Relating Gauge Theories via the Gauge/Bethe Correspondence (II)

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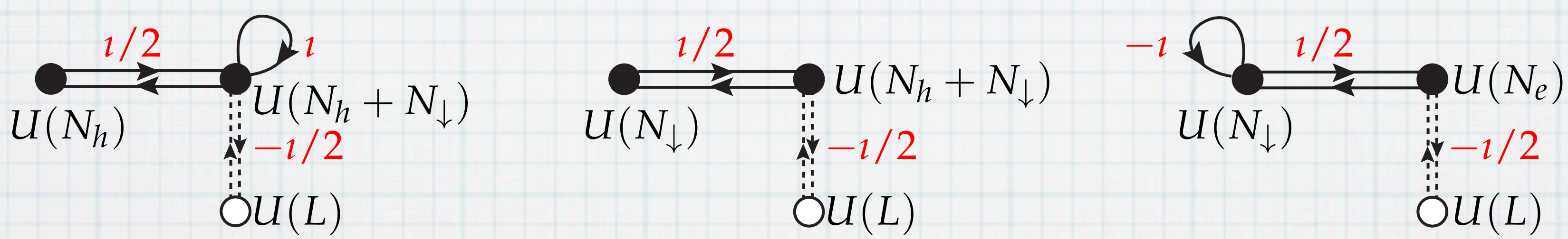
based on arXiv:1005.4445 with D. Orlando

Intro/Summary

Use techniques from integrable models to relate susy gauge theories.

Tool: Gauge/Bethe correspondence as stated by Nekrasov/Shatashvili.

Statement: 3 different N=(2,2) quiver gauge theories in 2d have the same susy ground states:



(a) Case A (b) Case B (c) Case C

Correspondence to an integrable spin chain: tJ model

Can apply this technique in a general context.

Outline Part I

Outline

1. *Why are we here?*
2. *Bethe Ansatz for the XXX spin chain*
   - Parameters in a spin chain
   - XXX
   - Algebraic Bethe Ansatz
3. *Effective twisted superpotential in two dimensions*
   - Basics of $\mathcal{N} = (2,2)$ field theories
   - Effective theory in the Coulomb branch
4. *The Gauge/Bethe Correspondence*

Outline Part II

- Gauge/Bethe correspondence
- The tJ model
- Supergroup symmetry
- Bethe ansatz
- The Dictionary
- Quiver gauge theories
- Relation via brane cartoons
- Generalizations/Open questions
- Summary

The Gauge/Bethe Correspondence

Relates N=(2,2) gauge theories in 2d to integrable spin chains.

The susy vacua of the gauge theory correspond to the Bethe spectrum of the spin chain.

Generators of chiral ring correspond to commuting Hamiltonians. Nekrasov, Shatashvili

Integrable model: spectrum determined by Bethe equations.

Gauge theory: ground states determined by eff. twisted superpotential.

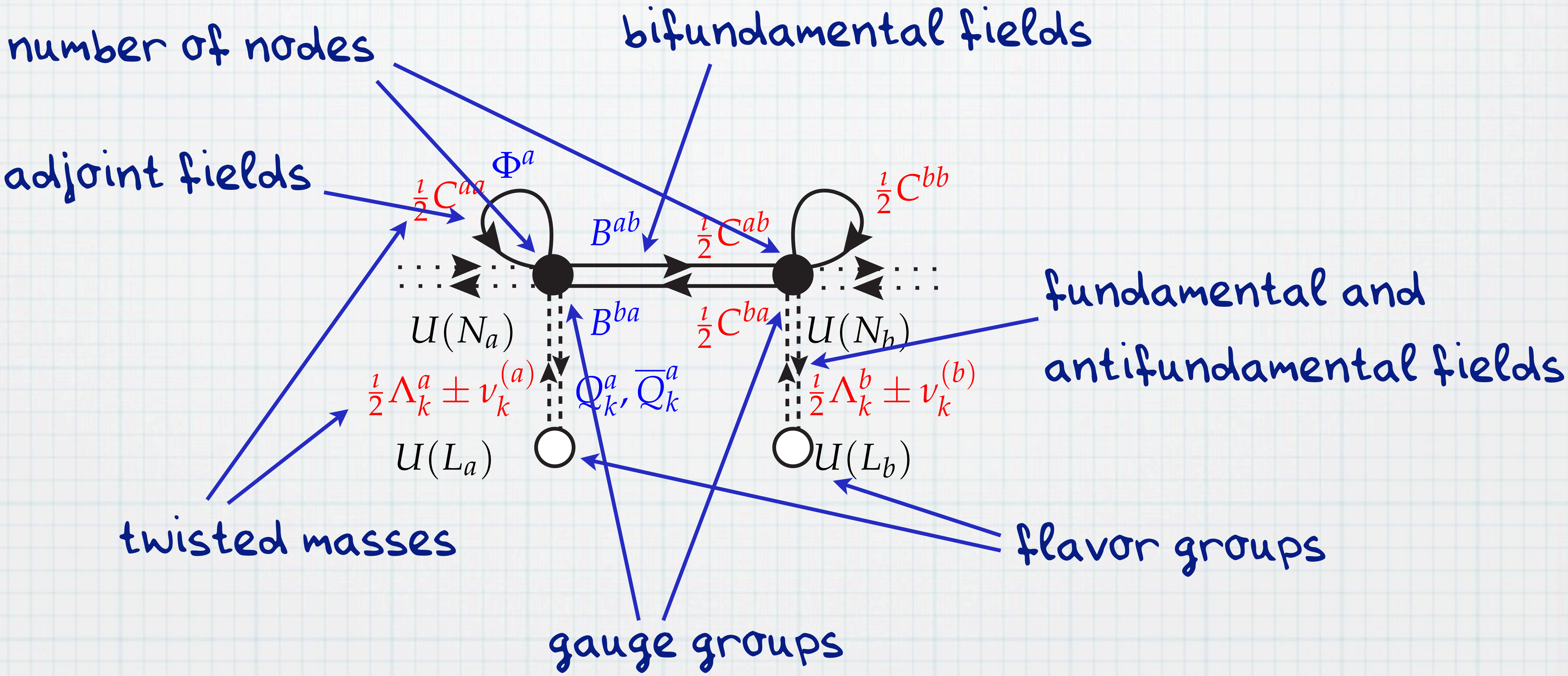
Correspondence works for all Bethe solvable spin chains.

