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Linear Sigma Models

2d (2,2) (← dim reduction of 4d $\mathcal{N}=1$)

1 Abelian LSM

2 O / USp exact results

3 Non-abelian LSM

1 Abelian LSM

Ref
Witten '91 "Phases …"

$G = U(1)$ $V \to \Sigma = \bar{D}_+ D_- V$

matter $P,\ X_1 \cdots X_5$ ← chiral

charge $-5\quad 1 \cdots 1$

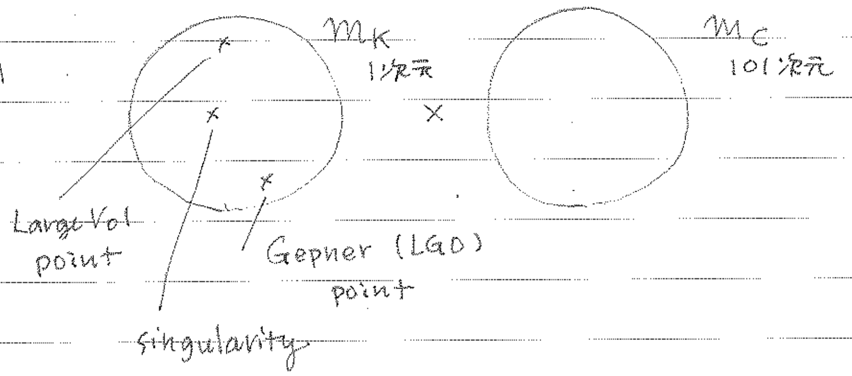
$W = P \cdot f(X_1 \cdots X_5)$ f: deg 5 (101 パラメーター)

FI parameter r, Theta θ ($t = r - i\theta$)

e : gauge coupling (mass-dim 1)

→ flows in the IR to $\hat{c} = \frac{c}{3} = 3$ (2,2) SCFT

moduli空間



4d $V \leftarrow (A_\mu, \lambda)$

2d $V \leftarrow (A_\mu\ (\mu=0,1),\ \sigma = A_2 + iA_3,\ \lambda)$

$$U(\sigma, p, x) = \frac{e^2}{2}\left(-5|p|^2 + |x|^2 - r\right)^2 + |f(x)|^2 + \sum_{i=1}^{5}\left|p\frac{\partial f}{\partial x_i}\right|^2$$

$$+ |-5\sigma p|^2 + |\sigma x|^2$$

ポテンシャルの零点

- $r \gg 0$

D term $\longrightarrow$ $X \neq 0$ $\quad U(1) \xrightarrow{\text{broken}} \{1\}$

$\sigma = 0,\ f(X) = 0,\ p\frac{\partial f}{\partial X_i} = 0 \quad (i = 1 \cdots 5) \quad (\Rightarrow p = 0)$

解 $\{\sigma = 0,\ X \neq 0,\ p = 0,\ f(X) = 0,\ |X|^2 = r\}/U(1)$

$$\boxed{X_f = \{f(X) = 0\} \subset \mathbb{CP}^4 \qquad \text{Quintic CY3} \begin{cases} h^{1,1} = 1 \\ h^{2,1} = 101 \end{cases}}$$

低エネルギー理論 = X_f を target とする NLSM

- $r \ll 0$

D-term $\longrightarrow$ $p \neq 0$ $\quad U(1) \xrightarrow{\text{broken}} \mathbb{Z}_5$

$\sigma = 0,\ f(X) = p\frac{\partial f}{\partial X_i} = 0 \longrightarrow X = 0$

bottom locus は 1点だが X は massless に留まる

低エネルギー理論 : X のみの理論

LG model $\quad W = \langle p \rangle f(X)$

modulo $\quad \mathbb{Z}_5 \ni \omega : (X_1 \cdots X_5) \longmapsto (\omega X_1, \cdots \omega X_5)$

- $r = 0$

D, F-term $\longrightarrow$ $X = p = 0 \quad \sigma$: free

$r \gg 0$, $r \ll 0$ の場合は X_f のコンパクト性 or LG potential のため discrete spectrum

$r = 0$ の場合は continuous spectrum "singular"

量子論 $\quad$ θ-parameter は b.g. electromagnetic flux (Coleman)

