Dilatonic Black Holes in String Theories with Gauss-Bonnet Correction and Their Physical Applications

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- **I. Asymptotically flat solution without Λ: Prog. Theor. Phys. 120 (2008) 581 [arXiv:0806.2481 [gr-qc]].**
- **Ia. The same but in the string frame: Phys. Rev. D80 (2009) 104032 [arXiv:0908.4151 [hep-th]].**
- **Ib. Charged extreme solution: Phys. Rev. D 81 (2010) 024002 [arXiv:0910.3488 [hep-th]].**
- **II. Asymptotically AdS with negative Λ and $k = 0$: Prog. Theor. Phys. 121 (2009) 253 [arXiv:0811.3068 [gr-qc]].**
- **III. Asymptotically AdS with negative Λ and $k = \pm 1$: Prog. Theor. Phys. 121 (2009) 959 [arXiv:0902.4072 [hep-th]].**
- **IV. Case of positive Λ: Prog. Theor. Phys. 122 (2009) 1477 [arXiv:0908.3918 [hep-th]].**
- **V. Global Structure: arXiv:1004.2779 [hep-th]], to appear in Prog. Theor. Phys. 124 (2010)**
- **VI. Application to shear viscosity: Phys. Rev. D 79 (2009) 066004 [arXiv:0901.1421 [hep-th]].**

1 Introduction

- **Check of the predictions of superstring theories**

The situation where the effects of quantum gravity become important

$\Rightarrow$ **Black holes (singularity)**
Early universe (singularity)

It is urgent to see whether and how these problems are resolved and if superstrings can give realistic models of particles and their interaction including gravity, not to mention finding evidence of string theories.

Here we consider black holes. ———————————

- **We need dilaton!!**

Many studies of black holes have been performed by using low-energy effective theories inspired by string theories, which typically involve not only the metric but also the dilaton field (as well as several gauge fields).

There are studies of such solutions in Einstein theories with dilaton.

- **What about higher order corrections?**

It is known that there are correction terms of higher orders in the curvature to the lowest effective supergravity action coming from superstrings. The simplest correction is the Gauss-Bonnet (GB) term coupled to the dilaton field.

Black holes in Einstein-GB theories have been studied much but WITH-OUT DILATON!

In order to understand properties of black holes in string theories, we should include dilaton!

Incorporation of each of these are tractable. But when both are included, the problem becomes enormously difficult! We need numerical study. (Actually complicated but not difficult numerically.)

- **Another motivation:**

Many people consider the application to the calculation of shear viscosity in strongly coupled gauge theories using black hole solutions in five-dimensional Einstein-GB theory via AdS/CFT correspondence, but without dilaton. In order to see this in the context of superstrings, we should again include dilaton.

We obtained asymptotically flat solution for spherically symmetric space (curvature of the space $k = +1$) without cosmological constant, and then extended the work to all possible cases in various dimensions up to 10, with possible AdS/CFT application in mind (for negative cosmological constant)!

There are several sources of (negative) cosmological constant in superstrings. e.g. RR 10-form. Don't worry about the CC.

2 Dilatonic Einstein-GB theory

2.1 Basic equations

The action:

$$S = \frac{1}{2\kappa_D^2} \int d^D x \sqrt{-g} \left[R - \frac{1}{2} (\partial_\mu \phi)^2 + \alpha_2 e^{-\gamma \phi} R_{\rm GB}^2 - \Lambda e^{\lambda \phi} \right],$$

R**: the scalar curvature,** ϕ**: a dilaton field,**
$R^2_{\rm GB} = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2$**: the GB combination,**
$\kappa_D^2 = 8\pi G_D$**: a** D**-dimensional gravitational constant,**
$\alpha_2 = \alpha'/8$**:** α **is the Regge slope parameter** α'**,** $\gamma = 1/2$**,**
Λ**: (zero, negative or positive) cosmological "constant."**

Line element in D**-dimensional static spacetime**

$$ds_D^2 = -B e^{-2\delta} dt^2 + B^{-1} dr^2 + r^2 h_{ij} dx^i dx^j,$$

where $h_{ij} dx^i dx^j$ **represents the line element of a** $(D-2)$**-dimensional hypersurface with constant curvature of signature** k **and volume** Σ_k **for** $k = \pm 1, 0$**.**