Spin chains with supergroup symmetry correspond to quiver gauge theories.

The Gauge/Bethe Correspondence

Here: quiver gauge theories

What are the parameters?



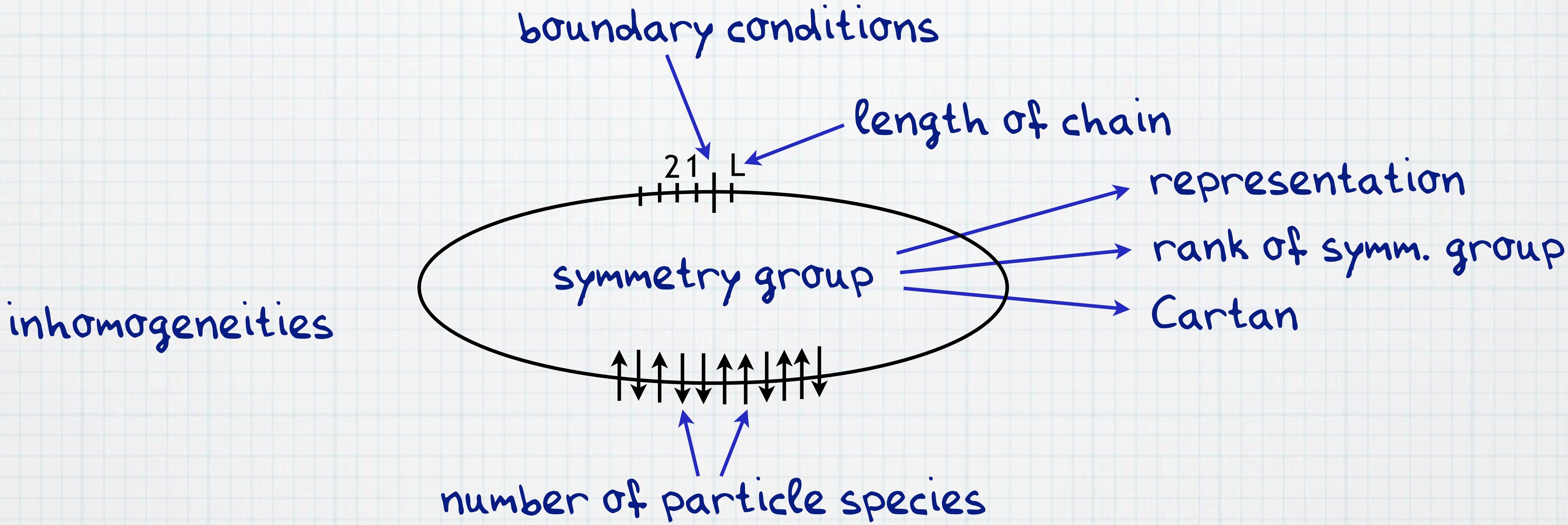
Vacuum equation:

low energy eff. twisted superpotential

$$\exp\left[2\pi\frac{\partial\widetilde{W}_{\text{eff}}(\sigma)}{\partial\sigma_i}\right] = 1$$

The Gauge/Bethe Correspondence

What are the parameters of a spin chain?