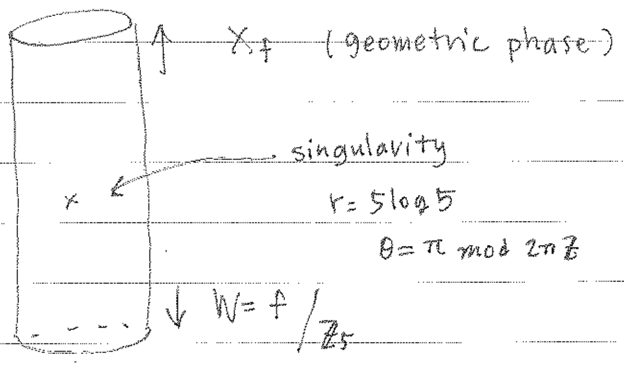
nonzero energy 生じる。

$$U_{eff}(\sigma) = \frac{e^2}{2} r^2 + \cdots = \frac{e_{eff}^2(\sigma)}{2}\left(r_{eff}(\sigma)^2 + \theta_{eff}^2\right)$$

$$r_{eff} = r - 5\log 5$$

$$\theta_{eff} = \min_{n \in \mathbb{Z}} \{(\theta - \pi + 2\pi n)^2\}$$

singularity : $r = 5\log 5$, $\theta = \pi \pmod{2\pi\mathbb{Z}}$



拡張 $U(1)^k$ ゲージ理論 ⟶ トーリック多様体中の complete intersection

e.g. $G=U(2)$, $\underbrace{P^1,\cdots,P^7}_{\det^{-1}}$, $\underbrace{X_1\cdots X_7}_{\mathbf{2}}$ $\quad X_i=(X_i^a)^{a=1,2}$

$$W=\sum_{i,j,k} A^{ij}_k P^k [X_iX_j] \qquad [X_iX_j]=X_i^1X_j^2-X_i^2X_j^1$$

$$A^{ij}_k=-A^{ji}_k$$

他の coupling FI, r, Theta θ $\quad e_{U(1)}, e_{SU(2)}$ ($\overset{\text{簡単のため}}{\longrightarrow}$ e)

KH - David Tong (2004-2006)

$$U(\sigma,p,x)=\frac{1}{2e^2}\mathrm{Tr}\left([\sigma,\sigma^\dagger]^2\right)+\frac{e^2}{2}\left(-|p|^2+|x|^2+r\right)^2+\frac{e^2}{2}\left(xx^\dagger-\frac{1}{2}\mathrm{tr}(xx^\dagger)\right)^2$$
$$+\sum_k\left|\sum_{i,j}A^{ij}_k[X_iX_j]\right|^2+\sum_{i,a}\left|A^{ij}_k p^k X_j^a\right|^2$$

$\underline{r\gg 0}$ $\quad X\neq 0$: $U(2)\longrightarrow$ ない
$P=0,\ \sigma=0$

$$\text{vac mfd}=\left\{\begin{matrix}p=\sigma=0\\ X\end{matrix}\ \middle|\ \begin{matrix}XX^\dagger=r\\ \sum_{i,j}A^{ij}_k[X_iX_j]=0\quad k=1\ldots 7\end{matrix}\right\}\Big/U(2)$$

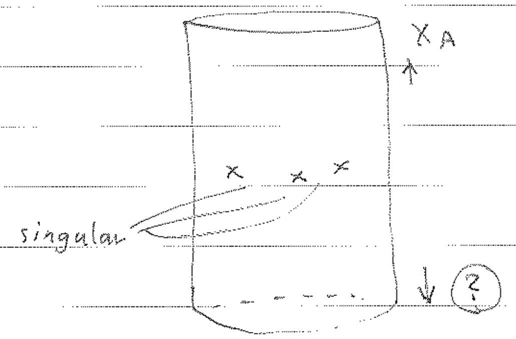
$$=\left\{\sum_{i,j}A^{ij}_k[X_iX_j]=0\quad k=1\ldots 7\right\}\subset \underbrace{G(2,7)}_{10次元}$$

3次元 CY $\quad h^{1,1}=1,\ h^{2,1}=50$

$r \ll 0$ のとき

$P \neq 0$: $U(2) \xrightarrow{\text{broken}} SU(2)$ （あとで説明）

$r \approx 0$. similar analysis $\to U_{eff}(\sigma)$



例 2 　 2011 Hosono - Takagi
2011 Hori

$$G = \frac{U(1) \times O(2)}{\{(\pm 1, \pm 1)\}} \longleftarrow \text{non-abelian} \qquad O(2) \quad g^T g = 1$$

