# Integrability in AdS/CFT correspondence

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3 Higher-loop corrections
4 Finite-size problem
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## 1 Introduction

AdS/CFT correspondence

- gauge/string duality

Type IIB string theory on $AdS_5 \times S^5$ vs 4d $\mathcal{N}=4$ SYM

gauge theory $\lambda \ll 1$

string theory $\lambda \gg 1$

$$E(\lambda) = E_0\sqrt{\lambda} + E_1 + E_2\frac{1}{\sqrt{\lambda}} + \cdots$$

Quantitatively $E(\lambda) \overset{?}{=} \Delta(\lambda)$

$$\Delta(\lambda) = \Delta_0 + \Delta_1\lambda + \Delta_2\lambda^2 + \cdots$$

Planar $\mathcal{N}=4$ SYM
Free string theory
) are integrable.

can use many techniques in spin chains & 2D QFT

## 2 One-loop dilatation op and BA method

$\mathcal{N}=4$ SYM : Fields

$A_\mu$ : gluons

$\Phi_i$ $(i=1\cdots6)$ $SO(6)$ scalars

$\psi^A$ $(A=1\cdots4)$ Weyl fermions

} all belong to Adj $SU(N)$ gauge group

$$\langle O_n(x) O_m(y) \rangle = \frac{C_n \delta_{nm}}{|x-y|^{2\Delta_n}} \qquad \Delta_n \ \text{(conformal dim)}$$

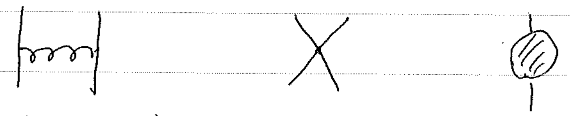
is an eigenvalue of $D$ (dilatation op)

$$D O_n = \Delta_n O_n$$

Here consider $SU(2)$ sector $\subset SO(6)_R$

$$Z = \Phi_1 + i\Phi_2, \quad W = \Phi_3 + i\Phi_4$$

at 1-loop level,



gluon exchange, 4pt scalar interaction, self-energy

$$D_1 = 2 \sum_{k=1}^{L} \left( I_{k,k+1} - P_{k,k+1} \right) = H_{XXX\,1/2}$$

use BA method to diagonalize $D_1 = H_{XXX\,1/2}$

operator $\leftrightarrow$ state

$Z \quad \uparrow$

$W \quad \downarrow$

$\mathrm{Tr}(ZZZWWZW) \to \uparrow\uparrow\uparrow\downarrow\downarrow\uparrow\downarrow$

$\mathrm{Tr}(Z^L)$ : ground state (ferromag vacuum)

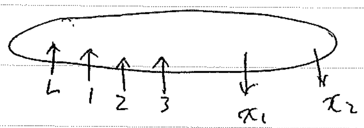
$D_1 \mathrm{Tr}(Z^J) = 0$

Excitations : magnons

## Bethe ansatz method

- Coordinate BA : easy to understand phys meaning (need other way to show the integrability)
- Algebraic BA : slightly complicated, easy to see the integrability

## Coordinate BA




$\Leftrightarrow |x_1, x_2 \cdots \rangle$ excitation at $x_1, x_2, \cdots x_M$-th cites

One-magnon state $\quad |p\rangle = \sum_{x=1}^{L} e^{ipx} |x\rangle$

$$D_1 |p\rangle = \epsilon(p) |p\rangle \qquad \epsilon(p) = 8 \sin^2\left(\frac{p}{2}\right)$$

Remark spin-chain with periodic BC

$p = 0 \mod 2\pi$

Two-magnon state

$$\begin{cases} |p_1, p_2\rangle = \sum_{1 \le x_1 < x_2 \le L} \psi(x_1, x_2) |x_1, x_2\rangle \\ \psi(x_1, x_2) = e^{i(p_1 x_1 + p_2 x_2)} + S(p_1, p_2) e^{i(p_1 x_2 + p_2 x_1)} \end{cases}$$