spectrum is given by solutions of

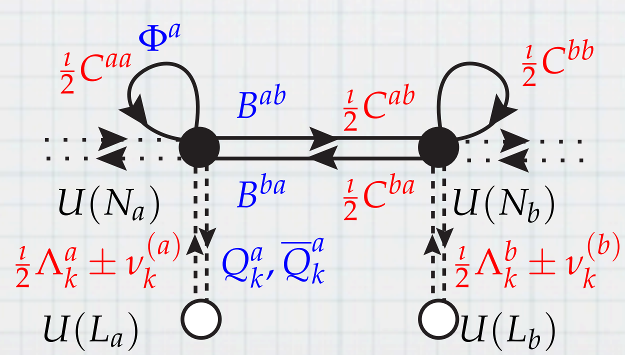
rapidities

$$e^{2\pi i \, \mathrm{d}Y(\lambda)} = 1$$

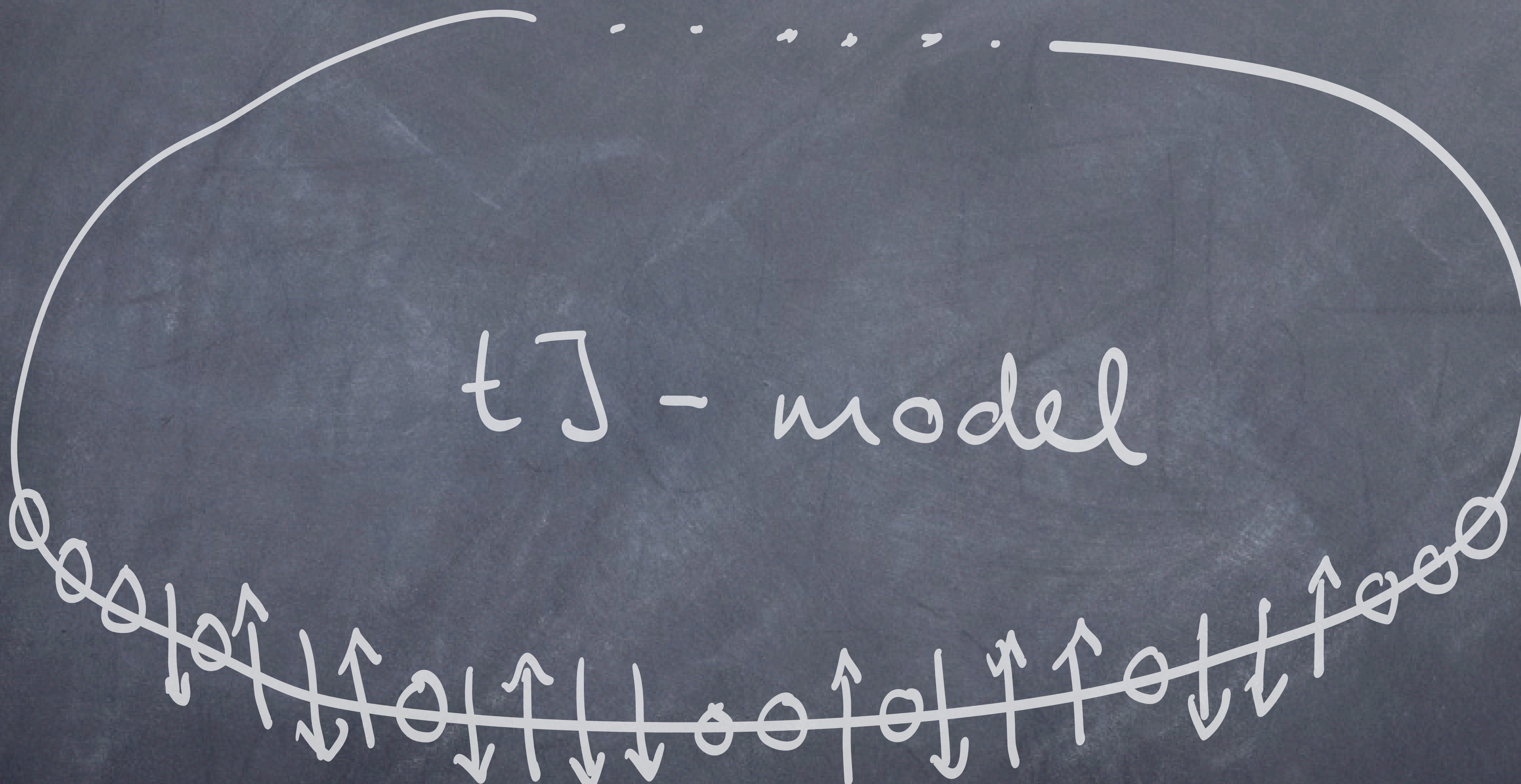
Yang counting fn (potential for Bethe equations)

