Nonperturbative functional renormalization group for disordered systems: The case of the random field Ising model

G. Tarjus and M. Tissier
(LPTMC, CNRS-Univ. PARIS 6)
Physics of “disordered systems”

- Systems in the presence of **quenched disorder** (due to impurities, dislocations, random environment, etc, frozen on the relevant time scale) pose new challenges to statistical physics:
  - new phases and phase transitions (spin glass, glassy phases, Griffiths phases,...)
  - new phenomena (localization, pinning,...)
  - slow relaxation, aging and hysteresis.

- One often needs new theoretical tools
  => Nonperturbative functional RG (NP-FRG)

- **Here**: focus on the equilibrium behavior of classical systems.
Random field model

• Prototypical model in theory of "disordered systems"

In the field-theoretical description (RFIM):

\[ S_h[\phi] = S_B[\phi] - \int_x h(x)\phi(x); \quad S_B = \int d^d x \left\{ \frac{1}{2} (\partial_\mu \phi(x))^2 + \frac{\tau}{2} \phi(x)^2 + \frac{u}{4!} \phi(x)^4 \right\} \]

with a quenched random field drawn from a given probability distribution (e.g., Gaussian)

\[ \overline{h(x)} = 0, \quad \overline{h(x)h(y)} = \Delta_B \delta^{(d)}(x - y) \]

• Physical realizations in soft and hard condensed matter:
  * Near critical fluids in disordered porous materials
  * Dilute antiferromagnets in a uniform magnetic fluid
  * Hysteresis in dirty magnets
  * Vortex phases in disordered type-II superconductors
Generic difficulties of disordered systems

• Due to quenched disorder \((h)\), one loses translational invariance.
  
  Way out: average over disorder, but what ?, how ?

• Presence of many low-energy (low-action) “metastable states”.

• Possible influence of rare events, rare spatial regions or rare samples.
Average over the disorder
[“self-averaging”, “replica trick”, etc.]

• RFIM equilibrium partition function in a given random-field sample $h$:

\[
Z_h[J] = e^{Wh[J]} = \int \mathcal{D}\phi \, e^{-S_h[\phi] + \int_x J(x)\phi(x)}
\]

• $Wh[J]$ is a random functional of the source $\Rightarrow$
  * in principle, one needs its whole probability distribution
  * or equivalently, the infinite set of its disorder-averaged cumulants:

\[
W_1[J] = \overline{Wh[J]}, \quad W_2[J_1, J_2] = \overline{Wh[J_1]Wh[J_2]}_c, \ldots
\]
Known results about the RFIM

• Existence of a $Z_2$ symmetry breaking transition for $d>2$ for the Ising version [transition for $d<4$ for the $O(N>2)$ version]. The upper critical dimension is $d_{uc}=6$.

• The critical behavior is associated with a zero-temperature fixed point (thermal fluctuations are formally irrelevant) and one can directly work at $T=0$.

• For a given realization of the disorder $h(x)$, the ground state is unique [except for rare values/configurations of the external source $J(x)$].
Known results about the RFIM (contd.)

Zero-temperature fixed point and its consequences

- **Additional exponent for the temperature flow:** \( \theta > 0 \)

- **Two distinct pair correlation functions:**
  
  \[
  \langle \phi(x) \rangle \langle \phi(x') \rangle \sim \frac{1}{|x - x'|^{d-4+\eta}}, \quad \text{with} \quad \theta = 2 + \eta - \bar{\eta},
  \]
  
  \[
  \langle \phi(x) \phi(x') \rangle - \langle \phi(x) \rangle \langle \phi(x') \rangle \sim \frac{T}{|x - x'|^{d-2+\eta}}
  \]

- **For \( T>0 \):** very slow “activated” critical dynamics, \( \tau \sim \exp(c \xi^\theta) \), with \( \xi \) the correlation length (that diverges at the critical point).
• At zero temperature, the equilibrium behavior of the RFIM is determined by the ground state configuration [absolute minimum of $S_h = S_B - (h + J)\phi$], which is solution of the stochastic field equation:

$$\frac{\delta S_B[\phi]}{\delta \phi(x)} = h(x) + J(x)$$

• However, for low disorder strength and in the region of interest (near the critical point), the equation has many solutions => many minima of the bare action ("metastable states").

