

imbalance parameter

\[
P = \frac{N_{\uparrow} - N_{\downarrow}}{N_{\uparrow} + N_{\downarrow}}
\]

M. Tezuka and M. Ueda, PRL 100, 110403 (2008)
Pair correlation and density distribution

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M. Tezuka and M. Ueda, PRL 100, 110403 (2008)
Pair correlation and density distribution

Pair correlation

$$\langle \psi_0^{(N)} | \hat{c}_{i,\downarrow} \hat{c}_{i,\uparrow} \hat{c}_{j,\uparrow} \hat{c}_{j,\downarrow} | \psi_0^{(N)} \rangle \approx \Delta(z_i) \Delta(z_j)$$

imbalance parameter

$$P = \frac{N_\uparrow - N_\downarrow}{N_\uparrow + N_\downarrow}$$

$P = 0.7$

M. Tezuka and M. Ueda, PRL 100, 110403 (2008)
Condensate? – two-body density matrix

$N$ Fermions in $M$ states:
Maximum possible eigenvalue = $N(M-N+2)/M \sim N$
(C.N. Yang, RMP 1962)

⇒ Measure of pair condensation

$\rho^{(2)}_{ii',jj'} \equiv \langle \psi_0 | \hat{c}_{i',\downarrow}^{\dagger} \hat{c}_{i,\downarrow}^{\dagger} \hat{c}_{j,\uparrow} \hat{c}_{j',\downarrow} | \psi_0 \rangle$

$L^4$ matrix elements for $L$-site chain

Diagonalize to obtain eigenvalue distribution

Eigenfunction: state occupied by ($\uparrow$, $\downarrow$) pairs

cf. Condensate fraction for Bose gas ⇐ one-body DM
Most of minority atoms contribute to quasi-condensate
LO condensate at trap center

Pair correlation: periodic sign change

Population difference: constant + oscillation

\[ \Delta k_F = \pi \Delta n \] in 1D


\( P = 0.1 \)

\( P = 0.3 \)
Phase diagram

No up-only region

Up-only; Phase sep.

LO condensate + equal-population

LO + spin-polarized (phase separation)
1D: LDA (local density approximation) results

Exact solution for system without trap
(Yang’s generalied Bethe ansatz $\Rightarrow$ Gaudin’s integral equation)

Orso, PRL 98, 070402 (2007)  

$\Rightarrow$ Consistent with our DMRG results
1D Experiment (Rice group)

Density at central tube reconstructed

\[ T/T_F \sim 0.15 \text{ (fit to thermodynamic Bethe ansatz with LDA)} \]

Liao et al.: Nature 467, 567 (Sep 2010)

Majority

Minority

\[ P = 0.015 \]

\[ P = 0.055 \]

\[ P = 0.10 \]

\[ P = 0.33 \]

Our data at \( T=0 \)

(interaction strength NOT tuned)

Low enough for condensation?
Conclusion: (1)
Population imbalance + harmonic trap

- Pairing? \(\Rightarrow\) LO (quasi-) condensate
- Phase separation? \(\Rightarrow\) Yes (LO at center)
- Upper limit in imbalance \(P\) for condensation? \(\Rightarrow\) not observed

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see also:
Feiguin and Heidrich-Meisner: PRB 76, 220508R (2007); PRL 102, 076403 (2009) (Ladder);
Machida, Yamada, Okumura, Ohashi, and Matsumoto: PRA 77, 053614 (2008);

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M. Tezuka and M. Ueda:
PRL 100, 110403 (2008); NJP 12, 055029 (2010)
2) Optical lattice with disorder

Disorder: enhance? suppress?

Superconductivity

Superfluidity

Population imbalance = “magnetic field” : FFLO

Interaction: pairing force? other phase?

