Hidden symmetry of supergravity black holes

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We want to solve the higher-dimensional gravitational theories and explicitly construct the solutions that have physically and mathematically interesting properties.

But, it is difficult to solve them in general.

Investigating known solutions, we consider a generalization of them.

We focus on hidden symmetries of black holes.
**Exact solutions – vacuum black holes**

with $S^n$ horizon topology

\[ \text{vacuum Einstein’s Eq.} \]

\[ Ric(g) = \lambda g \]

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<thead>
<tr>
<th><strong>Four dimensions</strong></th>
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</tr>
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In 1963, Kerr discovered a solution describing rotating black holes in a vacuum.

$$ds^2 = -\frac{\Delta}{\Sigma} \left( dt - a \sin^2 \theta d\phi \right)^2 + \frac{\sin^2 \theta}{\Sigma} \left( a dt - (r^2 + a^2) d\phi \right)^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2$$

where

$$\Delta = r^2 - 2Mr + a^2 , \quad \Sigma = r^2 + a^2 \cos^2 \theta$$
Geometry of Kerr spacetime

Kerr’s metric

\[ ds^2 = -\frac{\Delta}{\Sigma} \left( dt - a \sin^2 \theta d\phi \right)^2 + \frac{\sin^2 \theta}{\Sigma} \left( a \, dt - (r^2 + a^2) d\phi \right)^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 \]

where \( \Delta = r^2 - 2Mr + a^2 \), \( \Sigma = r^2 + a^2 \cos^2 \theta \)

- Two parameters
  - mass \( M \)
  - angular momentum \( J = Ma \)
- Two isometries
  - time translation \( \partial/\partial t \)
  - axial symmetry \( \partial/\partial \Phi \)
- Ring singularity at \( \Sigma = 0 \), i.e., \( r = 0, \theta = \pi/2 \)
- Two horizons at \( r = r_\pm \) s.t. \( \Delta(r_\pm) = 0 \)
In 1968, Cater demonstrated that the Hamilton-Jacobi equation for geodesics

$$\partial_\lambda S + g^{ab} \partial_a S \partial_b S = 0$$

for the Kerr’s metric can be separated for a solution

$$S = -\kappa_0 \lambda - Et + L\phi + R(r) + \Theta(\theta)$$

and then the functions $R(r)$ and $\Theta(\theta)$ follow

$$(R')^2 - \frac{W_r^2}{\Delta^2} - \frac{V_r}{\Delta} = 0 \ , \ (\Theta')^2 + \frac{W_\theta^2}{\sin^2 \theta} - V_\theta = 0$$

where

$$W_r = -E(r^2 + a^2) + aL \ , \ W_\theta = -aE \sin^2 \theta + L$$

$$V_r = \kappa + \kappa_0 r^2 \ , \quad V_\theta = -\kappa + \kappa_0 a^2 \cos^2 \theta$$
Scalar fields in the Kerr spacetime

He also demonstrated that the massive Klein-Gordon equation

$$(\nabla^2 - m^2)\Phi = 0$$

for the Kerr’s metric can be separated for a solution

$$\Phi = e^{-i\omega t + in\phi} R(r) \Theta(\theta)$$

and then the functions $R(r)$ and $\Theta(\theta)$ follow

$$\frac{1}{R \, dr} \left( \frac{\Delta}{\Delta} \frac{dR}{dr} \right) + \frac{U_r^2}{\Delta} - m^2 r^2 - \kappa = 0,$$

$$\frac{1}{\Theta \, \sin \theta \, d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) - \frac{U_\theta^2}{\sin^2 \theta} - m^2 a^2 \cos^2 \theta - \kappa = 0$$

where

$$U_r = an - \omega (r^2 + a^2), \quad U_\theta = n - a\omega \sin^2 \theta$$
Separation of variables in various equations for the Kerr’s metric

- Hamilton-Jacobi equation for geodesics
  \[ \partial_{\lambda} S + g^{ab} \partial_a S \partial_b S = 0 \]

- Klein-Gordon equation
  \[ (\nabla^2 - m^2) \Phi = 0 \]  
  \[ \text{Carter (1968)} \]