• What is their effect on the long-distance properties?
  [Also known to go with slow relaxation, hysteresis and "glassiness"]
At $T=0$, generating functional of the correlation functions:

$$\mathcal{Z}_h[J, \hat{J}] = \int \mathcal{D}\phi \delta\left[ \frac{\delta S_B[\phi]}{\delta \phi} - h - J \right] \left| \frac{\delta^2 S_B[\phi]}{\delta \phi \delta \phi} \right| e^{\int_x \hat{J}(x) \phi(x)}$$

If there is a unique solution of the stochastic field equation, usual manipulations:

- Introduce auxiliary fields $\hat{\phi}(x), \psi(x), \bar{\psi}(x)$,
- average over disorder $h$ (Gaussian probability distribution),
- introduce a superspace with 2 Grassmann coordinates $\underline{x} = (x, \bar{\theta}, \theta)$
- and supermetric $d\underline{x}^2 = dx^2 + \frac{4}{\Delta_B} d\bar{\theta}d\theta$,
- a superLaplacian $\Delta_{SS} = \partial_\mu^2 + \Delta_B \partial_\theta \partial_{\bar{\theta}}$,
- a superfield $\Phi(x) = \phi(x) + \bar{\theta}\psi(x) + \bar{\psi}(x)\theta + \bar{\theta}\theta \hat{\phi}(x)$, super-etc...
Parisi-Sourlas supersymmetric approach of the RFIM (contd.)

• The generating functional $\mathcal{Z}_h$ can then be obtained from a superfield theory with action:

$$S_{SUSY}[\Phi] = \int_x \left\{ -\frac{1}{2} \Phi(x) \Delta_{SS} \Phi(x) + \frac{\tau}{2} \Phi(x)^2 + \frac{u}{4!} \Phi(x)^4 \right\}$$

• Invariant under SUSY (super-rotations in superspace)
  => leads to "dimensional reduction": RFIM in $d$ dimensions is equivalent to the pure theory in $d-2$.

$$\left[ \int d^d x d\theta d\bar{\theta} f(x^2 + \frac{4}{\Delta_B} \theta \bar{\theta}) = \left( \frac{4\pi}{\Delta_B} \right) \int d^{d-2} x f(x^2) \right]$$

Beautiful, but wrong!!
Problem with multiple solutions!!
Rare events: toy model (d=0 RFIM)

Stochastic equation: \[ \frac{\delta S_B(\phi)}{\delta \phi} = J + h \quad \text{with} \]

- **For zero temperature** \( T=0 \), select the ground state:

  The pair correlation function for slightly different sources,

  \[
  \frac{\phi_{GS,h}(J + \delta J)\phi_{GS,h}(J - \delta J)}{\phi_{GS,h}(J)^2 + A(J)|\delta J| + O(\delta J^2)} =
  \]

  has a nonanalytic behavior (a “cusp”) when \( J \rightarrow 0 \) due to the “avalanches”.

- **For a small** \( T>0 \), Boltzmann weighting of the minima ( \( e^{-\frac{S_B(\phi) - (J+h)\phi}{T}} \)):

  The cusp is rounded in a “thermal boundary layer”

  \[
  < \phi(J + \delta J + h) > < \phi(J - \delta J + h) > = < \phi(J + h) >^2 + T f(J, \frac{|\delta J|}{T}) + \ldots
  \]

  **ISSUE:** Does this persist at long distance when \( d>0 \)?
Long-standing puzzles concerning random-field systems

- Critical behavior: what is the way out of dimensional reduction?
- What is the phase diagram of the $d$-dimensional random-field $O(N)$ model in the whole $(N,d)$ plane?
Why does one need a nonperturbative functional RG?

- **RG**, because one is interested in the long-distance properties near to the critical point; in particular, the “metastable states” of potential relevance are not those of the bare action but those of a scale-dependent renormalized functional;

- **Functional**, because the influence of the rare events (avalanches and droplets) can only be described through a singular dependence of the cumulants of the renormalized disorder on their arguments;

- **Nonperturbative**, because standard perturbation theory completely fails (dimensional reduction), relevant dimensions are far from $d=6$, disorder grows strong under coarse graining.
Program for RG study of RFIM  
[Search for the proper T=0 IR (critical) fixed point]

• Select with high probability the ground state at the running IR scale \( k \) among the solutions if several of them and ensure that only the ground state is considered when \( k \rightarrow 0 \).

