”Elliptic genus of K3 surface and Mathieu group M24”

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♣ Elliptic genus of K3

Elliptic genus in string theory is expressed as

$$Z_{elliptic}(z;\tau) = Tr_{\mathcal{H}_L \times \mathcal{H}_R}(-1)^{F_L+F_R} e^{4\pi i z J^3_{L,0}} q^{L_0-\frac{c}{24}} \bar{q}^{\bar{L}_0-\frac{c}{24}}$$

and describes topological invariants of the manifold. Here L_0 denotes the zero mode of the Visasoro operators and F_L and F_R are left and right moving fermion numbers. $q = e^{2\pi i\tau}$. In elliptic genus the right moving sector is frozen to the supersymmetric ground states (BPS states) while in the left moving sector all the states in the Hilbert space $\mathcal{H}_L$ contribute.

Elliptic genus of K3 surface is known: EOTY

$$Z_{K3}(z;\tau) = 8\left[\left(\frac{\theta_2(z;\tau)}{\theta_2(0;\tau)}\right)^2 + \left(\frac{\theta_3(z;\tau)}{\theta_3(0;\tau)}\right)^2 + \left(\frac{\theta_4(z;\tau)}{\theta_2(0;\tau)}\right)^2\right]$$

$$Z_{K3}(z=0)=24, \quad Z_{K3}(z=\frac{1}{2})=16+O(q),$$
$$Z_{K3}(z=\frac{1+\tau}{2})=2q^{-\frac{1}{2}}+O(q^{\frac{1}{2}})$$

Elliptic genus of a complex D-dimensional manifold is a Jacobi form of weight=0 and index=D/2. When D=2, space of Jacobi form is one-dimensional and given by the above formula.

String theory on K3 has an N=4 superconformal symmetry and its states fall into representations of N=4 superconformal algebra (SCA). N=4 SCA contains an affine $SU(2)_k$ symmetry and has a central charge $c = 6k$. $k = n$ case decsribes complex-$2n$ dimensional hyperKähler manifolds.

We would like to study the decomposition of the elliptic genus in terms of irreducible characters of N=4 SCA. In N=4 SCA, hightest-weight states $|h, I\rangle$ are charactered by

$$\boldsymbol{L_0|h,\ell\rangle = |h,\ell\rangle, \qquad J_0^3|h,\ell\rangle = \ell|h,\ell\rangle}$$

and the theory possesses two different type of representations, BPS and non-BPS representations. In the case of $k = 1$

there are representations (in Ramond sector)

$$\text{BPS rep.} \qquad h = \frac{1}{4}; \quad \ell = 0,\ \frac{1}{2}$$

$$\text{non-BPS rep.} \qquad h > \frac{1}{4}; \quad \ell = \frac{1}{2}$$

The index of a representation is given by the value of its character at $z = 0$, $Tr_{\mathcal{R}}(-1)^F q^{L_0} e^{4\pi i z J_0^3}|_{z=0}$. BPS representations have a non-vanishing index

$$\text{index (BPS, } \ell = 0) = 1$$

$$\text{index (BPS, } \ell = \frac{1}{2}) = -2$$

Character function of $\ell = 0$ BPS representation has the form

$$ch^{\tilde{R}}_{k=1,h=\frac{1}{4},\ell=0}(z;\tau) = \frac{\theta_1(z;\tau)^2}{\eta(\tau)^3}\mu(z;\tau)$$

where

$$\mu(z;\tau) = \frac{-ie^{\pi iz}}{\theta_1(z;\tau)}\sum_n (-1)^n \frac{q^{\frac{1}{2}n(n+1)}e^{2\pi inz}}{1-q^n e^{2\pi iz}}$$

On the other hand the character of non-BPS representations are given by

$$ch^{\tilde{R}}_{k=1,h>\frac{1}{4},\ell=\frac{1}{2}} = q^{h-\frac{3}{8}}\frac{\theta_1(z;\tau)^2}{\eta(\tau)^3}$$

