WALL CROSSING, FREE FERMIONS & MATRIX MODELS

1° Introduction

$$
\begin{tikzcd}
\text{Topological strings} \arrow[rr, leftrightarrow, "\substack{\text{G-V} \\ \text{O-S-V}}"] \arrow[dr, "\text{D-V}"'] & & \text{BPS counting} \arrow[dl, leftrightarrow, "?"] \\
& \text{Matrix Models} &
\end{tikzcd}
$$

Topological strings

- A-model $\quad F_g = \{ \# \; \Sigma_g \rightsquigarrow \text{Calabi-Yau 3fold} \}$

$$= \sum_\beta N_{g,\beta} e^{-\beta t}$$

$N_{g,\beta}$: G-W inv $\qquad t$: Kähler moduli

$$F = \sum F_g g_s^{2g-2} \Rightarrow Z_{top} = e^F$$

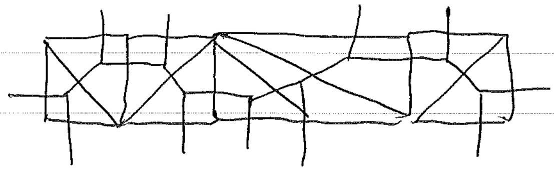
G-V: $$Z_{top} = \prod_{n=1}^{\infty} \prod_{\beta, m} \left(1 - q^{n+m} Q^\beta\right)^{m \underline{\underline{N_m^\beta}}}$$

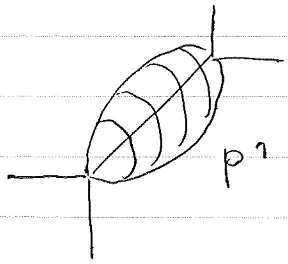
G-V inv $\in \mathbb{Z}$

Consider Toric local manifolds

(with no compact 4-cycles)

TORIC

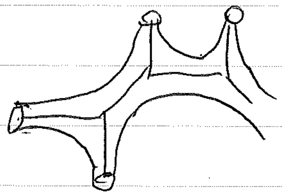




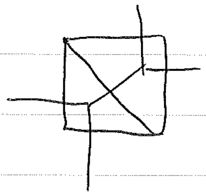
topological vertex

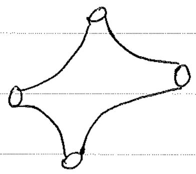
B-model : mirror manifold : $xy + H(e^v, e^u) = 0$

mirror curve : $H(e^u, e^v) = 0$



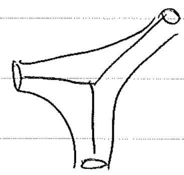
eg Resolved conifold $x_1 x_2 - x_3 x_4 = 0$





$e^u + e^v + e^{u-v-t} + 1 = 0$

eg $\mathbb{C}^3$



$e^u + e^v + 1 = 0$

eg. $\mathbb{C}^3/\mathbb{Z}_N$

4-cycle $\leftrightarrow$ loops in toric web

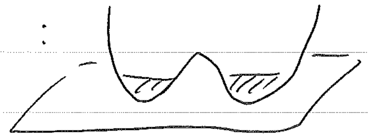
Matrix Models

$$Z_{MM} = \int DM\, e^{-\frac{1}{\hbar}\mathrm{Tr}\, V(M)} \longrightarrow e^{\sum \hbar^{2g-2} F_g}$$

– Hermitian : $DM \longrightarrow \prod_i dx_i \prod_{k<l} (x_k - x_l)^2$

Unitary : $DM \longrightarrow \prod_i du_i \prod_{k<l} \sin^2(u_k - u_l)$

Potential :



$N \to \infty$

$F_0 \longleftrightarrow$ spectral curve

Resolvent $\omega(x) = \left\langle \mathrm{Tr} \frac{1}{x-M} \right\rangle$

<u>eg</u> $V(M) = \frac{M^2}{2}$ $\longrightarrow$ spectral curve $x^2 - y^2 \sim T = N\hbar$

M: Hermite ('t Hooft param)

M: Unitary

$e^x + e^y + e^{x-y-T} + 1 = 0$

(Mirror of resolved conifold)

$$\left(\begin{aligned} p &= \omega_+ - \omega_- \\ y &= \omega_+ + \omega_- \end{aligned} \right)$$

Eynard - Orantin

Recursion Relations $\longrightarrow$ Matrix Model F_g

spectral curve

$$\check{Z}_{MM} = Z_{EO}$$

REMODELING CONJECTURE (BKMP)

$$Z_{top}[xy - H(e^u, e^v) = 0] = Z_{EO}[H(e^u, e^v) = 0]$$

Motivation : Relate $Z_{MM} \longleftrightarrow Z_{BPS}$

$Z_{BPS} \rightarrow Z_{top} \Rightarrow$ Prove Remodeling conjecture

— symplectic invariant

— BCOE Hol Anomaly eq

— Integrability

BPS counting

1 D6 / D2 / D0 on local toric mfd without 4 cycles

IIA $S_t \times \mathbb{R}^3 \times CY$ FIND: $Z_{BPS} = \sum_{\beta, n} \Omega_{\underset{D0}{n}, \underset{D2}{\beta}} Q^{\beta} q^{n}$