Optical trap shape

Feshbach resonance

Now widely controllable in cold Fermi atomic gases

- Optical lattice ← laser standing wave
  - Bichromatic lattice [Roati et al.: Nature 453, 895 (2008)]

- Holographic potential imprinting in 2D
Experimental realization

- Optical lattice \( \Leftrightarrow \) laser standing wave
  - Bichromatic lattice
- Another possibility: holographic potential imprinting in 2D
  - M. Greiner’s group
    - J. I. Gillen et al., PRA 80, 021602(R) (2009)
    - W.S. Bakr et al., Nature 462, 74 (2009)

Our motivation: what happens in 1D?

- Quantum fluctuation suppresses true long range order (even for \( T=0 \))
- Finite system: can have condensate (superfluid)
- Is coherence length \( O(\text{system size}) \)?
  
  Can be studied with numerically exact low-energy methods
  (Here we use DMRG)
Existing results

• Speckle potential (Gaussian random)
  – All eigenstates exponentially localized
    • L. Sanchez-Palencia et al.: PRL 98, 210401 (2007)
    • A.M. García-García and E. Cuevas: PRB 79, 073104 (2009)

• Fibonacci potential ABAABABA ...
  – Critical irrespective of the strength of $\lambda$

• Bichromatic potential
  “Aubry-Andre model”
  \[ V(n) \equiv \lambda \cos(2\pi\omega n + \theta) \]
  – non-interacting: metal-insulator transition at $\lambda=2J$ ($J$: hopping)
  – Numerical studies of interacting systems
    – Spinless Fermions: Chaves and Satija (ED, PRB 55, 14076 (1997)); Schuster et al. (PBC DMRG, PRB 65, 115114 (2002))
Quasiperiodic potential

Modeled by a single-band Hubbard model with site level modification
Formulation: Hubbard model + quasiperiodic potential

\[ \hat{H} = -J \sum_{i=1}^{L-1} \left( \hat{c}_{i+1,\sigma}^\dagger \hat{c}_{i,\sigma} + \text{h.c.} \right) + U \sum_{i=0}^{L-1} \hat{n}_{i,\uparrow} \hat{n}_{i,\downarrow} + \sum_{i=0}^{L-1} V(i) \hat{n}_i, \]

\[ V(n) = \lambda \cos(2\pi \omega n + \theta), \quad \omega = \frac{F_k}{F_{k+1}} \approx \frac{\sqrt{5} - 1}{2}, \quad L = F_{k+1} + 1 \]

\[ (F_0 = F_1 = 1, F_{k+2} = F_{k+1} + F_k) \]

- Ratio of consective Fibonacci numbers → golden ratio (=irrational number) as \( k \to \infty \)

- \((N_\sigma, L) = (10, 90), (26, 234), (42, 378) : \nu = 2/9\)

- Non-interacting case: all eigenstates become critical at \( \lambda = 2J \)

\( J \): unit of energy (=1)
\( U \): negative for attractive interaction
One-electron level scheme (non-interacting)

$\lambda=1$

$\lambda=2$

$\lambda=3$

Fermions localize for $\lambda>2$ for $|U|=0$

(Energy spectrum is fractal and changes smoothly as $\lambda$ is increased)
Interaction strength

Negative $U$: $|interaction\ energy|$ linearly increase as $|U|$ is increased.
Positive $U$: it has a peak because double occupancy is suppressed as $U \to$ large.

$t=1$
(band width=4)

$|U|=1$ : weakly interacting
$|U|=6$ : strongly interacting

band width

$\lambda=0$  
$0.3$  
$1$
How to detect pairing and delocalization?

**Pairing**

On-site pair correlation function:
easy to calculate with DMRG
depends on the site potentials of the site pair

Averaged equal-time pair structure factor

Sum of pair correlation for all lengths
average over sites

cf. Hurt et al.: PRB 72, 144513 (2005);
Mondaini et al.: PRB 78, 174519 (2008)

\[
\Gamma(i, r) \equiv \left\langle \hat{c}_{i+r}^\dagger \hat{c}_{i}^\dagger \hat{c}_{i} \hat{c}_{i+r} \right\rangle
\]

\[
P_s \equiv \left\langle \sum_r \Gamma(i, r) \right\rangle_i
\]