These have a vanishing index

$$\text{index (non-BPS rep)} = 0$$

At the unitarity bound non-BPS representation splits into a sum of BPS representations

$$q^{-\frac{1}{8}} \frac{\theta_1(z;\tau)^2}{\eta(\tau)^3} = ch^{\tilde{R}}_{k=1,h=\frac{1}{4},\ell=\frac{1}{2}} + 2ch^{\tilde{R}}_{k=1,h=\frac{1}{4},\ell=0}$$

Function $\mu(z;\tau)$ is a typical example of the so-called Mock theta functions (Lerch sum or Appell function). Mock theta functions look like theta functions but they have anomalous modular transformation laws. Recently there has been a development in the understanding the nature of Mock theta functions by Zwegers and he has developed a way to improve their modular properties. We will adopt his method of handling Mock theta functions.

It is possible to derive the following idenities

$$ch^{\tilde{R}}_{k=1,h=\frac{1}{4},\ell=0}(z;\tau) = \left(\frac{\theta_2(z;\tau)}{\theta_2(0;\tau)}\right)^2 + \mu_2(\tau)\frac{\theta_1(z;\tau)^2}{\eta(\tau)^3}$$
$$ch^{\tilde{R}}_{k=1,h=\frac{1}{4},\ell=0}(z;\tau) = \left(\frac{\theta_3(z;\tau)}{\theta_3(0;\tau)}\right)^2 + \mu_3(\tau)\frac{\theta_1(z;\tau)^2}{\eta(\tau)^3}$$
$$ch^{\tilde{R}}_{k=1,h=\frac{1}{4},\ell=0}(z;\tau) = \left(\frac{\theta_4(z;\tau)}{\theta_4(0;\tau)}\right)^2 + \mu_4(\tau)\frac{\theta_1(z;\tau)^2}{\eta(\tau)^3}$$

where

$$\mu_2(\tau)=\mu(z=\frac{1}{2};\tau),\ \mu_3(\tau)=\mu(z=\frac{1+\tau}{2};\tau),\ \mu_4(\tau)=\mu(z=\frac{\tau}{2};\tau)$$

$$\mu(z;\tau) = \frac{-ie^{\pi iz}}{\theta_1(z;\tau)}\sum_n(-1)^n\frac{q^{\frac{1}{2}n(n+1)}e^{2\pi inz}}{1-q^ne^{2\pi iz}}$$

Then by making use of these formulas we can rewrite the ellitpic genus as

$$Z_{K3} = 24ch^{\tilde{R}}_{k=1,h=\frac{1}{4},\ell=0}(z;\tau) - 8\sum_{i=2}^{4}\mu_i(\tau)\frac{\theta_1(z;\tau)^2}{\eta(\tau)^3}$$

q-expansion of functions μ_i

$$8\left(\mu_2(\tau)+\mu_3(\tau)+\mu_4(\tau)\right) = \quad 2q^{-\frac{1}{8}} \; -2\sum_{n=1} A_n q^{n-\frac{1}{8}}$$

$\Uparrow$

polar term

$A_n \; (n=1,2,\cdots)$ are positive integers. Using

$$q^{-\frac{1}{8}}\frac{\theta_1(z;\tau)^2}{\eta(\tau)^3} = ch^{\tilde{R}}_{k=1,\ell=\frac{1}{2}} + 2ch^{\tilde{R}}_{k=1,\ell=0}$$

we can rewrite

$$Z_{K3} = 20ch^{\tilde{R}}_{k=1,\ell=0} - 2ch^{\tilde{R}}_{k=1,\ell=\frac{1}{2}} + 2\sum_{n=1} A_n\, q^{n-\frac{1}{8}}\frac{\theta_1(z;\tau)^2}{\eta(\tau)^3}$$

This is the character expansion of K3 elliptic genus. Type II string theory compactified on K3 possesses

$$1 : \text{graviton multiplet}$$
$$20 : \text{tensor, hyper multiplets}$$

At smaller values of n, Fourier coefficients A_n may be obtained by direct expansion. We find

n	1	2	3	4	5	6	7	8	$\cdots$
A_n	45	231	770	2277	5796	13915	30843	65550	$\ldots$

A surprize: Dimensions of some irreducible reps. of Mathieu group M_{24} appear

dimension	45	231	770	2277	5796	3520	10395	5544	$\cdots$

$$A_6 = 3520 + 10395,$$
$$A_7 = 10395 + 5796 + 5544 + 5313 + 3312 + 2024 + 1774,$$
$$\cdots$$