The Gauge/Bethe Correspondence

gauge theory			integrable model
number of nodes in the quiver	r	r	rank of the symmetry group
gauge group at a–th node	$U(N_a)$	N_a	number of particles of species a
effective twisted superpotential	$\widetilde{W}_{\text{eff}}(\sigma)$	$Y(\lambda)$	Yang–Yang function
equation for the vacua	$e^{2\pi \mathrm{d}\widetilde{W}_{\text{eff}}} = 1$	$e^{2\pi\imath \mathrm{d}Y} = 1$	Bethe ansatz equation
flavor group at node a	$U(L_a)$	L_a	effective length for the species a
lowest component of the twisted chiral superfield	$\sigma_i^{(a)}$	$\lambda_i^{(a)}$	rapidity
twisted mass of the fundamental field	$\widetilde{m}^{\mathrm{f}}{}_k^{(a)}$	$\frac{1}{2}\Lambda_k^a + \nu_k^{(a)}$	highest weight of the representation and inhomogeneity
twisted mass of the anti–fundamental field	$\widetilde{m}^{\bar{\mathrm{f}}}{}_k^{(a)}$	$\frac{1}{2}\Lambda_k^a - \nu_k^{(a)}$	highest weight of the representation and inhomogeneity
twisted mass of the adjoint field	$\widetilde{m}^{\mathrm{adj}\,(a)}$	$\frac{1}{2}C^{aa}$	diagonal element of the Cartan matrix
twisted mass of the bifundamental field	$\widetilde{m}^{\mathrm{b}\,(ab)}$	$\frac{1}{2}C^{ab}$	non–diagonal element of the Cartan matrix
FI–term for $U(1)$–factor of gauge group $U(N_a)$	τ_a	$\hat{\vartheta}^a$	boundary twist parameter for particle species a



tJ - model



The tJ model

Properties:

- spin chain of length L
- periodic boundary conditions
- 3 particle species: spin up, spin down, empty
- sl(1|2) symmetry
- most important feature: 3 inequivalent choices of Cartan matrix

Essler, Korepin

- 3 different (but equivalent) sets of Bethe equations
- spectrum the same for all 3 cases
- fundamental rep. (spin 1/2) at each lattice point
- no inhomogeneities

The tJ model

Electrons on a lattice of length L, $\uparrow, \downarrow, \circ$

Hilbert space: $\mathscr{H}_k = \mathbb{C}^{(1|2)}$

Fundamental representation of $sl(1|2)$

creation/annihilation operators: $c^\dagger_{k,s}, c_{k,s}\,, \quad s = \{\uparrow, \downarrow\}$

$$|s\rangle_k \quad = c^\dagger_{k,s}|\circ\rangle_k$$

vacuum

projector on single occupancy

$$S^-_k = c^\dagger_{k,\uparrow}c_{k,\downarrow}\,, \quad S^+_k = c^\dagger_{k,\downarrow}c_{k,\uparrow}\,, \quad S^z_k = \frac{1}{2}(n_{k,\uparrow} - n_{k,\downarrow})$$

$$n_{k,s} = c^\dagger_{k,s}c_{k,s}$$
$$n_k = n_{k,\uparrow} + n_{k,\downarrow}$$

$$\mathcal{H} = \sum_{k=1}^{L-1}\left[-t\mathcal{P}\sum_{s=\uparrow,\downarrow}\left(c^\dagger_{k,s}c_{k+1,s} + \text{h.c.}\right)\mathcal{P} + J\left(\mathbf{S}_k\cdot\mathbf{S}_{k+1} - \frac{1}{4}n_k n_{k+1} + 2\,n_k - \frac{1}{2}\right)\right]$$

nearest neighbor hopping

spin interaction

The tJ model

Number of holes, up and down spins:

$$N_h = \sum_{k=1}^{L}(1-n_k)\,, \quad N_\uparrow = \sum_{k=1}^{L} n_{k,\uparrow}\,, \quad N_\downarrow = \sum_{k=1}^{L} n_{k,\downarrow}\,, \quad N_e = N_\uparrow + N_\downarrow\,.$$

Single occupancy: $L = N_h + N_\uparrow + N_\downarrow$

tJ model has supergroup symmetry!

A superalgebra can be decomposed into an even and an odd part: $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$

The even part of $sl(1|2)$ is $\mathfrak{g}_0 = gl(1) \oplus sl(2)$ and is generated by $S^\pm, S^z, Z$

Fermionic generators: $Q^\pm_s\,,\ s = \{\uparrow, \downarrow\}$

$$S_k^- = c^\dagger_{k,\uparrow} c_{k,\downarrow}\,, \quad S_k^+ = c^\dagger_{k,\downarrow} c_{k,\uparrow}\,, \quad S_k^z = \frac{1}{2}\left(n_{k,\uparrow} - n_{k,\downarrow}\right)\,,$$

$$Q^+_{k,\downarrow} = (1-n_{k,\uparrow})\, c^\dagger_{k,\downarrow}\,, \quad Z_k = 1 - \frac{1}{2} n_k\,.$$

$$Q^-_{k,\uparrow} = (1-n_{k,\downarrow})\, c_{k,\uparrow}\,, \quad Q^+_{k,\uparrow} = (1-n_{k,\downarrow})\, c^\dagger_{k,\uparrow}\,, \quad Q^-_{k,\downarrow} = (1-n_{k,\uparrow})\, c_{k,\downarrow}\,,$$

Supergroup Symmetry

Root decomposition. Lie algebra: reflections of positive root systems are conjugate to each other.