Denef - Moore

central charge $\zeta(\Gamma) = \int_{CY} e^{-(\underset{\text{B-field}}{B} + i \underset{\text{Kähler}}{J})} \cdot \Gamma$

$$\langle \Gamma_1, \Gamma_2 \rangle \, \mathrm{Im}\left(\zeta(\Gamma_1) \overline{\zeta(\Gamma_2)}\right) > 0$$

IIA : $S_t \times \mathbb{R}^3 \times CY$

M-theory $S_t \times TN_1 \times \underbrace{CY}_{M2}$ $\left(S^1_{TN} \rightarrow TN_1 \downarrow \mathbb{R}_3 \right)$

$$R(S^1_{TN}) \longrightarrow \infty \Longrightarrow S_t \times \mathbb{R}^4 \times \underbrace{CY}_{M2}$$

$$\phi(z_1, z_2) = \sum_{i_1, i_2} \alpha_{i_1, i_2} z_1^{i_1} z_2^{i_2}$$

$$Z_{BPS} = \mathrm{Tr}\left(Q^{\Theta_2} q^{\Theta_0}\right) \qquad \zeta = n + \beta B + i\,\underline{[J]} \qquad > 0$$

$$\longrightarrow 0$$

$$= \prod_{n=1}^{\infty} \prod_{\beta, m} \left(1 - q^{n+m} Q^{\beta}\right)^{n N_\beta^m}$$

$$(n + \beta B > 0)$$

◎ Relation to statistical Mechanics models

<u>QUIVER</u> $\rightarrow$ BIPARTITE GRAPH on T^2

Dimer $\Leftrightarrow$ Crystal Model

$$\boxed{Z^{(B)}_{BPS} \Leftrightarrow Z_{CRYSTAL}}$$

2° Crystal Melting & Free fermions

$$Z^{\mathbb{C}^3} = M(q) = \prod \frac{1}{(1-q^i)^i} = Z^{\mathbb{C}^3}_{BPS}$$

Special cases

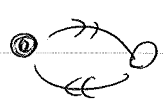
$1 \gg B > 0 \longrightarrow$ take into account all possible β's

$$Z_{BPS} = Z_{top}(Q)\, Z_{top}(Q^{-1}) = |Z_{top}(Q)|^2$$

non-commutative chamber

$$B \gg 1 \longrightarrow Z_{BPS} = M(1)^{\chi/2} \cdot Z_{top}(Q)$$

commutative chamber

Klebanov - Witten quiver 

$$Z_{crystal} = \sum N_{k\ell} (q_{red})^k (q_{yellow})^\ell = Z_{BPS} = Z_{BPS}(Q, q; B)$$

$n \leftarrow$ chamber

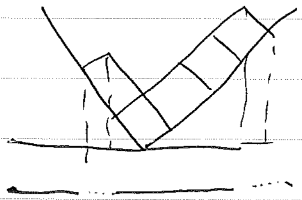
$n=1$ ncDT chamber

$n=\infty$ .. topological string

$q = q_{yellow} q_{red}$

$Q = -(q_{yellow} q_{red})^n q_{yellow}$

$$Z_{BPS} = Z_{crystal} = \langle \begin{matrix}\text{free fermion}\\ \text{vertex op}\end{matrix} \rangle = \left(\begin{matrix}\tau\text{-func of}\\ \text{integrable hierarchy}\end{matrix} \right)$$



$|0\rangle \to |R\rangle = \prod \psi\psi^* |0\rangle$

Interlacing condition $R_k \succ R_{k+1}$

$$\Gamma_\pm(x) = e^{\sum_{n=1}^{\infty} \frac{\alpha_{\pm m}}{m} x^m}$$

$$\Gamma_-^{(-1)}(1)|R\rangle = \sum_{P^t \succ R^t} |P\rangle$$

QUIVER

$$Z_{BPS}^{\mathbb{C}^3} = \langle 0| \Gamma_+(1) q^{L_0} \cdots \Gamma_-(1) q^{L_0} \Gamma_-(1)|0\rangle = \prod \frac{1}{1-q^{i+j-1}} = M(1)$$

$$\Gamma_+(y)\Gamma_-(x) = \Gamma_-(x)\Gamma_+(y)\frac{1}{1-xy}$$

conifold $A_\pm(1) = \Gamma_\pm(1)\, \hat{Q}_{yellow} \cdot \Gamma_\pm^{-1}(1)\, Q_{red}$

$$Z_{BPS}^{n=1} = \langle 0| \prod A_+(1) \,|\, \prod A_-(1) |0\rangle$$

$$Z_{BPS}^{n} = \langle 0| \prod A_+(1) \,|\, \underline{W^n} \,|\, \prod A_-(1) |0\rangle$$

Wall-crossing op.