"Bethe ansatz"

$$\Rightarrow D_1 |p_1, p_2\rangle = \sum_{x_1+1 < x_2} A(x_1, x_2) |x_1, x_2\rangle + \sum_{x} B(x, x+1) |x, x+1\rangle$$

$$A(x_1, x_2) = 2\{4\psi(x_1, x_2) - \psi(x_1-1, x_2) - \psi(x_1+1, x_2) - \psi(x_1, x_2-1) - \psi(x_1, x_2+1)\}$$

$$B(x, x+1) = 2\{2\psi(x, x+1) - \psi(x-1, x+1) - \psi(x, x+2)\}$$

$$A(x_1, x_2) = \epsilon(p_1, p_2) \psi(x_1, x_2)$$

$$B(x, x+1) = e^{i(p_1+p_2)x} \{ e^{ip_2}(2 - e^{-ip_1} - e^{ip_2}) + S(p_2, p_1) e^{ip_1}(2 - e^{ip_1} - e^{-ip_2}) \}$$

$$\epsilon(p_1, p_2) = \epsilon(p_1) + \epsilon(p_2) = 8\left[\sin^2\left(\frac{p_1}{2}\right) + \sin^2\left(\frac{p_2}{2}\right)\right]$$

If $B(x, x+1) = \epsilon(p_1, p_2) \psi(x, x+1)$,

$\Rightarrow |p_1, p_2\rangle$ is eigenstate of $D_1$

$$\Rightarrow B(x, x+1) = \epsilon(p_1, p_2) \psi(x, x+1)$$

$$\Rightarrow \underbrace{S(p_1, p_2)}_{\text{S-matrix}} = \frac{u(p_1) - u(p_2) + i}{u(p_1) - u(p_2) - i} \qquad \underbrace{u(p)}_{\text{rapidity}} = \frac{1}{2}\cot\left(\frac{p}{2}\right)$$

periodic BC $\longrightarrow$ $\psi(x_1, x_2) = \psi(x_2, x_1 + L)$

$$\begin{cases} e^{ip_1 L} = S(p_1, p_2) \\ e^{ip_2 L} = S(p_2, p_1) = \dfrac{1}{S(p_1, p_2)} \end{cases}$$

Bethe ansatz equation

$$e^{i(p_1+p_2)L} = 1, \quad p_1 + p_2 = 0 \mod 2\pi$$

<u>M-magnon state</u>

BAE $$\underbrace{e^{ip_j L} = \prod_{\substack{k=1 \\ k \neq j}}^{M} S(p_j, p_k)}_{= \prod_k \text{(each phase shift)}}$$

$$\sum_{k=1}^{M} p_j = 0 \mod 2\pi$$

$$\epsilon(p_1, \cdots p_M) = \sum_{j=1}^{M} \epsilon(p_j) = \sum_j 8 \sin^2\left(\frac{p_j}{2}\right)$$

<u>Example:</u> $O_K = Tr(WZWZ) - Tr(Z^2W^2)$

$D_1 O_K = 12 O_K \qquad L = 4 \quad M = 2$

$$e^{4ip} = \frac{u(p) - u(-p) + i}{u(p) - u(-p) - i}, \qquad \underline{\underline{p = \frac{2\pi}{3}}}$$

$$\epsilon(p, -p) = 2\epsilon\left(\frac{2\pi}{3}\right) = 12$$

3, Higher-loop corrections

straightforward way $\quad g = \frac{\sqrt{\lambda}}{4\pi}$

$$D = D_0 + g^2 D_1 + g^4 D_2 + g^6 D_3 + \cdots$$

In SU(2) sector, $D$ has been fixed

Another strategy : $D$ can be diagonalized By BAM

$$e^{ip_j L} = \prod_{\substack{k=1 \\ k\neq j}}^{M} S(p_j, p_k; \lambda) \quad , \quad \Delta = L - M + \sum_{j=1}^{M} E(p_j, \lambda)$$

problem : to determine the exact S-matrix $S(p_1, p_2, \lambda)$
dispersion $E(p, \lambda)$

Results : SU(2) sector $\quad S(p_1, p_2, \lambda) = \frac{U(p_1) - U(p_2) + i}{U(p_1) - U(p_2) - i} \underline{\sigma^2(p_1, p_2)}$