Superalgebra: reflections of odd roots lead to **new** positive root system (not conjugate).

Here: three choices for Cartan matrix:

C^{ab}	Dynkin diagram
$\begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}$	$\otimes$ — $\bigcirc$
$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	$\otimes$ — $\otimes$
$\begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}$	$\bigcirc$ — $\otimes$

even root ($\bigcirc$) — odd root ($\otimes$)

Bethe ansatz

Case A:

$$C^{ab} = \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}, \quad \Lambda = [0\ 1], \quad N_1 = N_h, \quad N_2 = N_h + N_\downarrow$$

Nested Bethe ansatz equations:

$$\left(\frac{\lambda_i^{(a)} + \frac{i}{2}\Lambda^a}{\lambda_i^{(a)} - \frac{i}{2}\Lambda^a}\right)^L = \prod_{\substack{(b,j)=(1,1)\\(b,j)\neq(a,i)}}^{(r,N_b)} \frac{\lambda_i^{(a)} - \lambda_j^{(b)} + \frac{i}{2}C^{ab}}{\lambda_i^{(a)} - \lambda_j^{(b)} - \frac{i}{2}C^{ab}}, \quad a = 1, 2, \ldots, r, \quad i = 1, \ldots, N_a,$$

$$\left(\frac{\lambda_p^{(2)} + \frac{i}{2}}{\lambda_p^{(2)} - \frac{i}{2}}\right)^L = \prod_{\substack{q=1\\q\neq p}}^{N_h+N_\downarrow} \frac{\lambda_p^{(2)} - \lambda_q^{(2)} + i}{\lambda_p^{(2)} - \lambda_q^{(2)} - i} \prod_{i=1}^{N_h} \frac{\lambda_p^{(2)} - \lambda_i^{(1)} - \frac{i}{2}}{\lambda_p^{(2)} - \lambda_i^{(1)} + \frac{i}{2}}, \quad p = 1, \ldots, N_h + N_\downarrow,$$

$$1 = \prod_{p=1}^{N_h+N_\downarrow} \frac{\lambda_p^{(2)} - \lambda_i^{(1)} - \frac{i}{2}}{\lambda_p^{(2)} - \lambda_i^{(1)} + \frac{i}{2}}, \quad i = 1, \ldots, N_h.$$

Yang Yang counting function:

$$Y_A(\lambda) = \frac{L}{2\pi} \sum_{p=1}^{N_h+N_\downarrow} \hat{x}(2\lambda_p^{(2)}) - \frac{1}{2\pi} \sum_{\substack{p,q=1\\p\neq q}}^{N_h+N_\downarrow} \hat{x}(\lambda_p^{(2)} - \lambda_q^{(2)}) + \frac{1}{2\pi} \sum_{p=1}^{N_h+N_\downarrow} \sum_{i=1}^{N_h} \hat{x}(2\lambda_p^{(2)} - 2\lambda_i^{(1)})$$

$$+ \sum_{i=1}^{N_h} n_i^{(1)} \lambda_i^{(1)} + \sum_{p=1}^{N_h+N_\downarrow} n_p^{(2)} \lambda_p^{(2)}.$$

$$\hat{x}(\lambda) = \lambda \arctan(\lambda^{-1}) + \frac{1}{2} \log(1 + \lambda^2)$$

Bethe ansatz

Case B:

$$C^{ab} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \Lambda = [0\ 1], \quad N_1 = N_\downarrow, \quad N_2 = N_h + N_\downarrow .$$

Nested Bethe ansatz equations:

$$\left(\frac{\lambda_p^{(2)} + \frac{i}{2}}{\lambda_p^{(2)} - \frac{i}{2}} \right)^L = \prod_{i=1}^{N_\downarrow} \frac{\lambda_i^{(1)} - \lambda_p^{(2)} - \frac{i}{2}}{\lambda_i^{(1)} - \lambda_p^{(2)} + \frac{i}{2}}, \quad p = 1, \ldots, N_h + N_\downarrow$$

$$1 = \prod_{p=1}^{N_h+N_\downarrow} \frac{\lambda_i^{(1)} - \lambda_p^{(2)} - \frac{i}{2}}{\lambda_i^{(1)} - \lambda_p^{(2)} + \frac{i}{2}}, \quad i = 1, \ldots, N_\downarrow .$$