Master equations:

$$\Big[(k-B)\tilde{r}^{D-3}\Big]'\frac{D-2}{\tilde{r}^{D-4}}h - \frac{1}{2}B\tilde{r}^2\phi'^2 - (D-1)_4\, e^{-\gamma\phi}\frac{(k-B)^2}{\tilde{r}^2}$$
$$+4(D-2)_3\,\gamma e^{-\gamma\phi}B(k-B)(\phi''-\gamma\phi'^2)$$
$$+2(D-2)_3\,\gamma e^{-\gamma\phi}\phi'\frac{(k-B)[(D-3)k-(D-1)B]}{\tilde{r}} - \tilde{r}^2\tilde{\Lambda}e^{\lambda\phi} = 0\,,$$

$$\delta'(D-2)\tilde{r}h + \frac{1}{2}\tilde{r}^2\phi'^2 - 2(D-2)_3\,\gamma e^{-\gamma\phi}(k-B)(\phi''-\gamma\phi'^2) = 0\,,$$

$$(e^{-\delta}\tilde{r}^{D-2}B\phi')' = \gamma(D-2)_3 e^{-\gamma\phi-\delta}\tilde{r}^{D-4}\Big[(D-4)_5\frac{(k-B)^2}{\tilde{r}^2} + 2(B'-2\delta'B)B'$$
$$-4(k-B)BU(r) - 4\frac{D-4}{\tilde{r}}(B'-\delta'B)(k-B)\Big] + e^{-\delta}\tilde{r}^{D-2}\lambda\tilde{\Lambda}e^{\lambda\phi},$$

3 eqs. for 3 unknown: B, δ, ϕ.
Parameters: k, D, Λ (for fixed γ and λ). Boundary conditions: ϕ_H, r_H.

$$\tilde{r} \equiv \frac{r}{\sqrt{\alpha_2}}, \quad \tilde{m} \equiv \frac{Gm}{\alpha_2^{(D-3)/2}}, \quad (D-m)_n \equiv (D-m)(D-m-1)(D-m-2)\cdots(D-n),$$
$$h \equiv 1 + 2(D-3)e^{-\gamma\phi}\Big[(D-4)\frac{k-B}{\tilde{r}^2} + \gamma\phi'\frac{3B-k}{\tilde{r}}\Big],$$
$$\tilde{h} \equiv 1 + 2(D-3)e^{-\gamma\phi}\Big[(D-4)\frac{k-B}{\tilde{r}^2} + \gamma\phi'\frac{2B}{\tilde{r}}\Big],$$

$$
\begin{aligned}
U(r) &\equiv \frac{1}{2\tilde{h}}\Bigg[(D-3)_4\frac{k-B}{\tilde{r}^2 B} - 2\frac{D-3}{\tilde{r}}\Big(\frac{B'}{B}-\delta'\Big) - \frac{1}{2}\phi'^2 \\
&\quad +(D-3)e^{-\gamma\phi}\Bigg[(D-4)_6\frac{(k-B)^2}{\tilde{r}^4 B} - 4(D-4)_5\frac{k-B}{\tilde{r}^3}\Big(\frac{B'}{B}-\delta'-\gamma\phi'\Big) \\
&\quad -4(D-4)\gamma\frac{k-B}{\tilde{r}^2}\Big(\gamma\phi'^2+\frac{D-2}{\tilde{r}}\phi'-\Phi\Big) + 8\frac{\gamma\phi'}{\tilde{r}}\Big\{\Big(\frac{B'}{2}-\delta' B\Big)\Big(\gamma\phi'-\delta'+\frac{2}{\tilde{r}}\Big) \\
&\qquad -\frac{D-4}{2\tilde{r}}B'\Big\} + 4(D-4)\Big(\frac{B'}{2B}-\delta'\Big)\frac{B'}{\tilde{r}^2} - 4\frac{\gamma}{\tilde{r}}\Phi(B'-2\delta' B)\Bigg]\Bigg], \\
\Phi &\equiv \phi'' + \Big(\frac{B'}{B}-\delta'+\frac{D-2}{\tilde{r}}\Big)\phi'.
\end{aligned}
$$

This is valid in all dimensions and for any value of cc.

2.2 Symmetries in asymptotically flat case with $\Lambda = 0$

If we set $B = 1 - \dfrac{2Gm(\tilde{r})}{\tilde{r}^{D-3}}$, the system has a symmetry under

$$
\phi \to \phi - \phi_\infty, \quad \tilde{r} \to e^{\frac{1}{2}\gamma\phi_\infty}\tilde{r}, \quad \delta \to \delta, \quad \tilde{m} \to e^{\frac{D-3}{2}\gamma\phi_\infty}\tilde{m}.
$$

⇒ the asymptotic value of the dilaton field = zero

Another shift symmetry

$$
\delta \to \delta - \delta_\infty, \quad t \to e^{-\delta_\infty} t,
$$

⇒ the asymptotic value of δ = 0.