Dressing phase

$$U(p) = \frac{1}{2}\cot\left(\frac{p}{2}\right)\sqrt{1 + 16g^2 \sin^2\left(\frac{p}{2}\right)}$$

$$E(p, \lambda) = \sqrt{1 + 16g^2 \sin^2\left(\frac{p}{2}\right)}$$

$\sigma(p_1, p_2)$ fixed by Beisert - Eden - Staudacher

4. Finite - size problem

Higher loop dil op has long-range ints

In general $\ell$-loop $D_\ell$ has ints betw $(\ell+1)$ site

wrapping problem

consider spin-chain with Length $L$

then $L$-loop $D_L$ contains $(L+1)$ site interaction

$\rightarrow$ cannot def asymp. state, S-matrix $\rightarrow$ ~~Bethe Ansatz~~

$\rightarrow$ A soln proposed "Y-system" (related to TBA)

[Gromov - Kazakov - Vieira]

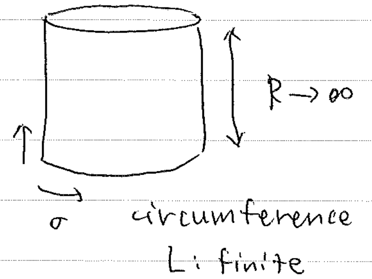
## 5. TBA for 2d Relativistic QFTs

TBA appears in
- Spectral problem in AdS/CFT
- Minimal surface problem in AdS space
- Wall crossing phenomenon [Gaiotto - Moore - Neitzke]
- Gauge - Bethe correspondence

TBA eq. derived by Yang - Yang '69

Al. B. Zamolodchikov, use TBA in 2D QFT

Idea




consider $\tau$: time

$\sigma$: space

$$Z = \sum_j e^{-RE_j(L)} \xrightarrow{R\to\infty} e^{-RE_0(L)}$$

$\tau$: space, $\sigma$: time $\quad T = \frac{1}{L}$

$$Z = e^{-LF(L)} \qquad E_0(L) = \lim_{R\to\infty} \frac{L}{R} F(L)$$

Space direction is non-compact $\Rightarrow$ spectrum determined by BAE

$$\text{BAE}: \quad e^{ip_j R} \prod_{\substack{k \neq j \\ k=1}}^{N} S(\theta_j - \theta_k) = 1$$

$$\longrightarrow \quad i p_j R + \sum_{k\neq j} \log S(\theta_j - \theta_k) = 2\pi i n_j$$

$$(E = m\cosh\theta, \; p = m\sinh\theta)$$

want to know the <u>solution</u> minimize the free energy $R\to\infty$

thermodynamic limit $R\to\infty$, $N\to\infty$, $\frac{N}{R}$ = finite

particle density $\rho_r(\theta) \equiv \sum_{j=1}^{N} \delta(\theta - \theta_j)$ rapidity density

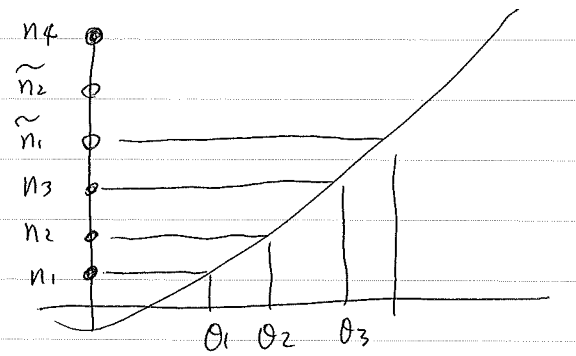
$$m R \sinh\theta_j + \int_{-\infty}^{\infty} d\theta' \, \delta(\theta_j - \theta') \rho(\theta') = 2\pi n_j \quad (j = 1 \cdots N)$$

$$\delta(\theta) = -i \log S(\theta)$$

$$J(\theta) = mR\sinh\theta + \int_{-\infty}^{\infty} d\theta' \delta(\theta-\theta') \rho_r(\theta')$$

$$J(\theta_j) = 2\pi n_j$$

For $\{n_1, n_2, \ldots n_N\}$ define $\{\tilde{n}\} = \mathbb{Z} \setminus \{n\}$