• Describe full functional dependence of cumulants of renormalized disorder and allow for nonanalytical dependence on their arguments.

• Start the RG flow with a “regularized” stochastic field equation having a unique solution.

• Use a nonperturbative truncation and be able to recover dimensional reduction if it has a range of validity.

=> NP-FRG in a superfield setting
Superfield formalism for the RFIM

- Several copies + a weighting factor => Generating functional:

\[
Z_h^{(\beta)} \{J_a, \hat{J}_a\} = \prod_a \int \mathcal{D}\phi_a \delta \left[ \frac{\delta S_B[\phi_a]}{\delta \phi_a} - h - J_a \right] \det \left( \frac{\delta^2 S_B[\phi_a]}{\delta \phi_a \delta \phi_a} \right) \\
\times e^{\int_x \hat{J}_a(x) \phi_a(x)} e^{-\beta \left( S_B[\phi_a] - \int_x [h(x) + J_a(x)] \phi_a(x) \right)}
\]

Average over disorder generates cumulants with full functional dependence:

\[
Z_h \{J_a, \hat{J}_a\} = \prod_a e^{\mathcal{W}_h[J_a, \hat{J}_a]} = e^{\sum_a \mathcal{W}_h[J_a, \hat{J}_a] + \frac{1}{2} \sum_{ab} \mathcal{W}_h[J_a, \hat{J}_a] \mathcal{W}_h[J_b, \hat{J}_b]_c + \ldots}
\]

- Introduce superfields and a “curved” Grassmannian space

\[
\Phi(\theta) = \phi + \bar{\theta} \psi + \bar{\psi} \theta + \bar{\theta} \bar{\psi} \phi; \quad \int_{\tilde{\theta}} = \int \int d\theta d\bar{\theta} (1 + \beta \bar{\theta} \theta)
\]

\[
\Rightarrow \quad S^{(\beta)} \{\Phi_a\} = \sum_a \int_{\tilde{\theta}} S_1[\Phi_a(\theta)] + \frac{1}{2} \sum_{ab} \int_{\tilde{\theta}_1 \tilde{\theta}_2} S_2[\Phi_a(\theta_1), \Phi_a(\theta_2)]
\]

\[
S_1 = \int_x \left[ \frac{1}{2} \left( \partial_{\mu} \Phi_a(\theta, x) \right)^2 + U_B(\Phi_a(\theta, x)) \right]; \quad S_2 = \int_x \Delta_B \Phi_a(\theta_1, x) \Phi_b(\theta_2, x)
\]
Add coupling to supersources \( \sum_a \int_{\theta,x} \mathcal{J}_a(\theta, x) \Phi_a(\theta, x) \rightarrow \mathcal{W}^{(\beta)}[\{\mathcal{J}_a\}] \)

+ Legendre transform \( \rightarrow \) Effective action \( \Gamma^{(\beta)}[\{\Phi_a\}] \)

The action is invariant under a large group of symmetries and supersymmetries (\( S_n \) between copies, global \( \mathbb{Z}_2 \) and Euclidean translations + rotations, isometries of the curved Grassmann subspace copy by copy).

The expansion in increasing number of sums over copies generates the “cumulant expansion” of the 1PI generating functional (effective action):

\[
\Gamma^{(\beta)}[\{\Phi_a\}] = \sum_{a_1} \Gamma_1^{(\beta)}[\Phi_{a_1}] - \frac{1}{2} \sum_{a_1, a_2} \Gamma_2^{(\beta)}[\Phi_{a_1}, \Phi_{a_2}] + \cdots
\]
NP-FRG in superfield formalism

• Add an IR regulator to the action:

\[ \Delta S^{(\beta)}_k[\{\Phi_a\}] = \frac{1}{2} \sum_{ab} \int_{x_1} \int_{x_2} \Phi_a(x_1) \mathcal{R}_{k,ab}(x_1, x_2) \Phi_b(x_2) \]

\[ \mathcal{R}_{k,ab}(x_1, x_2) = \delta_{\theta_1, \theta_2} \hat{R}_k(q^2) + \tilde{R}_k(q^2) \] : suppresses fluctuations of \( \Phi \) field and random field