2.3 Symmetries for $k = 0$ with a cosmological constant

1. **Scaling transf.** $B \to a^2 B, \quad \tilde{r} \to a\tilde{r}$, (**$a$: an arbitrary constant).**
   $\Rightarrow$ **generate solutions with different horizon radii $\tilde{r}_H$ but the same $\tilde{\Lambda}$.**
   $\Rightarrow$ **The mass scales like**

$$\tilde{M}_0 \propto \tilde{r}_H^{D-1}, \qquad \tilde{\Lambda}: \text{ fixed}$$

2. **Scaling of c.c.:**

$$\phi \to \phi - \phi_*, \quad \tilde{\Lambda} \to e^{(\lambda-\gamma)\phi_*}\tilde{\Lambda}, \quad B \to e^{-\gamma\phi_*} B\,,$$

   $\Rightarrow$ **generate solutions for different cosmological constants $\tilde{\Lambda}$ but with the same horizon radius $\tilde{r}_H$. $\Rightarrow$ The mass scales as**

$$\tilde{M}_0 \propto |\tilde{\Lambda}|^{\gamma/(\gamma-\lambda)}, \qquad \tilde{r}_H: \text{ fixed}$$

3. **Another shift symmetry**

$$\delta \to \delta - \delta_\infty, \quad t \to e^{-\delta_\infty} t,$$

   $\Rightarrow$ **the asymptotic value of $\delta = 0$.**

2.4 Boundary conditions

1. The existence of a regular horizon $\tilde{r}_H$:

$$B(\tilde{r}_H) = 0, \quad |\delta_H| < \infty, \quad |\phi_H| < \infty\,.$$

2. The nonexistence of singularities outside the event horizon ($\tilde{r} > \tilde{r}_H$):

$$B(\tilde{r}) > 0, \quad |\delta| < \infty, \quad |\phi| < \infty\,.$$ **This is important.**

3. "asymptotic behavior" ($\tilde{r} \to \infty$):

$$B \sim (\tilde{b}_2\tilde{r}^2 +)1 - \frac{2\tilde{M}}{\tilde{r}^\mu}, \quad (\tilde{r}^2 \text{ term for } \Lambda \neq 0), \quad \delta(r) \sim \delta_0 + \frac{\delta_1}{\tilde{r}^\sigma}, \quad \phi \sim \phi_0 + \frac{\phi_1}{\tilde{r}^\nu}\,,$$

with finite constants $\tilde{b}_2 > 0$, $\tilde{M}$, δ_0, δ_1, ϕ_0, ϕ_1 and positive constant μ, σ, ν.

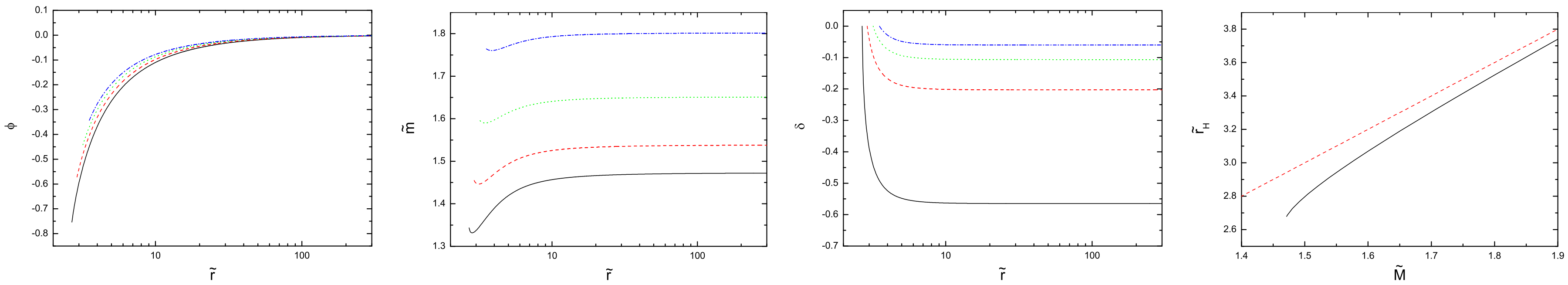
Given the b.c. at the horizon, ϕ'_H is determined ($k = 0$):

$$\begin{aligned}
&B_H = 0, \quad h_H = \tilde{h}_H = 1, \\
&B'_H = -\frac{\tilde{\Lambda}}{D-2}\tilde{r}_H e^{\lambda\phi_H}, \\
&\phi'_H = -\frac{1}{\tilde{r}_H}\Big[2\gamma(D-3)\tilde{\Lambda}e^{(\lambda-\gamma)\phi_H} + (D-2)\lambda\Big], \\
&\delta'_H = -\frac{1}{2(D-2)}\tilde{r}_H(\phi'_H)^2.
\end{aligned}$$