Mathieu moonshine? T.E.-Ooguri-Tachikawa

Monsterous moonshine:

$$J(q) = \frac{1}{q} + 196884 + 21493760q^2 + \cdots$$

q-expansion coeffcients of J-function are decomposed into a sum of irred. reps. of monster simple group.

Mukai: enumeration of eleven K3 surfaces with finite non-Abelian automorphism group. All these groups are sugb-groups of M_{24}.

fantasy: Is it possible that these automorphism groups at iso-letd points in K3 moduli space are enhanced to M_{24} over the whole moduli space when one cosnider the elliptic genus?

On the other hand, using the method of Rademacher expansion adapted to the case of Mock theta functions (Bringmann-Ono) we can determine the asymptotic behavior of coefficients A_n as

$$A_n \approx \frac{2}{\sqrt{8n-1}} e^{2\pi\sqrt{\frac{1}{2}(n-\frac{1}{8})}} \qquad \text{T.E.-Hikami}$$

More about Mathieu group

The Mathieu group M_{24} is a subgroup of A_{24} and an automorphism group of binary Golay code. There are 26 conjugacy classes and 26 irreducible representations. Characters

are is given in the table. ($e_p^{\pm}$ stands for $e_p^{\pm} = (\pm\sqrt{-p}-1)/2$).
The dimensions of the irreducible representations are

$$\mathbf{1, 23, 45, 45^*, 231, 231^*, 252, 253, 483, 770, 770^*,}$$
$$\mathbf{990, 990^*, 1035, 1035^*, 1035', 1265, 1771, 2024,}$$
$$\mathbf{2277, 3312, 3520, 5313, 5796, 5544, 10395.}$$

Here irreducible representations of dimensions

$$\mathbf{45, 231, 770, 990, 1035}$$

come in complex conjugate pairs. There is in addition an extra real 1035-dimensional irreducible representation.

Character table I

1A	2A	3A	5A	4B	7A	8A	6A	11A	15A	14A	23A	12B	6B	4C	3B	2B	10A	21A	4A
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
23	7	5	3	3	2	1	1	1	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1
252	28	9	2	4	0	0	1	-1	-1	0	-1	0	0	0	0	12	2	0	4
253	13	10	3	1	1	-1	-2	0	0	-1	0	1	1	1	1	-11	-1	1	-3
1771	-21	16	1	-5	0	-1	0	0	1	0	0	-1	-1	-1	7	11	1	0	3
3520	64	10	0	0	-1	0	-2	0	0	1	1	0	0	0	-8	0	0	-1	0
45	-3	0	0	1	e_7^+	-1	0	1	0	$-e_7^+$	-1	1	-1	1	3	5	0	e_7^-	-3
990	-18	0	0	2	e_7^+	0	0	0	0	e_7^+	1	1	-1	-2	3	-10	0	e_7^-	6
1035	-21	0	0	3	$2\,e_7^+$	-1	0	1	0	0	0	-1	1	-1	-3	-5	0	$-e_7^-$	3
1035	27	0	0	-1	-1	1	0	1	0	-1	0	0	2	3	6	35	0	-1	3
231	7	-3	1	-1	0	-1	1	0	e_{15}^+	0	1	0	0	3	0	-9	1	0	-1
770	-14	5	0	-2	0	0	1	0	0	0	e_{23}^+	1	1	-2	-7	10	0	0	2
483	35	6	-2	3	0	-1	2	-1	1	0	0	0	0	3	0	3	-2	0	3
1265	49	5	0	1	-2	1	1	0	0	0	0	0	0	-3	8	-15	0	1	-7
2024	8	-1	-1	0	1	0	-1	0	-1	1	0	0	0	0	8	24	-1	1	8
2277	21	0	-3	1	2	-1	0	0	0	0	0	0	2	-3	6	-19	1	-1	-3
3312	48	0	-3	0	1	0	0	1	0	-1	0	0	-2	0	-6	16	1	1	0
5313	49	-15	3	-3	0	-1	1	0	0	0	0	0	0	-3	0	9	-1	0	1
5796	-28	-9	1	4	0	0	-1	-1	1	0	0	0	0	0	0	36	1	0	-4
5544	-56	9	-1	0	0	0	1	0	-1	0	1	0	0	0	0	24	-1	0	-8
10395	-21	0	0	-1	0	1	0	0	0	0	-1	0	0	3	0	-45	0	0	3