Increasing function of \(L\)
if decay of correlation is slow

**Delocalization**

Phase sensitivity: requires (anti-)periodic condition [see e.g. Schuster et al.: PRB 65, 115114 (2002)]
Hard to calculate within DMRG (not open BC) in large systems (OK for small systems)

Inverse participation ratio (IPR)

Add 2 atoms \(\Rightarrow\) How uniformly is the population increase distributed?

\[
I_E \equiv \left( \sum_i \left( \langle \hat{n}_i \rangle_{N+1,N+1} - \langle \hat{n}_i \rangle_{N,N} \right)^2 \right)^{-1}
\]

Compare between different system sizes
The case without disorder ($\lambda=0$)

Pair structure factor
indicator of global (quasi long-range) superfluidity

Inverse participation ratio
indicator of atom delocalization

Both increase with $|U|$, and system size $L$
$U = -1$:
Quasi long-range pairing disappears ($\lambda_p \sim 0.95$) before localization ($\lambda_c \sim 1.00$)

Inverse participation ratio

Tezuka and García-García: PRA 82, 043613 (2010)
$U = -6$:
Quasi long-range pairing disappears at localization ($\lambda_c \approx 0.30$)
Spin gap

Minimum energy to break a pair by spin flipping
\[ \Delta_s \equiv E_0(n+1, n-1) - E_0(n, n) \]
Continues to increase even after \( \lambda_c \)

Delocalized, superconducting pairs \( \lambda \) below sc-metal transition

Localized, tightly bound pairs at large \( \lambda \)
Schematic phase diagram

- Effect of coexisting disorder (bichromatic potential) and short-range attractive interaction
  - Studied for 1D fermionic atoms on optical lattice

- For strong attraction ($|U| \gg J$), pairing decreases as disorder $\lambda$ is increased, and localizes at $\sim$ insulating transition $\lambda_c$

- For weaker attraction ($|U| \sim J$), pairing has a peak as a function of disorder $\lambda$, but localizes before $\lambda_c$
What about dynamics?

- Many experiments observe the dynamics of the atomic clouds after release from a trap


Subdiffusion observed in bichromatic lattice (3D)

\[ V(x) = V_1 \cos^2(k_1 x) + V_2 \cos^2(k_2 x), \quad k_1 = \frac{2\pi}{1064.4\text{nm}}, \quad k_2 = \frac{2\pi}{859.6\text{nm}} \]

50 thousand $^{39}\text{K}$ atoms, almost spherical trap switched off at $t=0$

Initially $a=280a_0$ (repulsive), $\lambda \sim 3J$ (localized) $\Rightarrow$ tuned to final value within 10 ms

- Initially $E_{\text{int}}=0$
- $E_{\text{int}}=1.8J$
- $E_{\text{int}}=2.3J$

**What happens for interacting fermions?**
Does the phase depend on filling? What do we see?

If the phase diagram is sensitive to the filling

Density decrease after release may induce (de)localization

Intuitively,

The density is decreased as the atoms flow to the outer side of the system; density of states at the Fermi surface changes (not monotonously)

But we checked

The ground state phase diagram: does not depend strongly up to filling ~ 0.31 (per spin per site)

Small change of density of states does not affect $\lambda_{\text{loc}}$ strongly

Gap at ~31% filling
One parameter scaling theory

Abrahams et al.: PRL 42, 673 (1979)
see also Garcia-Garcia and Wang: PRL 100, 070603 (2008)

$L$ : system size, $\delta$ : (one-particle) mean level spacing

Dimensionless conductance

$$g(L) = \frac{E_T}{\delta}$$

$E_T$ : Thouless energy
(related to typical time for particle to travel $L$)

$(d>2)$D Normal metal: $g(L) \propto L^{d-2} \to \infty$

$\therefore E_T \propto L^{-2}, \delta \propto L^{-d}$

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Metal-insulator transition: $g(L) = g_c$ ---