$p^1 \ldots p^5 \quad (-2, \mathbb{1})$

$X^1 \ldots X^5 \quad (1, 2) \qquad W = \sum_{i,j,k} S^{ij}_k P^k (X_i X_j) \qquad X_i X_j = \sum_a X_i^a X_j^a$

$$S^{ij}_k = S^{ji}_k$$

FI r, Theta θ

$e_{U(1)}$, $e_{O(2)}$

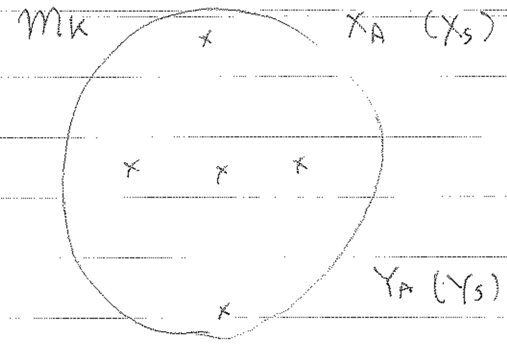
$r \gg 0 \quad G \to \{1\}$

vac mfd $\left\{ X \,\middle|\, \sum_{ij} S_k^{ij} (X_i X_j) = 0 \quad k = 1 .. 5 \right\} \subset \dfrac{\mathbb{CP}^4 \times \mathbb{CP}^4}{\mathbb{Z}_2}$

~~$h^{1,1} = 1 \quad h^{2,1} = 26$~~

note $G = U(1) \times U(2) \rtimes \mathbb{Z}_2$

$r \ll 0 \quad P \neq 0 . \quad G \to O(2)$ unbroken



B-brane全体のなす構造

"D-brane Category"

"Derived Category of coherent sheaves"

$$D^b(\mathrm{Coh}\,X_A) \cong D^b(\mathrm{Coh}\,Y_A)$$

→ Borisov–Căldăraru 2006

Models

O/SO $k=1,2,3,\cdots$ $N=0,1,2,\cdots$

$G = O(k)$, $SO(k)$

Matters N コの $\mathbb{C}^k$ $X_1 \cdots X_N$, $W=0$

$O(k) = SO(k) \rtimes \mathbb{Z}_2$

$\therefore$ $\underline{\mathbb{Z}_2 \text{ orbifold of } SO(k) \text{ theory}}$

2 possibilities $<$ $O_+(k)$, $O_-(k)$

$$(-1)^{F_S} = \begin{cases} +1 & \text{untwisted NSNS, twisted RR} \\ -1 & \text{twisted NSNS, untwisted RR} \end{cases}$$

. $k \geq 3$ $\pi_1(G) = \mathbb{Z}_2$ ... mod 2 theta angle, NO FI.

$k=2$ $SO(2)$: $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, FI $\in \mathbb{R}$ $\longrightarrow$ $(r=0,\ \theta=0,\pi)$

$O_\pm(2)$: $\theta = 0, \pi$, no FI.

—— . ——

$k = 2,4,6,\cdots$ $N=0,1,2,\cdots$

$G = USp(k)$ $\left(\begin{array}{l} SU(2) = USp(2) = Sp(1) \\ SU(2)_{\mathbb{C}} = SL(2\mathbb{C}) = Sp(2\mathbb{C}) \end{array} \right.$

Matter N コ $\mathbb{C}^k$, $X_1 \cdots X_N$, $W=0$

$\pi_0(G) = \{1\}$, $\pi_1(G) = \{1\}$ No θ. No FI.