$\Rightarrow$ **no solution without c.c.**

3 Asymptotically flat solutions ($D = 4$ for example)

Start with $\Lambda = 0, k = 1$.
Give b. c. on ϕ_H and δ_H at the horizon $\Rightarrow \phi'_H$ is determined.
Only the smaller solution gives regular BH.
Use the shift symmetry to set the asymptotic value of the dilaton to zero.
Regular black hole solutions exist only for $\tilde{r}_H \geq 1.47126$ in $D = 4$. (Gap)
(depends on the dilaton coupling and frame.)



dilaton ϕ | mass function $\tilde{m}$ | lapse function δ | mass versus horizon radius

dilaton (solid line) and non-dilaton (dashed line)

Figure 1: **Black hole solutions in the four-dimensional Einstein-GB-dilaton system with $\gamma = \frac{1}{2}$. The behaviours are for four different radii of event horizon: $\tilde{r}_H = r_H/\sqrt{\alpha_2} = 2.68697$ (solid (black) line), 2.90965 (dashed (red) line), 3.19148 (dotted (green) line) and 3.52851 (dash-dotted (blue) line). (d) The mass versus horizon radius in the dilatonic case (solid line) and in the non-dilatonic case (dashed line). The masses $\tilde{M}$ for these cases are found to be 1.47251, 1.53808, 1.65113, and 1.80161, respectively.**

Properties in general dimensions:

In four dim., solution disappears below certain radius, but the regular black hole solutions exist for all $\tilde{r}_H > 0$ beyond $D = 4$.

The mass of the dilatonic black holes approaches a non-zero constant in $D = 5$ but goes to zero in higher dims.

5 dimension is very special!

There is no solution for $k = 0$.

4 Asymptotically AdS solutions with $\Lambda < 0$

The most interesting case is $k = 0$.

Effective potential picture

The dilaton field equation

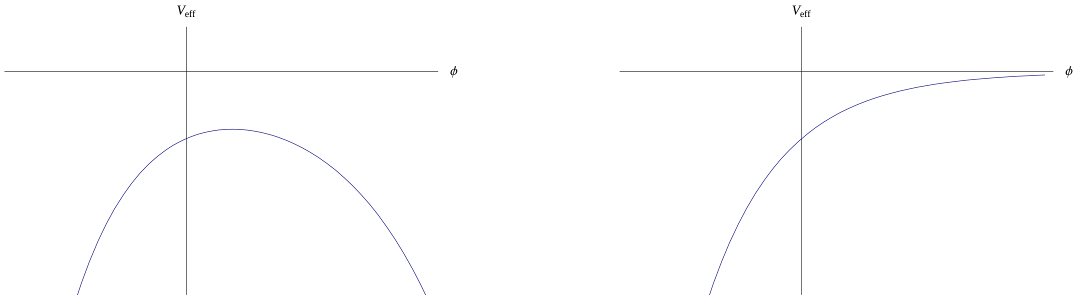
$$\Box\phi - \frac{d\tilde{V}_{\rm eff}}{d\phi} = 0,$$

with the "effective potential"

$$\tilde{V}_{\rm eff} = -e^{-\gamma\phi}\tilde{R}^2_{\rm GB} + \tilde{\Lambda}e^{\lambda\phi}.$$

For the asymptotic AdS behavior for B, this gives

$$\tilde{V}_{\rm eff} = -(D)_3\,\tilde{b}_2^2\;e^{-\gamma\phi} + \tilde{\Lambda}e^{\lambda\phi}. \quad (\Rightarrow \quad \lambda > 0 \textbf{ or } \lambda < 0\ ?)$$



(a) (b)

Figure 2: The effective potentials of the dilaton field in the Liouville potential case with (a) $\lambda > 0$ and (b) $\lambda < 0$.

When $\lambda > 0$, the effective potential has a maximum (Fig. 1 (a)), and the dilaton field would approach a finite constant ϕ_0 at $r = \infty$. Otherwise, the dilaton diverges, and we consider only the case of $\lambda > 0$.

For $\Lambda \neq 0$, the asymptotic forms of the fields are

$$\phi \sim \phi_0 + \frac{\phi_+}{\tilde{r}^{\nu_+}} + \cdots, \quad B \sim \tilde{b}_2 \tilde{r}^2 - \frac{2\tilde{M}_+}{\tilde{r}^{\nu_+ - 2}} - \frac{2\tilde{M}_0}{\tilde{r}^{D-3}} + \cdots, \quad \delta \sim \delta_0 + \frac{\delta_+}{\tilde{r}^{\nu_+}} + \cdots.$$

where

$$\nu_\pm = \frac{D-1}{2}\left[1 \pm \sqrt{1 - \frac{\tilde{m}^2}{\tilde{m}_{BF}^2}}\right], \quad \tilde{m}_{BF}^2 = -\frac{(D-1)^2}{4\tilde{\ell}_{AdS}^2} = -\frac{(D-1)^2}{4}\tilde{b}_2,$$

There is a term $1/r^{\nu_-}$ (non-normalizable); we tune the boundary condition such that this term disappears.