Insulator: $g(L) \propto \exp(-L/\xi) \to 0$

$\langle x^2(t) \rangle \propto t^\alpha$

Motion slowed down,
$\sim L^{2/\alpha}$ time to propagate $L$,

$E_T^{-1} \propto t^{-2/\alpha}, \alpha < 1$

Multifractal spectrum with
Hausdorff dimension $d_H$

$\Rightarrow \delta \propto L^{-d/d_H}$

MIT should occur at $\alpha = 2d_H/d = 2d_H$
(has been checked for the non-interacting case;
Artuso et al.: PRL 68, 3826 (1992);
Piechon et al.: PRL 76, 4372 (1996))

Localization length $\xi$: should diverge as $|\lambda-\lambda_c|^{-\nu}$
$(\nu=1$ at $U=0)$
Near metal–insulator transition

Disordered 1D system, $U < 0$

- $|U| \to 0$
  - Diffusion $\langle x^2(t) \rangle \propto t^\alpha$
  - $\alpha \sim 1$
  - Brownian motion

- Intermediate $|U|$
  - Hausdorff dimension of the spectrum $d_H$
  - $d_H \sim 0.5$
  - See e.g. Artuso et al.: PRL 68, 3826 (1992)

- $|U| \to \infty$
  - $\alpha \sim 2$
  - Ballistic motion

- Localization length close to transition
  - $\xi \propto |\lambda - \lambda_c|^{-\nu}$
  - $\nu \sim 1$

- One parameter scaling
  - $\alpha = 2d_H$ at MIT
  - $d_H \sim 1$

- Our conjecture
  - $\nu = 1/(2d_H) = 1/\alpha$?
  - $\nu \sim 1/2$
  - Mean-field like; similar to Cayley tree

→ Let us numerically check by studying the dynamics
Setup (1)

- Optical lattice + incommensurate potential (Aubry-André model)
- On-site attractive interaction
- Initially trapped in a harmonic potential

• Remove the harmonic potential but keep the incommensurate potential on: what happens?

⇒ Study by time-dependent DMRG
Finite system DMRG

Iterate until physical quantities (e.g. energy) converge
Time-dependent DMRG

Application of $\exp(-i\tau H_{ij})$ is almost exact if $H_{ij}$ only affects neighboring sites $i, j$

$T/\tau$ finite system iterations to reach time $T$

\[
\hat{H} = \hat{H}_{1,2} + \hat{H}_{2,3} + \cdots + \hat{H}_{L-2,L-1} + \hat{H}_{L-1,L}
\]

$$\exp(-i\tau \hat{H}) = \exp(-i\tau \hat{H}_{L/2,L/2+1}/2) \cdots \exp(-i\tau \hat{H}_{2,3}/2) \exp(-i\tau \hat{H}_{1,2})$$

$$\exp(-i\tau \hat{H}_{L-2,L-1}/2) \exp(-i\tau \hat{H}_{L/2,L/2+1}/2) + O(\tau^3)$$

Operator applied to the wavefunction at each step
Time-dependent DMRG: other schemes

Feiguin and White: PRB 72, 020404 (2005)
Dutta and Ramasesha: PRB 82, 035115 (2010)

Time-evolving block decimation (TEBD) has also been applied to cold atom systems

*e.g.* Macroscopic quantum tunneling between different supercurrent states in a ring
Danshita and Polkovnikov: PRB 82, 094304 (2010)
Applications of time-dependent DMRG on cold atom systems

Propagating 1D density waves of bosons in optical lattice
Kollath et al.: PRA 71, 053606 (2005)

Spin-charge separation in S=1/2 Hubbard model
Kollath et al.: PRL 95, 176401 (2005)

Spin-charge separation in two-component Bose-Hubbard model
Kleine et al.: PRA 77, 013607 (2008)
Applications of time-dependent DMRG on cold atom systems

Andreev-like reflection in bosons
Daley et al.: PRL 100, 110404 (2008)

Optical superlattice quenched
Yamamoto et al.: JPSJ 78, 123002 (2009)
Setup (1)

- Optical lattice + incommensurate potential (Aubry-André model)
- On-site attractive interaction
- Initially trapped in a harmonic potential

- Remove the harmonic potential but keep the incommensurate potential on: what happens?