Classical potential σ: vector, X: chiral

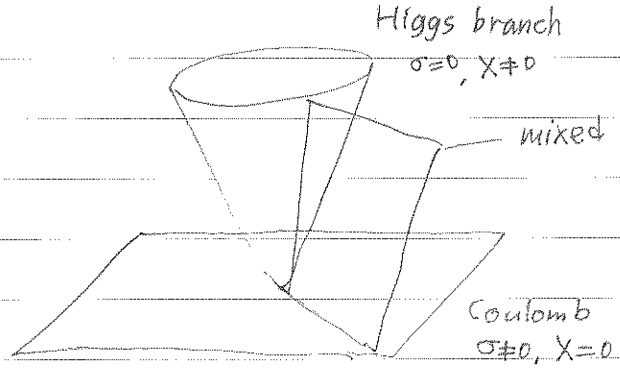
$$U(\sigma, X) = \frac{1}{2e^2}\left|[\sigma,\sigma^\dagger]\right|^2 + \frac{1}{2}|\sigma X|^2 + \frac{1}{2}|\sigma^\dagger X|^2 + \frac{e^2}{2}|\mu(X)|^2$$

$\mu(X) = $ "$XX^\dagger$"

$U=0 \rightarrow [\sigma,\sigma^\dagger]=0$, $\sigma X = 0$, $\sigma^\dagger X = 0$, $\mu(X) = 0$

σ: "diagonalizable"

eg. $O(k)$ $k=2l \longrightarrow \sigma = \begin{pmatrix} 0 & -\sigma_1 & & & \\ \sigma_1 & 0 & & & \\ & & 0 & \sigma_2 & \\ & & -\sigma_2 & 0 & \\ & & & & \ddots \end{pmatrix}$



Problem:

non-compactness of Coulomb branch

→ quantum correction in $U_{eff}(\sigma)$

$\sigma \neq 0$ のとき $G \longrightarrow U(1)^{\mathrm{rank}\, G}$

$$\Delta U = \frac{e_{eff}^2}{2}\theta_{eff}^2$$

$G = O(k),\ SO(k) \to \theta_{eff} \equiv \pi(N-k) + \underline{\theta_0}$ mod 2 theta angle (0 or π)

$G = USp(k)$ $\quad \theta_{eff} = \pi(N-k)$ mod 2π

~~We call the theory regular if $\theta_{eff} \neq 0$~~

Results

$O_\pm(k),\ SO(k)$ $\quad X_1 \cdots X_N,\ W=0$

$N \leq k-2$: ~~SUSY~~ $\left\{ \begin{array}{l} k=3 \text{ regular or not} \\ k=2 \text{ regular} \end{array} \right.$

$N = k-1$: Free theory of mesons. $(x_i x_j) = \sum_{a=1}^{k} x_i^a x_j^a$

$O_-(k),\ SO(k)$ 1 copy $\mathbb{C}^{\frac{N(N+1)}{2}}$

$O_+(k)$ 2 copies $\mathbb{C}^{\frac{N(N+1)}{2}} \cup \mathbb{C}^{\frac{N(N+1)}{2}}$

· $N \geq k$ $\exists$ duality $G = O_+(k) \longleftrightarrow \tilde{G} = SO(N-k+1)$

$SO(k) \longleftrightarrow O_+(N-k+1)$

$O_-(k) \longleftrightarrow O_-(N-k+1)$

with N コの $\mathbb{C}^{N-k+1}$ $\tilde{X}^1, \dots \tilde{X}^N$

$\frac{N(N+1)}{2}$ コの singlets S_{ij}

with

$$W = \sum_{i,j} S_{ij}(\tilde{X}^i \tilde{X}^j)$$

対応

· $(X_i X_j) \longleftrightarrow S_{ij}$

$G = SO(k) \leftrightarrow \tilde{G} = O_+(N-k+1)$

$\cancel{\mathbb{Z}_2 \text{ global sym}} \leftrightarrow \cancel{\text{Quantum } \mathbb{Z}_2 \text{ sym}}$

$\cancel{O(k)}/SO(k)$ ~~of the~~ $\cancel{\mathbb{Z}_2}$ ~~orbifold~~

$[X_{i_1} \cdots X_{i_k}]$ (baryon $\leftrightarrow$ twist operators

$G = O_-(k) \leftrightarrow \tilde{G} = O_-(N-k+1)$

quantum $\mathbb{Z}_2$ sym $\leftrightarrow$ quantum $\mathbb{Z}_2 \cdot (-1)^F$ sym

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$USp(k)$, $X_1 \cdots X_N$ in $\mathbb{C}^k$, $W=0$ $\qquad J_{ab} = \begin{pmatrix} 0 & -1_{k/2} \\ 1_{k/2} & 0 \end{pmatrix}$

- $N \le k$ ~~SUSY~~ (regular or not)
- $N = k+1$ Free theory of mesons $\qquad [X_i X_j] = \sum_{a,b} X_i^a J_{ab} X^b_j$