def Holes $J(\tilde{\theta}_j) = 2\pi \tilde{n}_j$

$$\rho_h(\theta) \equiv \sum_k \delta(\theta - \tilde{\theta}_k)$$

$$\rho(\theta) \equiv \rho_r(\theta) + \rho_h(\theta)$$

$$2\pi\rho(\theta) = \frac{dJ(\theta)}{d\theta} = mR\cosh\theta + \int d\theta' \varphi(\theta-\theta') \rho_r(\theta')$$

$$\varphi(\theta) \equiv \frac{1}{i} \frac{d}{d\theta} \log S(\theta)$$

$$F = E - TS, \quad E = \sum_{j=1}^{N} m\cosh\theta_j \to \int_{-\infty}^{\infty} d\theta \rho_r(\theta) m\cosh\theta$$

$$S = \int_{-\infty}^{\infty} d\theta \left[ \rho\log\rho - \rho_r\log\rho_r - (\rho-\rho_r)\log(\rho-\rho_r) \right]$$

$F[\rho, \rho_r] = E(\rho_r) - \frac{1}{L} S[\rho, \rho_r]$ $\longrightarrow$ find $(\rho, \rho_r)$ minimizing $F$

@ $\rho_r \to \rho_r + \delta\rho_r$
$\rho \to \rho + \delta\rho$

@ $2\pi\delta\rho = \int_{-\infty}^{\infty} d\theta' \varphi(\theta-\theta') \delta\rho_r$

$$\delta F = F[\rho+\delta\rho, \rho_r+\delta\rho_r] - F[\rho, \rho_r]$$

$$= \frac{1}{L}\int_{-\infty}^{\infty} d\theta\, \delta\rho_r(\theta) \Big[ mL\cosh\theta - \log\frac{\rho(\theta)-\rho_r(\theta)}{\rho_r(\theta)}$$

$$- \frac{1}{2\pi} \int_{-\infty}^{\infty} d\theta' \varphi(\theta-\theta') \log\frac{\rho(\theta')}{\rho(\theta')-\rho_r(\theta')} \Big] = 0$$

$$\frac{\rho_r(\theta)}{\rho(\theta)} = \frac{1}{e^{\epsilon(\theta)}+1}, \quad \epsilon(\theta) = mL\cosh\theta - \frac{1}{2\pi}\int_{-\infty}^{\infty} d\theta' \varphi(\theta-\theta') \log(1+e^{-\epsilon(\theta')})$$

$$F = -\frac{mR}{2\pi L}\int_{-\infty}^{\infty} d\theta \cosh\theta \log(1+e^{-\epsilon(\theta)}), \quad E_0(L) = \frac{LF}{R},$$

## 6. Comments on AdS/CFT case

2d Relativistic QFT

$\rightarrow$ 2d worldsheet sigma-model on cylinder

differences · dispersion relation

$$\epsilon(p)^2 - p^2 = m^2 \quad \longleftrightarrow \quad \epsilon(p)^2 - 16 g^2 \sin^2\left(\frac{p}{2}\right) = 1$$

· S-matrix cannot be written in terms of difference of rapidity

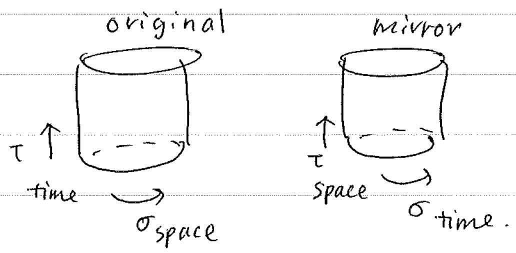
$$S(\theta_j - \theta_k) \longleftrightarrow S_{AdS/CFT}(u_j, u_k)$$

· S-matrix is non-diagonal

BAE $\longrightarrow$ nested BAE

TBA eqn for AdS/CFT.

◎ Start with the nested BAEs in Mirror theory




◎ thermodynamic limit

◎ consider minimization condition for Free energy

$\rightarrow$ TBA eq for ground state

$\mathrm{Tr}[2^J)$ ... does not receive quant corr

$\Delta = J$

Finite size effects appear for excited states

$\rightarrow$ TBA eq for excited states

(analytic contin of TBA for ground states)

TBA eq for AdS / CFT

… have been derived by
- Bombardelli et al 0902.3930
- Gromov et al 0902.4458
- Arutyunov. Frolov, 0903.0141