# String Theory on CY manifolds and Mirror Symmetry



## 1 Introduction

type IIA / IIB string on $CY_3 \times \mathbb{R}^4$

low energy eff theory $\mathcal{N}=2$ SUGRA in 4 dim

IIA $h^{1,1}$ vectormultiplets
$1+h^{2,1}$ hypermultiplets

IIB $h^{2,1}$ vector
$h^{1,1}+1$ hyper

Topological A-model
$\alpha'$-correction

Topological B-model
"easy" no $\alpha'$-correction

Mirror symmetry

$(Y, \tilde{Y})$ certain pairs of CY

A-model on $Y$ = B-model on $\tilde{Y}$

In this talk, take $(Y, \tilde{Y})$ = (quintic, mirror quintic)

B-model on $\tilde{Y}$ $\Rightarrow$ A-model on $Y$

(2875 holomorphic lines in $Y$)

## 2. CY

~~2.1~~ Kähler, Ricci $= 0$

① Kähler form $k$ $dk = 0$

$\omega \in H^{1,1} \Longrightarrow k \Rightarrow CY$

unique if exists $\quad [k] = \omega$

② Holomorphic 3-form

$\Omega \in \Gamma(\Lambda^{3,0} T^* Y)$ unique up to constant multiplication

String theory on $CY_3 \times \mathbb{R}^4$

metric deformation $\delta g$ $\mathrm{Ricci}(g + \delta g) = 0$

$\longrightarrow$ massless scalar fields

| | | IIA | IIB |
|---|---|---|---|
| · Kähler deformation | $\delta k \in H^{1,1}$ | vector | hyper |
| · Complex str deformation | $\delta\Omega \in H^{2,1} \oplus H^{3,0}$ | hyper | vector |

Kähler $\ni$ volume deformation.

### 2.2 cpx str moduli

$\mathcal{M}_{cs} \sim$ deformation of $\Omega$

coordinates $z^i$, $\bar{z}^i$ $\quad i = 1, \cdots h^{2,1}$

we choose "normalization" of $\Omega$ st $\partial_{\bar{i}} \Omega = 0$

still redundant $\Omega \to e^{f(z)} \Omega$

"gauge transformation"

Derivatives of $\Omega$

$$\partial_i \Omega \in H^{2,1} \oplus H^{3,0}$$

$$\partial_i \partial_j \Omega \in H^{1,2} \oplus H^{2,1} \oplus H^{3,0}$$

$$\partial_i \partial_j \partial_k \Omega \in H^{30} \oplus H^{21} \oplus H^{12} \oplus H^{03}$$

$$\int \Omega \wedge \partial_i \Omega = 0 \qquad \int \Omega \wedge \partial_i \partial_j \Omega = 0 \qquad \int \Omega \wedge \partial_i \partial_j \partial_k \Omega = K_{ijk}.$$

yukawa coupling

Basis of $H_3$

$$(A^I, B_I) \qquad A^I \cap B_J = \delta^I{}_J. \qquad A \cap A = B \cap B = 0$$

$$I = 0, \cdots h^{2,1}$$

Basis of $H^3$

~~Basis~~ $(\alpha_I, \beta^I)$

$$\int_{A^I} \alpha_J = \delta^I{}_J. \qquad \int_{B_I} \beta^J = -\delta_I{}^J \qquad \int \alpha_I \wedge \beta^J = \delta_I{}^J$$

$$\Omega = X^I \alpha_I - F_J \beta^J$$

$$X^I = \int_{A^I} \Omega. \quad F_J = \int_{B_J} \Omega$$

$$0 = \int \Omega \wedge \frac{\partial}{\partial X^I} \Omega = \int (X^J \alpha_J - F_J \beta^J) \wedge \left(\alpha_I - \frac{\partial F_K}{\partial X^I} \beta^K\right)$$

$$= F_I - X^J \frac{\partial F_J}{\partial X^I}$$

$$2F_I = F_I + X^J \frac{\partial F_J}{\partial X^I} = \frac{\partial}{\partial X^I}(X^J F_J)$$

$$F_I = \frac{\partial}{\partial X^I} F \qquad F = \frac{1}{2} X^I F_I$$

$$\Pi = \begin{pmatrix} X^I \\ F_I \end{pmatrix} \qquad \Sigma = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$$

$$K_{ijk} = \int \Omega \wedge \partial_i \partial_j \partial_k \Omega = \Pi^T \Sigma \partial_i \partial_j \partial_k \Pi$$

## 3. An Example

### 3.1 Quintic vs Mirror Quintic

$P(x) = 0$ in $\mathbb{CP}^4$ $\quad (x_1 : x_2 : x_3 : x_4 : x_5)$

$h^{1,1} = 1$

Mirror $\{ P(x) = 0$ in $\mathbb{CP}^4 \} / \mathbb{Z}_5^3$

$P(x)$: $\mathbb{Z}_5^3$ deg-5 polynom

generated by $\quad \alpha_A : x_A \to e^{\frac{2\pi i}{5}} x_A$

$x_5 \to e^{-\frac{2\pi i}{5}} x_5$

$\alpha_1 \alpha_2 \alpha_3 \alpha_4 = id$

$$P(x) = a_1 x_1^5 + \cdots a_5 x_5^5 - b\, x_1 x_2 \cdots x_5$$

$a_1 \cdots a_5, b$ : parameters $\quad$ 1 cpx str modulus

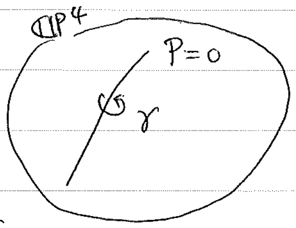
### 3.2 Picard - Fuchs equation

4-form in $\mathbb{C}^5$

$$\Xi = - x_1\, dx_2 \cdots dx_5 + dx_1 \cdot x_2\, dx_3 \cdots dx_5 + \cdots \quad \underline{\deg 5}$$