$$\chi_{R_i}^{C_j} = Tr_{R_i}\, g_j, \qquad g_j \in C_j$$

Character table II

conj. class$\backslash n$	0	1	2	3	4	5	6	7	8	9
M_{23}										
1A, 1^{24}	24	2x45	2x231	2x770	2x2277	2x5796	2x13915	2x30843	2x65550	2x132825
2A, $1^8 2^8$	16	-6	14	-28	42	-56	86	-138	188	-238
3A, $1^6 3^6$	6	0	-6	10	0	-18	20	0	-30	42
5A, $1^4 5^4$	4	0	2	0	-6	2	0	6	0	-10
7A, $1^3 7^3$	3	-1	0	0	4	0	-2	2	-3	0
4B, $1^4 2^2 4^4$		2	-2	-4	2	8	-2	-10	4	10
6A, $1^2 2^2 3^2 6^2$		0	2	2	0	-2	-4	0	2	2
8A, $1^2 \cdot 2 \cdot 4 \cdot 8^2$		-2	-2	0	-2	0	2	-2	0	-2
11A, $1^2 11^2$		2	0	0	0	-2	0	-2	2	0
14A,$1 \cdot 2 \cdot 7 \cdot 14$		1	0	0	0	0	2	2	-1	0
15A, $1 \cdot 3 \cdot 5 \cdot 15$		0	-1	0	0	2	0	0	0	2
23A, $1 \cdot 23$		-2	2	-1	0	0	0	0	0	0
M_{24}										
2B, 2^{12}		10	-18	20	-38	72	-90	118	-180	258
4A, $2^4 4^4$		-6	-2	4	-6	-8	6	6	-4	-14
4C, 4^6		1	3	-2	-3	0	3	-1	-6	5
3B, 3^8		6	0	-14	12	0	-16	30	0	-42
6B, 6^4		-2	0	2	4	0	0	-2	0	6
12B, 12^2		2	0	2	0	0	0	-2	0	-2
10A, $2^2 10^2$		0	2	0	2	2	0	-2	0	-2
12A, $2 \cdot 4 \cdot 6 \cdot 12$		0	-2	-2	0	-2	0	0	2	-2
21A, $3 \cdot 21$		-1	0	0	-2	0	-2	2	0	0

Twisted elliptic genera

$$Z_{pA}(\tau,z)=\frac{2}{p+1}\phi_{0,1}(\tau,z)+\frac{2p}{p+1}\phi_{-2,1}(\tau,z)\phi_2^{(p)}(\tau),\qquad p=2,3,5,7$$

$$Z_{4B}(\tau,z)=\frac{1}{3}\phi_{0,1}(\tau,z)+\phi_{-2,1}(\tau,z)\left(-\frac{1}{3}\phi_2^{(2)}(\tau)+2\phi_2^{(4)}(\tau)\right),$$

$$Z_{6A}(\tau,z)=\frac{1}{6}\phi_{0,1}(\tau,z)+\phi_{-2,1}(\tau,z)\left(-\frac{1}{6}\phi_2^{(2)}(\tau)-\frac{1}{2}\phi_2^{(4)}(\tau)+\frac{5}{2}\phi_2^{(6)}(\tau)\right),$$

$$Z_{8A}(\tau,z)=\frac{1}{6}\phi_{0,1}(\tau,z)+\phi_{-2,1}(\tau,z)\left(-\frac{1}{2}\phi_2^{(4)}(\tau)+\frac{7}{3}\phi_2^{(8)}(\tau)\right),$$