We choose the following parameters in our numerical analysis:

$$\gamma = \frac{1}{2}, \quad \lambda = \frac{1}{3}, \quad \tilde{\Lambda} < 0, \quad \phi_- = 0, \quad \delta_0 = 0,$$

In $D = 4$, for the horizon radius $\tilde{r}_H = 1$ and $\tilde{\Lambda} = -3/2$ ($\tilde{\ell} = 2$) with the additional boundary conditions, we find $\phi_H = 2.33422$ in order to obtain $\phi_- = 0$, and $\delta_H = -0.02893$, $\phi_0 = 2.43279$ and $\tilde{M}_0 = 0.28014$.

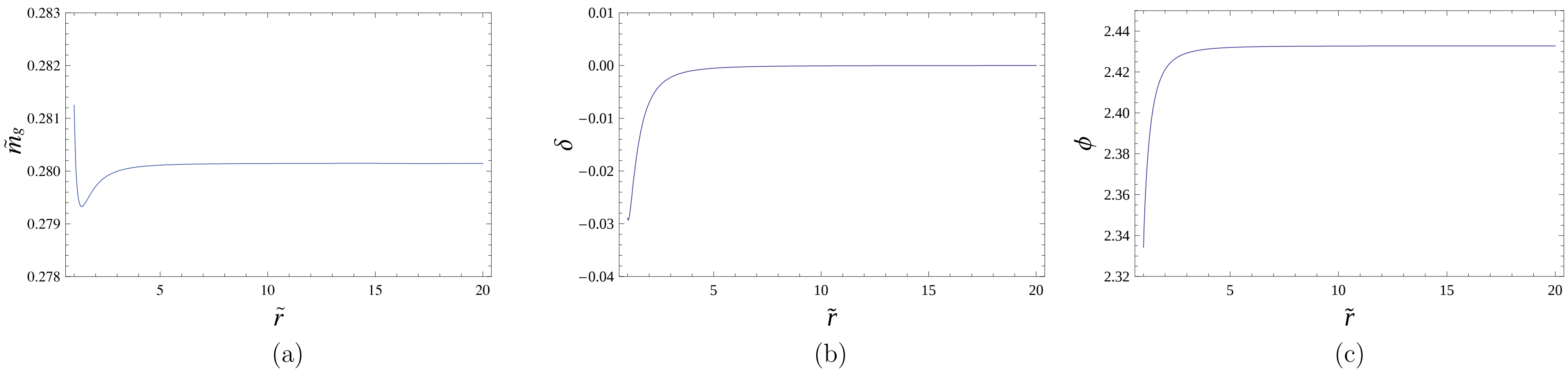


Figure 3: The configurations of the field functions (a) $\tilde{m}_g$, (b) δ and (c) ϕ in four dimensions for $\tilde{r}_H = 1$ and $\tilde{\Lambda} = -3/2$.

By using the symmetry, we can generate solutions for other $\tilde{\Lambda}$ and $\tilde{r}_H$ and the gravitational mass $\tilde{M}_0$:

$$\tilde{M}_0 = 0.28014 \left(\frac{2|\tilde{\Lambda}|}{3} \right)^3 \tilde{r}_H^{\,3} \,.$$

For $D = 5, 6, 10$ solutions, the configurations of the field functions are similar.

$D = 5$**: For the horizon radius $\tilde{r}_H = 1$ and $\tilde{\Lambda} = -3$ $(\tilde{\ell} = 2)$ with the additional boundary conditions at the horizon, we find $\phi_H = 9.35869$, $\delta_H = -0.02188, \phi_0 = 9.43249$ and $\tilde{M}_0 = 3.78189$.**

$$\tilde{M}_0 = 3.7819 \left(\frac{|\tilde{\Lambda}|}{3} \right)^3 \tilde{r}_H^{\,4} .$$

$D = 6$**:**

$$\tilde{M}_0 = 19.933 \left(\frac{|\tilde{\Lambda}|}{5} \right)^3 \tilde{r}_H^{\,5} .$$

$D = 10$**: The gravitational mass $\tilde{M}_0$ is given by** $\tilde{M}_0 = 771.68 \left(\frac{|\tilde{\Lambda}|}{18} \right)^3 \tilde{r}_H^{\,9}$.

The mass of the black hole approaches 0 as $\tilde{r}_H \to 0$.