$\dfrac{\Xi}{P}$, deg zero. 4-form in $\mathbb{CP}^4$

$$\hat{\Omega} = \oint_\gamma \frac{\Xi}{P(x)}$$




3-form in mirror quintic

$$\hat{\Omega}_I = \int_I \hat{\Omega} = \int_I \oint_\gamma \frac{\Xi}{P(x)}$$

$$\hat{\Omega}_I (a_1 \cdots a_5, b) = \frac{1}{b} \omega_I (z) \qquad z = \frac{a_1 \cdots a_5}{b^5}$$

$$\left( \partial_{a_1} \partial_{a_2} \cdots \partial_{a_5} + \partial_b^5 \right) \hat{\Omega}_I = 0$$

$$\theta\left[\theta^4 - 5z(5\theta+4)(5\theta+3)(5\theta+2)(5\theta+1)\right]\omega_{\mathrm{I}}(z) = 0$$

Periods known to satisfy $\quad \theta^4 - 5z(5\theta+4)(5\theta+3)(5\theta+2)(5\theta+1) = 0$

$$\theta = z\frac{\partial}{\partial z}$$

3.4 Yukawa coupling

$$K_{zzz} = \int \Omega \wedge \partial_z^3 \Omega = \int \Omega \wedge \theta^3 \Omega \cdot \frac{1}{z^3} = \frac{1}{z^3} \Pi^T \Sigma \theta^3 \Pi$$

PF eq $\Rightarrow$ diff eq for $K_{zzz}$

$$W_k = \int \Omega \wedge \theta^k \Omega \qquad W_1 = W_2 = 0 \qquad K_{zzz} = \frac{1}{z^3} W_3$$

$$\Pi^T \Sigma \left[\theta^4 - 5z(5\theta+4)(5\theta+3)\cdots(5\theta+1)\right]\Pi = 0$$

$$(1-5^5 z)W_4 - 5^5 z \cdot 2W_3 = 0$$

$$\theta W_3 = \int \theta\Omega \wedge \theta^3 \Omega + W_4$$

$$0 = \theta^2 W_2 = \underbrace{\int \cancel{\theta^2\Omega \wedge \theta^2 \Omega}}_{0} + 2\int \theta\Omega \wedge \theta^3\Omega + W_4$$

$$\rightarrow W_4 = 2\theta W_3$$

$$\frac{\partial}{\partial z} W_3 = \frac{5^5}{1-5^5 z} W_3 \qquad W_3 = \frac{A}{1-5^5 z}$$

$$K_{zzz} = \frac{A}{z^3(1-5^5 z)}$$

3.4 Mirror Map

$$[\theta^4 - 5z(5\theta+4)\cdots(5\theta+1)]\,\omega = 0$$

$\omega_0$ : power series $\sum_{n=0}^{\infty} c_n z^n$ , $c(0)=1$ $\quad c(n) = \frac{(5n)!}{(n!)^5}$

$$\omega(z,\rho) := \sum_n c(n+\rho) z^{n+\rho}$$

Log soln $\quad \omega_1(z) = \frac{1}{2\pi i} \frac{\partial}{\partial\rho} \omega(z,\rho) \Big|_{\rho=0}$

$$= \frac{1}{2\pi i}\left[\omega_0 \log z + \sum_n c'(n) z^n\right]$$

$$c'(0)=0 \qquad c'(1)=770$$

A-model $\quad \Pi' = \begin{pmatrix} 1 \\ t \\ \vdots \end{pmatrix} \quad t = B + ik \quad$ $B \sim B+1$, $t \sim t+1$

monodromy

$K_{ttt} = 5 + $ (inst correction)

$$\Pi = \begin{pmatrix} \omega_0 \\ \omega_1 \\ \vdots \end{pmatrix} \longrightarrow \Pi' = \begin{pmatrix} 1 \\ \frac{\omega_1}{\omega_0} \\ \vdots \end{pmatrix} \qquad t = \frac{\omega_1}{\omega_0} = \frac{1}{2\pi i}(\log z + \cdots)$$

monodromy $t \to t+1$

$$K_{ttt} = K_{zzz}\, \omega_0^{-2} \left(\frac{dz}{dt}\right)^3$$

$$= \frac{1}{z^3(1-5^5 z)} (1+120z+\cdots)^{-2} (2\pi i z)^3 \times (1+770z+\cdots)^{-3}$$

$$= 5 + 2875 z + \cdots$$

$$q := e^{2\pi i t} = z(1+770z+\cdots)$$

$$K_{ttt} = 5 + 2875 q + \cdots$$

$$K_{ttt} = 5 + \sum_{k=1}^{\infty} \frac{n_k k^3 q^k}{1-q^k} \qquad n_k : \text{integer}$$

$n_1 = 2875$ Gopakumar-Vafa invariant

$n_2 = 609250$

$\vdots$