3°
$$\mathbb{1} = \sum_R |R\rangle\langle R| = \sum_{P,R} |P\rangle\langle R|\, \delta_{P,R}$$

$$= \int dU \cdot \sum_{PR} s_P(z_1, z_2)\, s_R\left(\frac{1}{z_1}\ \frac{1}{z_2}\right) |P\rangle\langle R|$$

$$= \int dU \prod_k \Gamma_-(z_1, z_2 \ldots)|0\rangle\langle 0|\Gamma_+(z_1^{-1}\ z_2^{-1} \ldots)$$

$$Z_{BPS} = \langle \prod \Gamma^{(\pm)} | \mathbb{1} | \prod \Gamma^{(\pm)} | 0\rangle$$

$$= C_n \int dM \prod_i \left(e^{-\frac{1}{g_s} V(z_i)}\right) = C_n Z_{MM}$$

eg $\mathbb{C}^3$
$$e^{-\frac{1}{g} V(z)} = \theta(z, q) = \prod_{j=0}^{\infty} (1 + z q^j)\left(1 + \frac{q^{j+1}}{z}\right)$$

$$z = e^u$$

$$= e^{-\frac{1}{2}\frac{u^2}{g_s}} \left(1 + O(e^{1/g_s})\right)$$

4° conifold

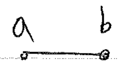
$$Z^{BPS}_{n=1} \Rightarrow e^{-\frac{V}{g_s}} = \frac{\theta(z, q)}{\theta(Qz, q)}$$

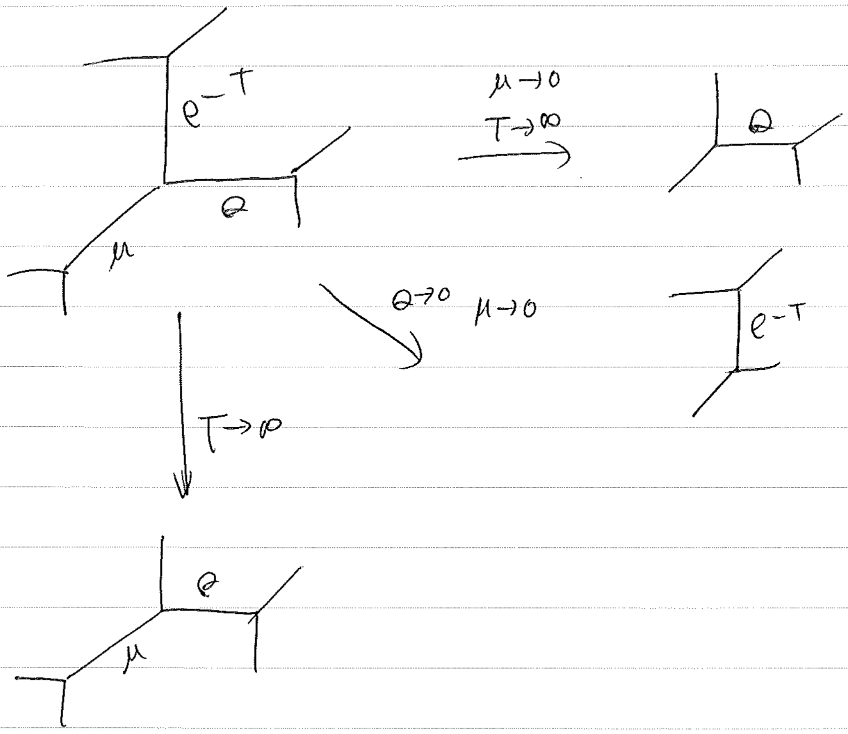
$$Z^{BPS}_n = C_n \int dM \prod_k \prod_{j=0}^{\infty} \frac{(1 + z_k q^{j+1})\left(1 + \frac{q^j}{z_k}\right)}{(1 + z_k q^{j+n+1})\left(1 + q\frac{j Q}{z}\right)}$$

$$V = Li_2(-z) + Li_2\left(-\frac{1}{z}\right) - Li_2\left(-\frac{z}{Qe^{\tau}}\right) - Li_2\left(\frac{Q}{z}\right)$$

$\tau = n g_s =$ fixed

$$\omega(p) = \left\langle Tr\left(\frac{1}{p - M}\right)\right\rangle = \frac{1}{2T} \oint \frac{dz}{2\pi i} \frac{\partial_z V(z)}{p - z} \sqrt{\frac{(p-a)(p-b)}{(z-a)(z-b)}}$$




e^{-T}
Q
μ
μ→0
T→∞
Q→0 μ→0
e^{-T}
Q
T→∞
Q
μ