5 Summary of the solutions

Table 1: **Summary of the black hole solutions in the dilatonic Einstein-Gauss-Bonnet theory in various dimensions. The head "existence" shows the existence of the black hole solution which satisfy appropriate boundary conditions. For the $\Lambda = 0$, $k = -1$ and the $\Lambda = 1$, $k = 1$ cases, "no" means that the non-existence of the solution cannot be proved exactly but we cannot find the solution numerically (most probebly no existence). For the $\Lambda = 1$, $k = -1$ case, there are solutions for $D = 5, 6$, and 10 but there is no solution for $D = 4$. The head "asymptotics" shows the asymptotic structure of the solution. There is no asymptotically dS solution generically. The head "lower bound" and "upper bound" show the existence of the lower and upper bounds for the horizon radius. For the $\Lambda = 1$, $k = 1$ case, there is the lower bound for $D = 4$, 5 but no bound for $D = 6$, 10. The head "paper" shows the number in the series of our paper where the model is discussed.**

	k	existence	asymptotics	lower bound	upper bound	paper
$\Lambda = 0$	$k = 1$	yes	flat	yes ($D = 4$, 5) no ($D = 6$, 10)	no	I
	$k = 0$	no	—	—	—	II
	$k = -1$	"no"	—	—	—	IV
$\Lambda = 1$	$k = 1$	"no"	—	—	—	IV
	$k = 0$	no	—	—	—	IV
	$k = -1$	yes ($D \geq 5$) no ($D = 4$)	AdS	yes	yes	IV
$\Lambda = -1$	$k = 1$	yes	AdS	yes	no	III
	$k = 0$	yes	AdS	no	no	II
	$k = -1$	yes	AdS	yes	no	III

6 Global Structure or Singularity

We study the internal structure of our black holes by integrating the field equations inward from the horizon.

We find that there appears non-central singularity between horizon and the center for lower-dimensional cases ($D = 4, 5$) in asymptotically flat solutions and in $k = +1$ and 0 asymptotically AdS solutions, where the metric does not diverge but the Kretschmann invariant does diverge. Hence this is a singularity, but we find the singularity is much milder than the Schwarzschild solution and the non-dilatonic one.

Table 2: **Summary of the divergent rate of the Kretschmann invariant $\mathcal{I}$ around the singularity in the $\Lambda = 0$ case. We also show those in GR and non-dilatonic cases for comparison.**

k	D	$\mathcal{I}$		$\mathcal{I}$ (GR)	$\mathcal{I}$ (non-dilatonic)
1	4	$(r - r_s)^{-4}$		$r^{-(2D-2)}$	$r^{-(D-1)}$
	5	$r^{-25.8}$	$(M_0 < 7.46)$		
		$(r - r_s)^{-4}$	$(M_0 > 7.46)$		
	6	$r^{-32.2}$	$(5.18 < M_0 < 120.9)$		
		$r^{-(D-1)}$	(otherwise)		
	10	$r^{-(D-1)} < \mathcal{I} < r^{-11.9}$			

We call this "fat singularity" because it is weakened due to the presence of the GB term and dilaton.

The formation mechanism of the fat singularity: the dilaton diverges.

Table 3: **Summary of the divergent rate of the Kretschmann invariant $\mathcal{I}$ around the singularity in the $\Lambda = -1$ case. We also show those in GR and non-dilatonic cases for comparison.**

<table>
<tr><th>k</th><th>D</th><th>$\mathcal{I}$</th><th>$\mathcal{I}$ (GR)</th><th>$\mathcal{I}$ (non-dilatonic)</th></tr>
<tr><td rowspan="4">1</td><td>4</td><td>$(r - r_s)^{-4}$</td><td rowspan="8">$r^{-(2D-2)}$</td><td>r^{-6}</td></tr>
<tr><td>5</td><td>$r^{-25.8}$ $(M_0 < 8.39)$
$(r - r_s)^{-4}$ $(M_0 > 8.39)$</td><td rowspan="3">$r^{-(D-1)}$</td></tr>
<tr><td>6</td><td>$r^{-32.2}$</td></tr>
<tr><td>10</td><td>$r^{-(D-1)} < \mathcal{I} < r^{-21.0}$ $(M_0 < 1.93)$
$r^{-57.6}$ $(M_0 > 1.93)$</td></tr>
<tr><td rowspan="4">0</td><td>4</td><td rowspan="2">$(r - r_s)^{-3}$</td><td>r^{-6}</td></tr>
<tr><td>5</td><td rowspan="3">$r^{-(D-1)}$</td></tr>
<tr><td>6</td><td>$r^{-32.2}$</td></tr>
<tr><td>10</td><td>$r^{-57.6}$</td></tr>
<tr><td rowspan="4">-1</td><td>4</td><td rowspan="4">$(r - r_s)^{-4}$ $(M_0 \neq 0)$
r^0 $(M_0 = 0)$</td><td rowspan="4">$r^{-(2D-2)}$ $(M_0 \neq 0)$
r^0 $(M_0 = 0)$</td><td>r^{-6} $(M_0 \neq 0)$, r^0 $(M_0 = 0)$</td></tr>
<tr><td>5</td><td rowspan="3">$(r - r_b)^{-3}$ $(M_0 < 0)$
r^0 $(M_0 = 0)$
$r^{-(D-1)}$ $(M_0 > 0)$</td></tr>
<tr><td>6</td></tr>
<tr><td>10</td></tr>
</table>

Other cases: **singularity at the center, much stronger than usual.**

All these singularities are spacelike, and our black hole solutions have only three different types of the global structures; the Schwarzschild, Schwarzschild-AdS and "regular AdS black hole" types.