- Maxwell equation
  \[ \nabla_\mu F^{\mu\nu} = 0 \]

- Linearized Einstein’s equation
  \[ \delta G_{\mu\nu} = 0 \]  
  \[ \text{Teukolsky (1972)} \]

- Neutrino equation
  \[ \gamma^\mu (\partial_\mu + \Gamma_\mu) \psi = 0 \]  
  \[ \text{Teukolsky (1973), Unruh (1973)} \]

- Dirac equation
  \[ (\gamma^\mu \nabla_\mu + m) \Psi = 0 \]  
  \[ \text{Chandrasekhar (1976), Page (1976)} \]
In order to give an account of such integrabilities and separabilities, a generalization of Killing symmetry has been studied since 1970s.

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<tr>
<th>vector</th>
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<th>conformal Killing vector</th>
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<tr>
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<td>anti-symmetric</td>
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<td>Yano (1952)</td>
<td>Tachibana (1969), Kashiwada (1968)</td>
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</table>
Plan of this talk

0. Introduction

1. Review
   – Hidden symmetry of Kerr black holes –

2. On spacetimes admitting CKY symmetry

3. A generalization of CKY symmetry

4. Summary & Outlook
1. Introduction
– Hidden symmetry of Kerr black holes –
Complete integrable system – Liouville integrability –

Geodesic equation

\[
\frac{dx^\mu}{d\tau} = \frac{\partial H}{\partial p_\mu}, \quad \frac{dp_\mu}{d\tau} = -\frac{\partial H}{\partial x^\mu}, \quad \text{for} \quad H = \frac{1}{2} g^{ab} p_a p_b.
\]

\[F(x, p) : \text{a constant of motion} \quad \Leftrightarrow \quad 0 = \frac{dF}{d\tau} = \frac{\partial F}{\partial x^\mu} \frac{\partial H}{\partial p_\mu} - \frac{\partial F}{\partial p_\mu} \frac{\partial H}{\partial x^\mu} =: \{F, H\}_P\]

Poisson’s bracket

Liouville integrability means that there exists a maximal set of Poisson commuting invariants.

\[\{\alpha_i, \alpha_j\}_P = 0, \quad i, j = 1, \ldots, D\]
Constants of motion and Killing tensors

\[ H = \frac{1}{2} g^{ab} p_a p_b \]

Assume \( C_K = K^{a_1 \cdots a_n} p_{a_1} \cdots p_{a_n} \)

\[ \{H, C_K\}_P = 0 \]

\[ \Leftrightarrow \nabla^{(a_1 K^{a_2 \cdots a_{n+1}})} p_{a_1} p_{a_2} \cdots p_{a_{n+1}} = 0 \]

= 0 ; Killing equation

**Def.** Killing-Stackel tensor (KS) is a rank-\( n \) symmetric tensor \( K \) obeying the Killing equation

Stackel (1895)
Hamilton-Jacobi approach

For a $D$-dimensional manifold $(M^D, g)$, a local coordinate system $x^a$ is called a separable coordinate system if a Hamilton-Jacobi equation in these coordinates

$$H(x^a, p_a) = \kappa_0, \quad p_a = \frac{\partial S}{\partial x^a}$$

where $\kappa_0$ is a constant, is completely integrable by (additive) separation of variables, i.e.,

$$S = S_1(x^1, c) + S_2(x^2, c) + \cdots + S_D(x^D, c)$$

where $S_a(x^a, c)$ depends only on the corresponding coordinate $x^a$ and includes $D$ constants $c=(c_1, \ldots, c_D)$. 
δr-Separability structure

**Theor.** A D-dimensional manifold \((M^D, g)\) admits separability of H-J equation for geodesics if and only if

1. There exist \(r\) indep. commuting Killing vectors \(X_{(i)}\):
   \[
   [X_{(i)}, X_{(j)}] = 0,
   \]

2. There exist \(D-r\) indep. rank-2 Killing tensors \(K_{(\mu)}\), which satisfy
   \[
   [K_{(\mu)}, K_{(\nu)}] = 0, \quad [X_{(i)}, K_{(\mu)}] = 0,
   \]