$$Z_{11A}(\tau,z)=\frac{1}{6}\phi_{0,1}(\tau,z)+\phi_{-2,1}(\tau,z)\left(\frac{11}{6}\phi_2^{(11)}(\tau)-\frac{22}{5}f_{11}(\tau)\right),$$

$$Z_{14A}(\tau,z)=\frac{1}{12}\phi_{0,1}(\tau,z)+\phi_{-2,1}(\tau,z)\left(-\frac{1}{36}\phi_2^{(2)}(\tau)-\frac{7}{12}\phi_2^{(7)}(\tau)+\frac{91}{36}\phi_2^{(14)}(\tau)-\frac{14}{3}f_{14}(\tau)\right),$$

$$f_{14}(\tau)=\eta(\tau)\eta(2\tau)\eta(7\tau)\eta(14\tau),$$

$$Z_{15A}(\tau,z)=\frac{1}{12}\phi_{0,1}(\tau,z)+\phi_{-2,1}(\tau,z)\left(-\frac{1}{16}\phi_2^{(3)}(\tau)-\frac{5}{24}\phi_2^{(5)}(\tau)+\frac{35}{16}\phi_2^{(15)}(\tau)-\frac{15}{4}f_{15}(\tau)\right),$$

$$f_{15}(\tau)=\eta(\tau)\eta(3\tau)\eta(5\tau)\eta(15\tau),$$

$$\cdots$$

$$\phi_{0,1}=1/2\times(\text{elliptic genus of K3}),\ \phi_{-2,1}=\frac{\theta_1^2}{\eta^6},\quad \phi_2^{(\mathrm{N})}=\frac{24}{\mathrm{N}-1}\mathrm{q}\partial_{\mathrm{q}}\log\frac{\eta(\mathrm{N}\tau)}{\eta(\tau)}$$

There is a strong evidence in favor of the Mathieu moonshine conjecture. M.Cheng,Gaberdiel-Hohenegger-Volpato

♠ Entropy of HyperKähler manifold

Since in the elliptic genus the right-moving sector is frozen to the ground (BPS) state, left-moving non-BPS states are half-BPS states when we consider both left-right moving sectors simultaneously. Thus A_n describes the multiplicity of half-BPS states at higher levels. When the multiplicity of states protected by supersymmetry increases like an exponential, we may interpret it as the entropy of the system.

We want to compute the entropy for higher dimensional hyperKähler manifolds, in particular symmetric products of K3 surfaces, $K3^{[k]}$ and check if the entropy considered here agrees with the entropy of black holes of string theory compactified on K3 surface.

Character expansion of elliptic genera at level k is given by

$$Z(H_{2k}) = \sum_{\ell=0}^{\frac{k}{2}} n_\ell \, ch^{\tilde{R}}_{k,h=\frac{k}{4},\ell}(z;\tau)$$

$$+ \sum_{\ell=0}^{\frac{k-1}{2}} \sum_{n=1}^{\infty} A_n^\ell q^{n-\frac{(\ell+\frac{1}{2})^2}{k+1}} \chi_{k-1,\ell}(z;\tau) \frac{\theta_1(z;\tau)^2}{\eta(\tau)^3}$$

Here $\chi_{k-1,\ell}$ is the character of affine $SU(2)$ at level-$(k-1)$.

Asymptotic growth of A_n^ℓ is determined by the polar parts of the series. Results are given as (for large k)

$$A_n^\ell \approx \exp(\pi\sqrt{4kn-\ell^2})$$

T.E.-Hikami

Dominant multiplicity A_n behaves as

$$A_n \approx \exp(\pi\sqrt{4kn})$$

and agrees with the entropy $S = 2\pi\sqrt{Q_1Q_5n}$ of the standard D_1D_5 black hole with $k = Q_1Q_5$.

K3 surface corresponds to the mini black hole $Q_1 = Q_5 = 1$. It will be very intersting if one can interprete the Mathieu group as acting on the micro-states of mini black holes.