The singularity in the theory exists for positive mass whereas that for pure GB term exists only for the unphysical case of negative mass.

So much for the solutions. Now application.

7 Shear viscosity

This solution has been used to study higher order corrections to shear viscosity to entropy density and the naive lower bound $1/4\pi$ as well as the new bound $4/25\pi$ from GB correction (without dilaton) may be violated.

R. G. Cai, Z. Y. Nie, N. O. and Y. W. Sun, Phys. Rev. D 79 (2009) 066004 [arXiv:0901.1421 [hep-th]].

Field theories behave hydrodynamically at large distances and time scales.

An interesting example is the ratio of shear viscosity to entropy density η/s, which is measurable in quark-gluon plasma (QGP) produced at RHIC. This value is unusually small, roughly $\sim 1/4\pi$.

The shear viscosity comes from hydrodynamic description of a field theory at larger distances and time-scales.

In the lowest order, the stress tensor is given by the familiar formula for ideal fluids:

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + P g^{\mu\nu}$$

At the next order, one has

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + P g^{\mu\nu} - \sigma^{\mu\nu}$$

where σ is proportional to derivatives of $T(x)$ (temp.) and $u_\mu(x)$ and is termed the dissipative part of $T_{\mu\nu}$.

According to the linear response theory, the shear viscosity can be calculated by the Kubo formula:

$$\eta = i \lim_{\omega \to 0} \frac{1}{\omega} \mathrm{Im} G^R_{xy,xy}(\omega, 0)$$

where G^R is the zero spatial momentum, low-frequency limit of the retarded Green's function of T_{xy}

$$G^R_{xy,xy}(\omega, 0) = \int dt d\boldsymbol{x} e^{i\omega t} \theta(t) \langle [T_{xy}(t, \boldsymbol{x}), T_{xy}(0,0)] \rangle = -i\eta\omega + O(\omega^2)$$

Consider a massless scalar field $\partial_\mu(\sqrt{-g} g^{\mu\nu} \partial_\nu \phi) = 0$ in AdS$_5$ black hole

$$ds^2 = \frac{(\pi T R)^2}{u^2}(-f(u) dt^2 + d\boldsymbol{x}^2) + \frac{R^2}{4u^2 f(u)} du^2 + R^2 d\Omega_5^2, \quad (f(u) = 1 - u^2)$$

The solution to this equation with the boundary condition $\phi = \phi_0$ at

$u=0$ **is** $\phi(p,u)=f_p\phi_0(p)$ **with**

$$f_p''-\frac{1+u^2}{uf}f_p'+\frac{w^2}{uf^2}f_p-\frac{q^2}{uf}f_p=0,\quad w=\frac{\omega}{2\pi T},\, q=\frac{k}{2\pi T}$$

The action of the scalar field:

$$S=\int\frac{d^4p}{(2\pi)^4}\phi_0(-p)\mathcal{F}(p,u)\phi_0(p)\Big|_{u=0}^{u=u_H}$$

The Green function:

$$G^R(p)=2\lim_{u\to 0}\mathcal{F}(p,u)$$

We can apply this idea to T_{xy}, which is dual to g_{xy}.

The equation of motion for $\phi=h_y^x$ is just the one of massless scalar filed:

$$\phi_p''-\frac{1+u^2}{uf}\phi_p'+\frac{w^2-q^2f}{uf^2}\phi_p=0$$

The solution, incoming at the horizon and normalized at $u=0$, is

$$f_p(z)=(1-u^2)^{-iw/2}+O(w^2,q^2)$$

From the action

$$S=-\frac{\pi^2N^2T^4}{8}\int du\frac{f}{u}\phi'^2\quad\Rightarrow\quad G^R_{xy,xy}(\omega,k)=-\frac{\pi^2N^2T^4}{4}iw\quad\Rightarrow\quad\eta=\frac{\pi}{8}N^2T^3$$

via the Kubo formula.

$$s=\frac{S}{V}=\frac{\pi^2}{2}N^2T^3 \quad \Rightarrow \quad \frac{\eta}{s}=\frac{1}{4\pi}$$