3. The Killing tensors \(K_{(\mu)}\) have in common \(D-r\) eigenvectors \(X_{(\mu)}\) s.t.
   \[
   [X_{(\mu)}, X_{(\nu)}] = 0, \quad [X_{(i)}, X_{(\mu)}], \quad g(X_{(i)}, X_{(\mu)}) = 0.
   \]

**Comments:**
- Some examples which are not separable but integrable are known.
  cf.) Gibbons-TH-Kubiznak-Warnick (2011)
Kerr spacetime admits a rank-2 irreducible Killing tensor. \( K_{ab} = K_{(ab)} \), \( \nabla_{(c} K_{ab)} = 0 \)

Comments:
- Kerr spacetime has 4 independent and mutually commuting constants of geodesic motion, which are corresponding to 2 Killing vectors and 2 rank-2 Killing tensors.

\[
\begin{align*}
(\partial_t)^a : & \quad E = (\partial_t)^a p_a \\
(\partial_\phi)^a : & \quad L = (\partial_\phi)^a p_a \\
 g_{ab} : & \quad \kappa_0 = g^{ab} p_a p_b \\
 K_{ab} : & \quad \kappa = K^{ab} p_a p_b
\end{align*}
\]

- One also finds that this Killing tensor admits the \( \delta_2 \)-separability structure of the H-J equation for geodesics.
The Killing tensor $K$ can be written as the square of a rank-2 Killing-Yano tensor $f$. 

\[ \exists f \text{ s.t. } K_{ab} = f_a^c f_{bc}, \quad f_{ba} = -f_{ab}, \quad \nabla_{(a}f_{b)c} = 0 \]

\[ \uparrow \text{ rank-2 KY equation} \]

Comments:

- Killing-Yano tensor (KY) is a rank-$p$ anti-symmetric tensor $f$ obeying $\nabla_{(a}f_{b_1) b_2 \ldots b_p} = 0$. 

- Having a Killing-Yano tensor, one can always construct the corresponding Killing tensor. On the other hand, not every Killing tensor can be decomposed in terms of a Killing-Yano tensor.
Moreover, the Killing-Yano tensor $f$ generates two Killing vectors.

$$\xi^a \equiv (\partial_t)^a = \frac{1}{3} \nabla_b (*f)^{ba}$$

$$\eta^a \equiv -a^2 (\partial_t)^a - a (\partial_\phi)^a = K^a{}_b \xi^b$$

In the end, all the symmetries necessary for complete integrability and separability of the H-J equation for geodesics can be generated by a single rank-2 Killing-Yano tensor.

$$g_{ab}, \quad f_{ab} \quad \rightarrow \quad K_{ab} = f_a{}^c f_{bc}$$

$$\xi^a \quad \rightarrow \quad \eta^a$$
The Killing-Yano tensor is derived from a 1-form potential $b$,

$$ f = *db $$

Carter (1987)

Comments:

- Obviously, $h = *f$ is closed 2-form.
- One finds that $h$ is a conformal Killing-Yano tensor (CKY) of rank-2, i.e., it follows

$$ \nabla_a h_{bc} + \nabla_b h_{ac} = 2g_{ab}\xi_c - g_{ac}\xi_b - g_{bc}\xi_a $$

where

$$ \xi^a = \frac{1}{3}\nabla_b h^{ba} $$

Tachibana (1969)
Hidden symmetry of Kerr spacetime V

Kerr’s metric

\[ ds^2 = -e^0 e^0 + e^1 e^1 + e^2 e^2 + e^3 e^3 \]

where

\[ e^0 = \sqrt{\frac{\Delta}{\Sigma}} \left( dt - a \sin^2 \theta d\phi \right), \quad e^2 = \sqrt{\frac{\sin^2 \theta}{\Sigma}} \left( a \, dt - (r^2 + a^2) d\phi \right) \]

\[ e^1 = \sqrt{\frac{\Sigma}{\Delta}} dr, \quad e^3 = \sqrt{\Sigma} d\theta \]