• ERGE for the effective average action at scale \( k \):

\[
\partial_k \Gamma^{(\beta)}_k[\{\Phi_a\}] = \frac{1}{2} \sum_{ab} \int_{x_1} \int_{x_2} \partial_k \mathcal{R}_{k,ab}(x_1, x_2) (\Gamma^{(2)}_k[\{\Phi_a\}] + \mathcal{R}_k)^{-1}(b, x_2)(a, x_1)
\]

• Through the expansion of \( \Gamma^{(\beta)}_k[\{\Phi_a\}] \) in increasing number of copies:

Hierarchy of coupled ERGE’s for the cumulants (functionals of the superfields):

\[ \partial_k \Gamma^{(\beta)}_{k,1}[\Phi_1] = \cdots, \partial_k \Gamma^{(\beta)}_{k,2}[\Phi_1, \Phi_2] = \cdots, \text{ etc} \]
“Grassmannian ultralocality” and superrotational invariance

• Property of the generating functionals when a unique solution of the stochastic equation is included:

“Grassmannian ultralocality”: \( \mathcal{W}_h^{(\beta)}[\mathcal{J}] = \int_\theta W[\mathcal{J}(\theta)] \)

• When \( \beta \to \infty \), “ultralocality” (UL) becomes exact, with the \( p \)th cumulant of the effective average action given by (more later!)

\[
\Gamma^{(\beta)}_{k,p}[\Phi_{a_1}, ..., \Phi_{a_p}] = \\
\int_{\theta_{a_1}} ... \int_{\theta_{a_p}} \left( \Gamma^{(UL)}_{k,p}[\Phi_{a_1}(\theta_{a_1}), ..., \Phi_{a_p}(\theta_{a_p})] + NUL \text{ corrections} \right)
\]

• When “Grassm. UL”, \( \beta \) drops out of the FRG equations.

Then, for supersources that reduce the theory to a 1-copy problem, the theory is invariant under superrotations (SUSY)

\( \Rightarrow \) Ward-Takahashi (WT) identities.
NP-FRG and SUSY breaking

• Grassm. ultralocality => hierarchy of ERGE’s for the cumulants with physical field arguments \((\Phi \equiv \phi)\):

\[
\begin{align*}
\partial_t \Gamma_{k1}[\phi] &= \frac{1}{2}\tilde{\partial}_t Tr\{ [\Gamma_{k1}^{(2)}[\phi] + \hat{R}_k]^{-1} [\Gamma_{k2}^{(11)}[\phi, \phi] - \hat{R}_k] \} \\
\partial_t \Gamma_{k2}[\phi_1, \phi_2] &= \cdots \\
\end{align*}
\]

!!! Recall: The auxiliary parameter \(\beta\) then drops out of the ERGE’s !!!

• As a result, superrotational invariance for 1 copy is \textit{a priori} preserved along the RG flow: From the WT identities, one can show that it leads (nonperturbatively) to dimensional reduction.

• \textbf{What can go wrong ?}

  * Spontaneous breaking of superrotation invariance: some 1PI vertex blows up when copy fields become equal.
  * Dimension reduction is broken when a cusp

\[
\Gamma_{k,2}^{(11)}(\varphi_1, \varphi_2) - \Gamma_{k,2}^{(11)}(\varphi_1, \varphi_1) \sim |\varphi_2 - \varphi_1| \quad \text{as} \quad \varphi_2 \to \varphi_1
\]

appears at a \textbf{finite} scale \(k_L\).
SUSY-compatible approximation and RG flow

• Ansatz for effective average action (under “Grassm. ultralocality”):

\[
\Gamma_{k1}[\phi] = \int_x \left[ U_k(\phi(x)) + \frac{1}{2} Z_k(\phi(x))(\partial_\mu \phi(x))^2 \right]
\]

\[
\Gamma_{k2}[\phi_1, \phi_2] = \int_x V_k(\phi_1(x), \phi_2(x)), \quad \Gamma_{k,p>2} = 0
\]

+ Regulators: \( \hat{R}_k = Z_k k^2 r(q^2/k^2) \), \( \tilde{R}_k = - (\Delta_k/Z_k) \partial_q^2 \hat{R}(q^2) \)

[ SUSY WT identity: \( \Delta_k = \Delta B Z_k \) ]