Yang Yang counting function:

$$Y_B(\lambda) = \frac{L}{2\pi} \sum_{p=1}^{N_h+N_\downarrow} \hat{x}(2\lambda_p^{(2)}) + \frac{1}{2\pi} \sum_{p=1}^{N_h+N_\downarrow} \sum_{i=1}^{N_\downarrow} \hat{x}(2\lambda_i^{(1)} - 2\lambda_p^{(2)}) + \sum_{i=1}^{N_\downarrow} n_i^{(1)} \lambda_i^{(1)} + \sum_{p=1}^{N_h+N_\downarrow} n_p^{(2)} \lambda_p^{(2)}$$

Bethe ansatz

Case C:

$$C^{ab} \; = \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}\,, \quad \Lambda = [0\ 1]\,, \quad N_1 = N_\downarrow\,, \quad N_2 \; = N_\uparrow + N_\downarrow\,.$$

Nested Bethe ansatz equations:

$$\left(\frac{\lambda_p^{(2)} - \frac{i}{2}}{\lambda_p^{(2)} + \frac{i}{2}}\right)^L \; = \; \prod_{i=1}^{N_\downarrow} \frac{\lambda_p^{(2)} - \lambda_i^{(1)} - \frac{i}{2}}{\lambda_p^{(2)} - \lambda_i^{(1)} + \frac{i}{2}}\,, \qquad p = 1, \ldots, N_e,$$

$$\prod_{p=1}^{N_e} \frac{\lambda_p^{(2)} - \lambda_i^{(1)} - \frac{i}{2}}{\lambda_p^{(2)} - \lambda_i^{(1)} + \frac{i}{2}} \; = \; \prod_{\substack{j=1\\ j\neq i}}^{N_\downarrow} \frac{\lambda_j^{(1)} - \lambda_i^{(1)} - i}{\lambda_j^{(1)} - \lambda_i^{(1)} + i}\,, \qquad i = 1, \ldots, N_\downarrow.$$

Yang Yang counting function:

$$Y_C(\lambda) = \frac{L}{2\pi}\sum_{p=1}^{N_e} \hat{x}(2\lambda_p^{(2)}) - \frac{1}{2\pi}\sum_{p=1}^{N_e}\sum_{i=1}^{N^\downarrow} \hat{x}(2\lambda_p^{(2)} - 2\lambda_i^{(1)}) + \frac{1}{2\pi}\sum_{\substack{i,j=1\\ i\neq j}}^{N_\downarrow} \hat{x}(\lambda_i^{(1)} - \lambda_j^{(1)})$$

$$+ \sum_{i=1}^{N_\downarrow} n_i^{(1)}\lambda_i^{(1)} + \sum_{p=1}^{N_e} n_p^{(2)}\lambda_p^{(2)}\,.$$

Dictionary

The Dictionary

gauge theory		integrable model	
number of nodes in the quiver	r	r	rank of the symmetry group
gauge group at a–th node	$U(N_a)$	N_a	number of particles of species a
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FI–term for $U(1)$–factor of gauge group $U(N_a)$	τ_a	$\hat{\vartheta}^a$	boundary twist parameter for particle species a

The Dictionary

The tJ model:

- spin chain of length L — U(L) flavor group
- periodic boundary conditions — unitary gauge groups
- 3 particle species: spin up, spin down, empty
- sl(1|2) symmetry — two nodes in quiver
- most important feature: 3 inequivalent choices of Cartan matrix — masses for adj. and bifund. fields
- 3 different (but equivalent) sets of Bethe equations — 3 gauge theories
- spectrum the same for all 3 cases — same susy ground states!
- fundamental rep. (spin 1/2) at each lattice point
- no inhomogeneities — tw. mass i/2 for fund. and antifund.