- KY 2-form

\[ f = a \cos \theta \, e^0 \wedge e^1 + re^2 \wedge e^3 \]

- CCKY 2-form

\[ h = re^0 \wedge e^1 + a \cos \theta e^2 \wedge e^3 \]

- rank-2 Killing tensor

\[ K = a^2 \cos^2 \theta (e^0 e^0 - e^1 e^1) + r^2 (e^2 e^2 + e^3 e^3) \]
**Symmetry operators**

\( \mathcal{O} : \) a sym. op. for \( \mathbf{D} \)

\( \iff [\mathcal{O}, \mathbf{D}] = 0 \) for a diff. op. \( \mathbf{D} \)

**Klein-Gordon equation**

For the scalar Laplacian \( \Box \),

\[
\hat{\eta}(j) = \eta(j)^a \nabla_a, \quad \hat{K}(j) = \nabla_a K(j)^{ab} \nabla_b,
\]

are symmetry operators, i.e.,

\[
[\hat{\eta}(j), \Box] = [\hat{K}(j), \Box] = 0 \quad \text{Carter (1977)}
\]

**Dirac equation**

For the Dirac operator \( \mathbf{D} \), the operator

\[
\hat{f} \equiv i \gamma_5 \gamma^a \left( f_a^b \nabla_b - \frac{1}{6} \gamma^b \gamma^c \nabla_c f_{ab} \right)
\]

is symmetry operator whenever \( f \) is a Killing-Yano tensor.

**Carter-McLenagahan (1979)**
Separability structures for Kerr black hole

Algebraic type of curvature is type-D.

Geodesic motion is completely integrable. Carter (1968)

Hamilton-Jacobi equation is separable. Carter (1968)

Klein-Gordon equation is separable. Carter (1968)

K-G symmetry operators exist. Carter (1977)

Dirac equation is separable. Chandrasekhar (1976)

Dirac symmetry operators exist. Carter-McLenaghan (1979)

Carter’s metric

Kerr’s metric

\[
\begin{align*}
    ds^2 &= -\frac{\Delta}{\Sigma} (dt - a \sin^2 \theta \, d\phi)^2 \\
    &\quad + \frac{\sin^2 \theta}{\Sigma} (a \, dt - (r^2 + a^2) \, d\phi)^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2
\end{align*}
\]

where \( \Delta = r^2 - 2M r + a^2 \), \( \Sigma = r^2 + a^2 \cos^2 \theta \)

coord. trasf. \( p = a \cos \theta \), \( \tau = t - a \phi \), \( \sigma = -\frac{\phi}{a} \)

\[
\begin{align*}
    ds^2 &= -\frac{Q}{r^2 + p^2} (d\tau - p^2 d\sigma)^2 \\
    &\quad + \frac{P}{r^2 + p^2} (d\tau + r^2 d\sigma)^2 + \frac{r^2 + p^2}{Q} dr^2 + \frac{r^2 + p^2}{P} dp^2
\end{align*}
\]

where \( Q = r^2 - 2M r + a^2 \), \( P = -p^2 + a^2 \)

Carter (1968)

The “off-shell” metric with \( Q \) and \( P \) replaced by arbitrary functions \( Q(r) \) and \( P(p) \) is said to be of Carter’s class.
Spacetimes admitting a Killing-Yano tensor

**Theor.** Let \((M^4, g)\) be a vacuum type-D space-time. The following conditions are equivalent:

1. \((M^4, g)\) is without acceleration.
2. \((M^4, g)\) is one of Carter’s class.
3. \((M^4, g)\) admits a \(\delta_2\)-separability structure.
4. \((M^4, g)\) admits a Killing-Yano tensor.