• Introduce scaling dimensions for T=0 fixed point (critical). Then,

\[
\partial_t u'_k(\phi) = \cdots
\]

\[
\partial_t z_k(\phi) = \cdots
\]

\[
\partial_t \delta_k(\phi_1, \phi_2) = \partial_t v_k^{(11)}(\phi_1, \phi_2) = \cdots
\]

\[
\eta_k = -\partial_t Z_k \quad \bar{\eta}_k = 2\eta_k + \partial_t \Delta_k
\]

• If no linear cusp in \( \delta_k(\phi_1, \phi_2) \), then \( \partial_t \delta_k(\phi, \phi) = \partial_t z_k(\phi) \) (WT id.)

and exact dim. reduction follows: found for \( d > d_{DR} \approx 5.1 \)

• Numerical resolution on a grid.
Above $d_{DR} \approx 5.1$: no cusp in $\delta_k(\varphi_1, \varphi_2)$.

Below $d_{DR}$: cusp in $\delta_k(\varphi_1, \varphi_2)$ and SUSY breaking in a finite RG time.

A second order derivative of $\delta_k$ blows up in a finite RG time for $d<d_{DR}$ (red curve), not for $d>d_{DR}$ (blue curve)

$\Rightarrow$ SUSY breaking

Flow of the dimensionless second cumulant $\delta_k$ in $d=4$
Results: Critical exponents $\eta$ and $\bar{\eta}$

Breakdown of dimensional reduction appears continuously in dimension $d$

- Dimensional reduction: $\bar{\eta} = \eta$ \[= \eta^{(pure,d-2)}\]
- Below $d_{DR}$: $\bar{\eta} > \eta$

Very good agreement with “best estimates”:

In $d=3$

$\eta = 0.57 \pm 0.05$

[0.51\(\pm\)0.04]

$\bar{\eta} = 1.08 \pm 0.05$

[1.02-1.10]
If one adds the first “non-ultralocal” corrections, one finds that they go to finite fixed-point values and that the flow of the ultralocal quantities is generically of the form

\[ \partial_t \delta_k(\varphi_1, \varphi_2) = \beta^{(UL)}_{\delta_k}(\varphi_1, \varphi_2) + \left( \frac{k \theta}{\beta} \right) \beta^{(NUL)}_{\delta_k}(\varphi_1, \varphi_2) \]

The second term drops out when \( (1/\beta) = 0 \) (zero auxiliary temperature) and one is back to the purely “ultralocal” contributions.

When \( \beta \) finite and \( k \to 0 \), nonuniform convergence to the “ultralocal” cuspy fixed point: cusp in \( \delta_k(\varphi_1, \varphi_2) \) is rounded in a thermal boundary layer in \( |\varphi_1 - \varphi_2|/k^\theta \).

The boundary layer is related to the presence of rare “droplet” excitations.
Region IV: Weak non-analyticity (at fixed pt.); dim. red. predictions O.K.

Regions I and II: Spontaneous SUSY breaking at finite RG scale; cusp in renormalized second cumulant; breakdown of dim. red. (II: QLRO)

Region III: No phase transition
Conclusion

• The description of the long-distance physics of systems with quenched disorder requires special theoretical tools to account for loss of translational invariance/average over disorder, rare events, metastable states, etc...

• NP-FRG in a superfield setting (with introduction of many copies and of a weighting factor to select the proper solution) solves the 30-year-old pending problems concerning the critical behavior in random field systems.

• It could be a useful formalism for other problems described by a stochastic field equation with multiple solutions (metastable states in glasses, shocks in fluid turbulence, Gribov copies in non-Abelian gauge theories,...).
Below $d_{DR} \approx 5.1$: cuspy fixed point for

$\delta_k(\varphi_1, \varphi_2)$

Dimensionless second cumulant at the fixed point in $d=3$
Results (contd)

Below $d_{DR}$: cusp in $\delta_k(\varphi_1, \varphi_2)$ & spontaneous SUSY breaking

Breakdown of the (SUSY) WT identity at the fixed point in $d=3$
Results (contd.)

Above $N_{\text{DR}} = 18$: no cusp and dimensional reduction
Below $N_{\text{DR}} = 18$: cusp and breakdown of dimensional reduction

Anomalous dimensions in $d=4+\varepsilon$ (1-loop exact FRG)
Results: Critical exponents $\eta$ and $\bar{\eta}$ (contd.)

Optimization of the cut-off (to ensure a better stability of the results)