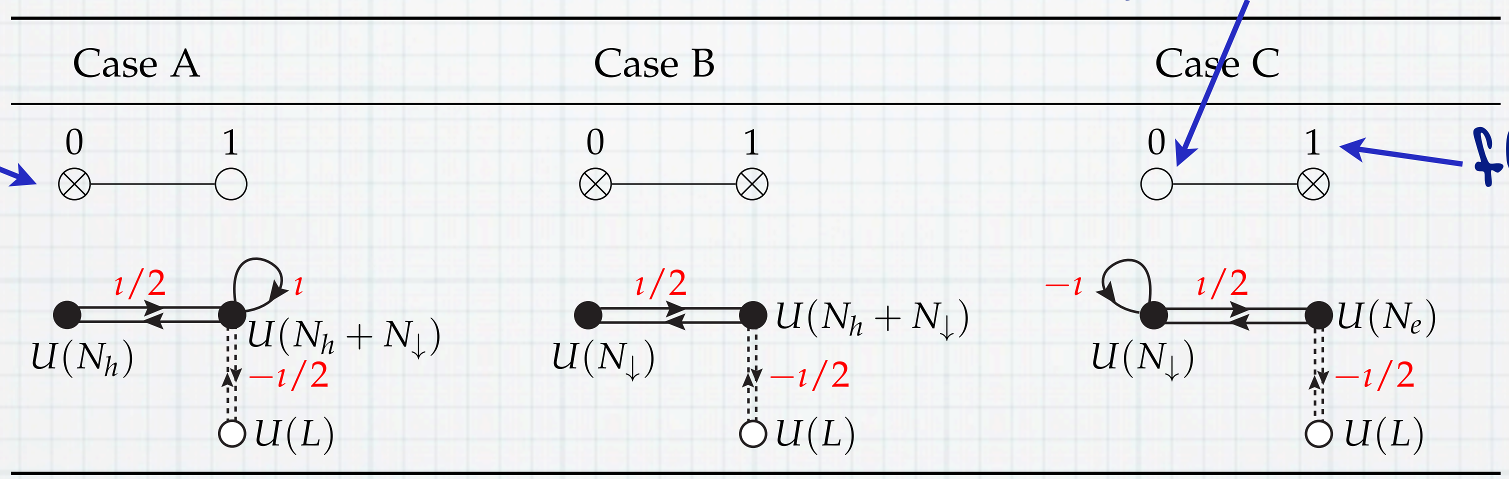
The Dictionary

Use Gauge/Bethe dictionary:

Kac Dynkin diagram

adjoint field

flavor group

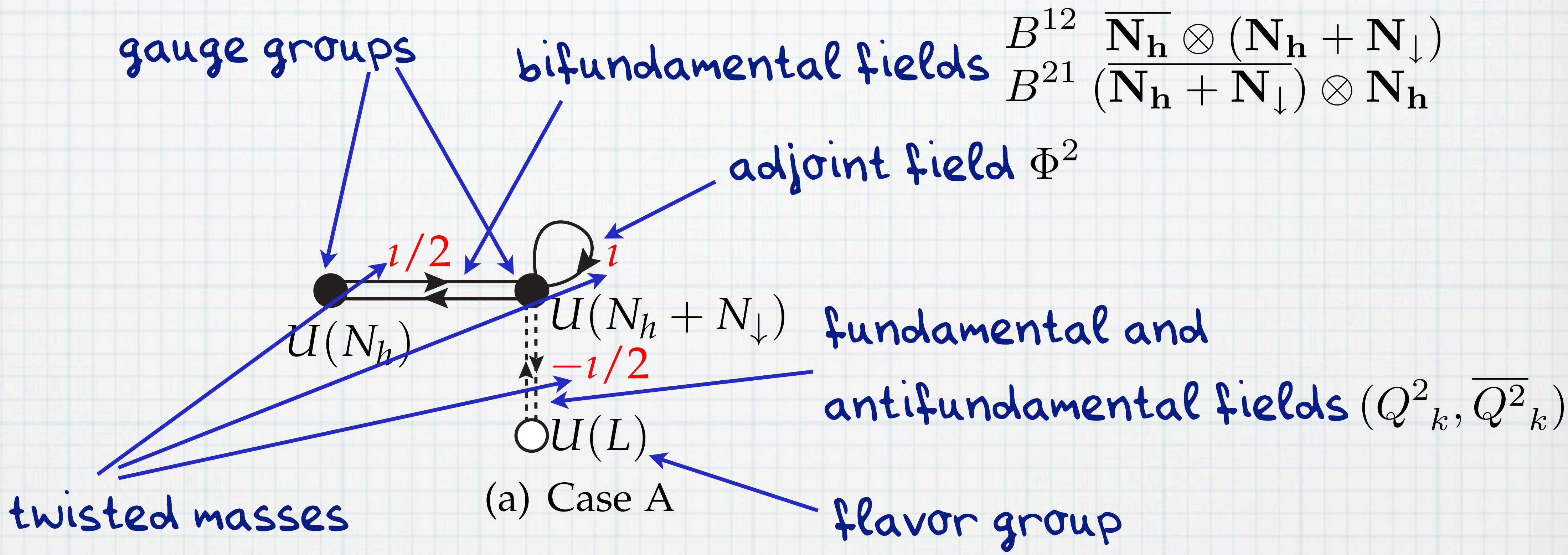


These 3 quiver gauge theories have the same ground states!

Conversely: any quiver gauge theories which can be associated to the same int. model have same ground states.

Quiver Gauge Theory

Case A: $C^{ab} = \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}$, $\Lambda = [0\ 1]$, $N_1 = N_h$, $N_2 = N_h + N_{\downarrow}$



(a) Case A

Global symmetry group:

$$U(L)_Q \times U(L)_{\widetilde{Q}} \times U(1)_B \times U(1)_{\widetilde{B}} \times U(1)_\Phi$$

Broken down to maximal torus by twisted masses.