  CFT on $\mathbb{C}^{\frac{N(N-1)}{2}}$

- $N \geq k+3$ $\quad \exists$ duality

$G = USp(k) \leftrightarrow \tilde{G} = USp(N-k-1)$.

$$\left.\begin{array}{l} N \ni \mathbb{C}^{N-k-1} : \tilde{X}^1 \ldots \tilde{X}^N \\ \frac{N(N-1)}{2} \ni \text{ singlets}: a_{ij} = -a_{ji} \end{array}\right] \quad W = \sum a_{ij} [\tilde{X}^i \tilde{X}^j]$$

$[X_i X_j] \longleftrightarrow a_{ij}$

Evidence of the results

~~$N = k-1$~~ (O, SO)

gauge-inv polynomial $\quad \mathbb{C}[X_i^a]^G \cong \mathbb{C}[(X_i X_j)]$ $\quad$ (no relation)

$N = k+1$ $\quad (G = USp(k))$

$\mathbb{C}[X_i^a]^G \simeq \mathbb{C}[[X_i X_j]]$ $\quad$ (no relation)

Test of duality

Symmetry

	$SU(N)$	$U(1)_B$	$U(1)_V$	$U(1)_A$
			R-sym	
X	$\square$	1	0	0
$\tilde{X}$	$\bar{\square}$	-1	1	0
$\tilde{S}$	$\square\!\square$	2	0	0

$\psi_\pm^X \;,\; \lambda_\pm$

't Hooft anomaly matching

$$\hat{c} = kN - \frac{k(k-1)}{2}$$

$$\hat{c} = \frac{N(N+1)}{2} - \frac{(N-k)(N-k+1)}{2}$$

Original theory

Higgs branch theory

$$H_{N,k} = \{(x_i{}^a)\} / G_{\mathbb{C}}$$

dual theory

$$F : \begin{cases} (\tilde{X}^i \tilde{X}^j) = 0 \\ S_{ij} \tilde{X}^j_{\tilde a} = 0 \end{cases}$$

D-term

$\tilde{X}^j_{\tilde a} = 0$

$\boxed{\{S_{ij}\}}$

Born - Oppenheimer 近似

S_{ij} : slow として. $\leftarrow$ mass matrix for $\tilde{X}^i_b$

$\tilde{V}$, $\tilde{X}^i_{\tilde a}$: fast

$$\tilde{N}_{eff} = N - rk(S) = \text{corank}(S)$$

$$\tilde{N}_{eff} \leq (N-k+1) - 2 \quad \therefore \cancel{\text{SUSY}}$$

$$= (N-k+1) - 1 \quad : \text{Free th of mesons} \quad rk(S) \leq k$$

$$\Rightarrow \tilde{N}_{eff} \geq N-k$$

$$\parallel$$

$$N - rk(S)$$

2d $(2,2)$ $\longrightarrow$ twisted mass $\longrightarrow$ vacuum counting

$$G \longrightarrow U(1)^{rk}$$

$$\tilde{W}_{eff}(\sigma) = -\sum_{X} X(\sigma) \left(\log X(\sigma) - 1\right)$$

$N \leq k-2$ $\quad O;SO$

$N \leq k$ $\quad USp$

$\longrightarrow$ No solution $\Rightarrow$ $\cancel{\text{SUSY}}$

Chiral Ring の比較

(c.c) ⟶ duality を使えば ~~比較~~ 決定できる

(a.c) ⟶ check できる

$O(1) = \mathbb{Z}_2$, $SO(1) = \{1\}$ $\Big\}$ これ自体の (dualityの) check できる

4d $SO(k), N \longleftrightarrow SO(N-k+4), N$ — Intriligator Seiberg

$USp(k), N \longleftrightarrow USp(N-k-4), N$ — Intriligator Pouliot

3d $O(k), N \longleftrightarrow O(N-k+2), N$ — Kapustin

$USp(k), N \longleftrightarrow USp(N-k-2), N$ — Aharony

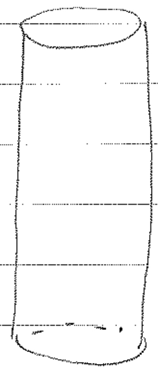
Application to LSM

$$G = U(2) = \frac{U(1)\times USp(2)}{\{(\pm 1, \pm \mathbb{1}_2)\}} \qquad \underbrace{p^1 \ldots p^7}_{\det^{-1}} ; \underbrace{X_1 \ldots X_7}_{\mathbf{2}}$$