Next order correction

$$W=C^{hmnk}C_{pmnq}C_h{}^{rsp}C^q{}_{rsk}+\frac{1}{2}C^{hkmn}C_{pqmn}C_h{}^{rsp}C^q{}_{rsk} \quad \Rightarrow \quad \frac{\eta}{s}=\frac{1}{4\pi}\Big(1+\frac{135\zeta(3)}{8(g^2N)^{3/2}}\Big)$$

To summarize the procedure:

1. **Solve the equation of motion with the following boundary conditions:**
   (a) **impose the infalling boundary condition at the horizon**
   (b) **at the infinity $\phi_a(r;k)|_{r=1/\epsilon}=J_a(k),\ \ k=(\omega,q)$ where $\epsilon\to 0$ gives an infrared cutoff near the infinity of spacetime, and $J_a(k)$ is an infinitesimal boundary source for the bulk source $\phi_a(r;k)$.**

2. **Put the solution into the action, reducing to pure surface term**

$$S=\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}J_a(-k)\mathcal{F}_a(k,r)J_a(k)\Big|_{r=1/\epsilon}$$

to find the retarded function $G_a(k)$ in momentum space for the boundary field dual to ϕ_a:

$$G_a(k)=\lim_{\epsilon\to 0}\mathcal{F}_a(k,r)\Big|_{r=1/\epsilon}$$

3. Use the Kubo formula to get the shear viscosity

$$\eta = i \lim_{\omega \to 0} \frac{1}{\omega} \mathrm{Im} G_{xy,xy}(\omega, 0)$$

We apply this to our dilatonic EGB theory:

$$S = \frac{1}{16\pi G} \int d^5x \sqrt{-g} \Big[R - \frac{1}{2} \nabla_\mu \phi_d \nabla^\mu \phi_d + \frac{\lambda l^2}{2} e^{-\gamma \phi_d} (R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2) - 2\Lambda e^{\tau \phi_d} \Big],$$

After some calculation, the effective action of the transverse gravitons ϕ for this gravity theory can be written as

$$S = \frac{1}{16\pi G} \left(-\frac{1}{2} \right) \int d^5x \sqrt{-g} \tilde{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi,$$

The shear viscosity is found to be

$$\eta = \frac{1}{16\pi G} \frac{r_+^3}{l^3} \left(1 - 4\lambda e^{(\tau - \gamma)\phi_d(1)} \left(1 + 2\gamma {\phi_d}'(1) \right) \right).$$

For this Ricci-flat black hole, the entropy still obeys the Bekenstein-Hawking entropy area law, so the entropy density is

$$s = \frac{1}{4G}\frac{r_+^3}{l^3}.$$

The ratio of shear viscosity over entropy density

$$\frac{\eta}{s} = \frac{1}{4\pi}\left(1 - 4\lambda e^{(\tau-\gamma)\phi_d(1)}\left(1 + 2\gamma\phi_d{}'(1)\right)\right).$$

pure GB case: obtained in the limit $\phi_d(1) \to 0$: $\frac{\eta}{s} = (1-4\lambda)/(4\pi)$, which gives lower bound $4/(25\pi)$ lower than $1/(4\pi)$.

We find that this new shear viscosity bound (due to GB term) could be weakly violated due to dilaton!!

8 Discussions

We find dilatonic black holes in string theory with GB correction in various dimensions, with and without cosmological constant.

The singularity structure exhibit new features due to the presence of GB term and dilaton. In particular, the dilaton significantly affects the singularity, sometimes taming it.

These solutions are expected to be useful to study strong coupling behaviors of field theories via AdS/CFT correspondence. We emphasize that, without dilaton there is no foundation to rely to AdS/CFT correspondence from string theory!

Remaining problems:

1. **The ambiguity of the frames: We have studied the solution in the Einstein frame.**

   There is, however, a possibility that the properties of solutions changes drastically by transforming to the string frame. In particular, the conformal transformation may become singular.

   ⇒ Work with Maeda and Sasagawa.

   ⇒ Existence of mass gap!

2. Charged solution:

It would be also interesting to extend our analysis to dilatonic black holes (large and small) with charges. $\Rightarrow$ Work with C.M. Chen et al.

We studied BPS solutions in the system of dilatonic Einstein-Maxwell-GB system for general dilaton coupling (common to Maxwell and GB), and found that BH solution exists for certain range of the dilaton coupling.

In particular, it cannot be too big; otherwise we encounter cusp type of singularity when we integrate from the horizon toward asymptotics.

In $D = 4$, there is no BPS solution for heterotic value. Beyond that, there are solutions.

One interesting feature is that from $D = 7$, the cusps appear in pairs, and solutions can be continued to infinity, in contrast to lower dimensions.

3. Stability:

The stability of our solutions is another important subject to study.

We hope that our solutions have more allpications!!

Thank you!