Quiver Gauge Theory

Effective twisted superpotential:

$$\widetilde{W}^A_{\text{eff}}(\sigma) = \frac{L}{2\pi}\sum_{p=1}^{N_h+N_\downarrow}\left[\left(\sigma_p^{(2)}+\frac{\imath}{2}\right)\left(\log(\sigma_p^{(2)}+\frac{\imath}{2})-1\right)-\left(\sigma_p^{(2)}-\frac{\imath}{2}\right)\left(\log(-\sigma_p^{(2)}+\frac{\imath}{2})-1\right)\right]$$

$$+\frac{1}{2\pi}\sum_{i=1}^{N_h}\sum_{p=1}^{N_h+N_\downarrow}\left[\left(\sigma_i^{(1)}-\sigma_p^{(2)}-\frac{\imath}{2}\right)\left(\log(\sigma_i^{(1)}-\sigma_p^{(2)}-\frac{i}{2})-1\right)\right.$$

$$\left.-\left(\sigma_i^{(1)}-\sigma_p^{(2)}+\frac{\imath}{2}\right)\left(\log(-\sigma_i^{(1)}+\sigma_p^{(2)}-\frac{\imath}{2})-1\right)\right]$$

$$+\frac{1}{2\pi}\sum_{\substack{p,q\\p\neq q}}^{N_h+N_\downarrow}\left(\sigma_p^{(2)}-\sigma_q^{(2)}-\imath\right)\left(\log(\sigma_p^{(2)}-\sigma_q^{(2)}-\imath)-1\right)-\imath\tau_1\sum_{i=1}^{N_h}\sigma_i^{(1)}-\imath\tau_2\sum_{p=1}^{N_h+N_\downarrow}\sigma_p^{(2)}\,.$$

Corresponds to Y.

Superpotential (compatible to eff. tw. superpotential):

$$W_A(Q^2,\overline{Q^2},\Phi^2,B^{12},B^{21})=\sum_k\left[a\,{Q^2}_k\Phi^2\overline{Q^2}_k+b\,{Q^2}_kB^{21}B^{12}\overline{Q^2}_k\right]$$

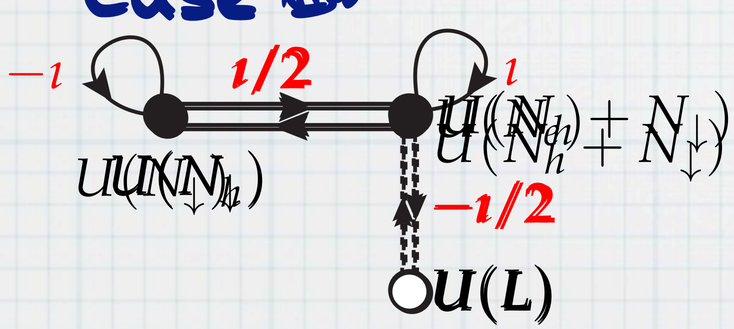
Relation via Brane Cartoons

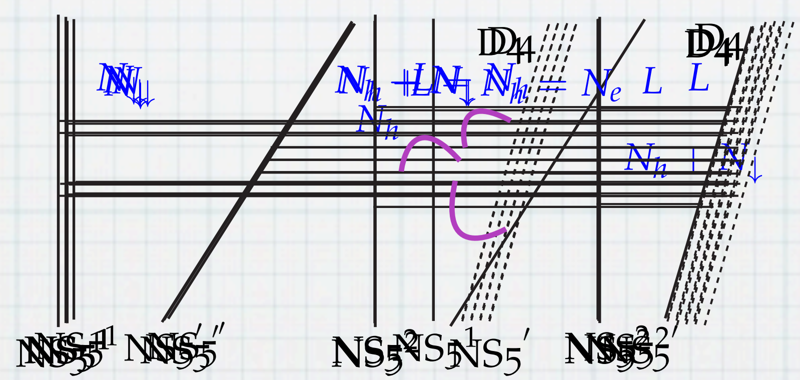
Different argument for relation: brane motions

Hanany, Witten;
Hanany, Hori

	0	1	2	3	4	5	6	7	8	9
$NS5^{1,2}$	×	×	×	×	×	×				
$NS5'$	×	×	×	×					×	×
D2	×	×					×			
D4	×	×						×	×	×

Case B:





Not surprising that the relation between the series can be seen in different ways.

But: difficult to turn on twisted masses!

Generalization/Open questions

Any spin chain with supergroup symmetry gives rise to several quiver gauge theories.

The supergroup $sl(m|n)$ gives rise to $\binom{m+n}{m}$ distinct quiver gauge theories.

Open questions:

Understand the meaning of twisted masses in gauge theory better.

Reproduce twisted masses in brane realizations.

Study and compare soliton solutions for the different quiver gauge theories corresponding to one spin chain.

Summary

The Gauge/Bethe correspondence of Nekrasov/Shatashvili relates the susy ground states of a 2d N=(2,2) gauge theory to the full spectrum of an integrable model.

We use this correspondence as a tool to relate different quiver gauge theories which correspond to the same integrable system.

Works for all integrable models with supergroup symmetry.

Window from the very well controlled integrable models into gauge theory.

Thank you for your attention!