$(-2, \mathbb{1})$ $(1, \mathbf{2})$

$$W = \sum A^{ij}_k p^k [X_i X_j]$$

r, θ, e



$r \gg 0$

X_A: σ-model

$r \ll 0$

$p \neq 0$, $G \xrightarrow{\text{broken}} USp(2)$

$\{p\} \cong \mathbb{CP}^6$

$W = \sum_{ij} A(p)^{ij} [X_i X_j]$ とみると

$\mathrm{rk}\, A(p) = 6 \;\ldots\; N_{eff} = 1$ ~~SUSY~~

$4 \;\ldots\; 3$ free theory of mesons

$$Y_A = \{ P \in \mathbb{CP}^6 \mid \mathrm{rk}\, A = 4 \} = CY3$$

dual LSM $\quad \tilde{G} = \dfrac{U(1)\times USp(4)}{\{(\pm 1, \pm \mathbb{1}_4)\}}$ $\quad p^1 \ldots p^7$, $\tilde{X}^1 \ldots \tilde{X}^7$, $a_{ij} = -a_{ji}$

$(-2, \mathbb{1})$ $(-1, \mathbf{4})$ $(2, \mathbb{1})$

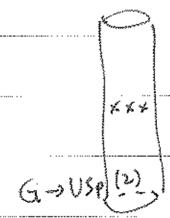
$$W = \sum_{ij} a_{ij} [\tilde{X}^i \tilde{X}^j] + \sum_{ijk} A^{ij}_k p^k a_{ij}$$

$$[\tilde{X}^i \tilde{X}^j] + A(p)^{ij} = 0$$

$$\text{vac mfd} = \Big\{ (p, \tilde{X}) \;\Big|\; \underset{p \neq 0}{[\tilde{X}^i \tilde{X}^j] + A(p)^{ij} = 0} \Big\} \Big/ \tilde{G}_{\mathbb{C}} = Y_A$$

$\mathrm{rk}\, A(p) \leq 4$ (unique soln for $\tilde{X}^i$ for given p)

Original LSM $\longrightarrow$ dual LSM



X_A

$G \to \{1\}$

quantum analysis より Y_A

$G \to USp(2)$

$\hat{G} \to USp(4)$

$\tilde{G} \to \{1\}$ classical に Y_A

$$G = U(2) = \frac{U(1)\times O_+(2)}{\{(\pm 1, \pm 1_2)\}} \text{ の方}$$

$p^1 .. p^5$, $X^1 .. X^5$ $\quad (-2, 1) \quad (1, 2)$

$$W = \sum S^{ij}_k p^k (X_i X_j) = \sum S^{ij}(p)(X_i X_j)$$

$\cancel{\mathrm{rk}\, S^{ij}(p) = 5}$	$N_{eff} = 0$	~~SUSY~~
4	1	free $X_i X_j$
3	2	
× 2		
× 1		
× 0		

$$Y_S = \{ p \in \mathbb{CP}^4 \mid \mathrm{rk}\, S(p) = 4 \}$$
$$\cup$$
$$C_S = \cancel{\{ p \mid \mathrm{rk}\, S(p) = 3\ 4}$$

~~dual LSM~~ $\tilde{G} = \cancel{\frac{U(1)\times SO(4)}{\{(\pm 1, \pm 1_4)\}}}$ $\quad$ $p^1 .. p^5$; $\tilde{X}^1 .. \tilde{X}^5$, $S_{ij} = S_{ji}$

$\quad (-2, 1) \quad (-1, 4) \quad (2, 1)$

$$W = \sum_{ij} s_{ij}(\tilde{X}^i \tilde{X}^j) + \sum_{ijk} \underbrace{S^{ij}_k p^k}_{S^{ij}(p)} s_{ij}$$

vac mfd $\quad \tilde{Y}_S = \{ (p, \tilde{X}) \mid (\tilde{X}^i \tilde{X}^j) + S^{ij}(p) = 0 \} / \tilde{G}_{\mathbb{C}}$

$p \neq 0$

$\uparrow$

$\mathrm{rk}\, S(p) \leq 4$