→ Study by time-dep. DMRG for Hubbard model
Non-interacting case

Smooth diffusion until bouncing back at (artificial) system boundary
\[ \lambda = 0.5 < \lambda_c(U=-1) \]

Atoms quickly flow to both sides.
\[ \lambda = 1.0 \sim \lambda_c(U=-1) \]

Significantly slower diffusion
\[ \lambda = 1.5 > \lambda_c(U=-1) \]

Atoms move only locally
Setup (2)

- Optical lattice + incommensurate potential (Aubry-André model)
- On-site attractive interaction
- Initially trapped in a box potential

- Remove the box potential but keep the incommensurate potential on: what happens?

→ Study by time-dep. DMRG for Hubbard model
Check: energy is nearly preserved

160 sites, 12+12 fermions, $\Delta t = 0.01$

$U=-1$

$U=-6$

Energy change

Time

Energy change

Time

$\lambda=0.75$

$\lambda=0$

$\lambda=0.65$

$\lambda=0.1$

$\lambda=0.95$

$\lambda=0.2$

$\lambda=1.05$

$\lambda=0.25$

$\lambda=1.15$

$\lambda=0.3$

$\lambda=0.35$

$t=100$

(10^4 steps)
Suppressed motion for $\lambda > \lambda_c$

200 sites, 18 up and 18 down fermions, $U=-1$

$\lambda=0.6$ $\lambda=0.9$ $\lambda=1.1$

$\lambda_c \sim 1.0$
3) Collision dynamics

Elongated two-component Fermi gas: up and down spins released from separate traps to collide (“Little Fermi Collider”)

cf. 1D Bosons: absence of thermalization

Spin diffusion constant minimum at Feshbach resonance; converges as $T \to 0$

What happens at 1D?

“Universal spin transport in a strongly interacting Fermi gas”
Sommer et al.: Nature 472 201 (2011)
Motivation

What kind of many-body effects are observed during a single collision between two one-dimensional fermion clusters?

- A spin-dependent harmonic trap
- Quenched to a shared potential
- The fermions collide at the trap center

Model: Hubbard model
Method: time-dependent DMRG
Example
(7+7 atoms, ~55% reflectance)

$U/J = 0.80$
Weak interaction: most of atoms are not reflected

Particle reflectance for $n + n$ atoms $R_n$: $R_1 \propto u^2 \ (u \to 0)$

Quasi-classical model

Quasi-classical model: a series of one-to-one collisions between two types of independent classical particles

(i) $u \to 0$:
$n^2$ times of independent spin-up and spin-down collisions

$\Rightarrow R_{qc} = n^2 R_1/n = nR_1$

Consistent with the simulation
Strong interaction: most of atoms are reflected back

Transmittance for \( n + n \) atoms \( T_n : T_1 \propto u^{-2} (u \to \infty) \), \( R_n + T_n = 1 \)

Quasi-classical model:

A series of one-to-one collisions between two types of *independent* classical particles

(ii) \( u \to \infty \):

\( n \) atoms collide successively against \( n \) atoms \( \Rightarrow T_{qcn} = nT_1/n = T_1 \)

Inconsistent with the simulation!

What is going on?
The work is in progress.
Summary

• Application of DMRG for static and dynamic behavior of Fermi cold atom gases in 1D
  – Population-imbalanced gas in harmonic trap: FFLO-like condensate
    \[ \text{PRL 100, 110403 (2008); New J. Phys 12, 055029 (2010)} \]
  – Quasiperiodic disorder
    • Can enhance condensation for weak attraction \[ \text{PRA 82, 043613 (2010)} \]
    • Trap-release dynamics close to metal-insulator transition: anomalous diffusion observed
      \[ \text{arXiv: 1107.0774} \]
  – Collision of spin clusters: more atoms pass through than quasi-classically expected = emergent many-body behavior

• There is much more to explore with powerful numerical methods!