Demianski-Francaviglia (1980)

**Theor.** A spacetime \((M^4, g)\) admits a rank-2 Killing-Yano tensor if and only if the metric is of Carter’s class, i.e.,

\[
ds^2 = - \frac{Q(r)}{r^2 + p^2} (d\tau - p^2 d\sigma)^2 \\
+ \frac{P(p)}{r^2 + p^2} (d\tau + r^2 d\sigma)^2 + \frac{r^2 + p^2}{Q(r)} dr^2 + \frac{r^2 + p^2}{P(p)} dp^2
\]

Dietz-Rudiger (1982), Taxiarchis (1985)
Carter’s metric in Einstein-Maxwell theory

The Carter’s metric

\[ ds^2 = -\frac{Q(r)}{r^2 + p^2}(d\tau - p^2 d\sigma)^2 \]
\[ + \frac{P(p)}{r^2 + p^2}(d\tau + r^2 d\sigma)^2 + \frac{r^2 + p^2}{Q(r)} dr^2 + \frac{r^2 + p^2}{P(p)} dp^2 \]

obeys the Einstein-Maxwell equations when provided that the functions take the form

\[ Q = -\frac{\lambda}{3} r^4 + \epsilon r^2 - 2mr + k + \epsilon^2 + g^2 \]
\[ P = -\frac{\lambda}{3} p^4 - \epsilon p^2 + 2np + k \]

and the vector potential reads

\[ A = -\frac{1}{r^2 + p^2}[er(d\tau - p^2 d\sigma) + gp(d\tau + r^2 d\sigma)] \]

This metric has six independent parameters.
Plebanski-Demianski metric

The important family of type D in four dimensions can be represented by the seven-parameter metric. 

\[ ds^2 = \frac{1}{(1 - pr)^2} \left\{ -\frac{Q}{r^2 + p^2} (d\tau - p^2 d\sigma)^2 + \frac{P}{r^2 + p^2} (d\tau + r^2 d\sigma)^2 + \frac{r^2 + p^2}{Q} dr^2 + \frac{r^2 + p^2}{P} dp^2 \right\} \]

Plebanski-Demianski (1976)

This metric obeys the Einstein-Maxwell equations provided that the functions take the form

\[ Q = -(k + \lambda/3)r^4 - 2nr^3 + \epsilon r^2 - 2mr + k + e^2 + g^2 \]

\[ P = -(k + e^2 + g^2 + \lambda/3)p^4 + 2mp^3 - \epsilon p^2 + 2np + k \]

and the vector potential reads

\[ A = -\frac{1}{r^2 + p^2} [er(d\tau - p^2 d\sigma) + gp(d\tau + r^2 d\sigma)] \]
Relationship b/w Carter’s metric and P-D metric

Plebanski-Demianski metric

\[ ds^2 = \frac{1}{(1 - pr)^2} \left\{ - \frac{Q}{r^2 + p^2} (d\tau - p^2 d\sigma)^2 
+ \frac{P}{r^2 + p^2} (d\tau + r^2 d\sigma)^2 + \frac{r^2 + p^2}{Q} dr^2 + \frac{r^2 + p^2}{P} dp^2 \right\} \]

where \( Q = -(k + \lambda/3) r^4 - 2nr^3 + \epsilon r^2 - 2mr + k + e^2 + g^2 \)
\( P = -(k + e^2 + g^2 + \lambda/3) p^4 + 2mp^3 - \epsilon p^2 + 2np + k \)

rescale \( p \rightarrow \sqrt{\alpha \omega p}, \ r \rightarrow \sqrt{\frac{\alpha}{\omega}} r, \ \tau \rightarrow \sqrt{\frac{\omega}{\alpha}} \tau, \ \sigma \rightarrow \sqrt{\frac{\omega}{\alpha^3}} \sigma \)

relabel \( m \rightarrow \left( \frac{\alpha}{\omega} \right)^{3/2} m, \ n \rightarrow \left( \frac{\alpha}{\omega} \right)^{3/2} n, \ e \rightarrow \frac{\alpha}{\omega} e, \ g \rightarrow \frac{\alpha}{\omega} g, \ \epsilon \rightarrow \frac{\alpha}{\omega} \epsilon, \ k \rightarrow \alpha^2 k \)

\[ ds^2 = \frac{1}{(1 - \alpha \omega r)^2} \left\{ - \frac{Q}{r^2 + \omega^2 p^2} (d\tau - \omega^2 p^2 d\sigma)^2 
+ \frac{P}{r^2 + \omega^2 p^2} (\omega d\tau + r^2 d\sigma)^2 + \frac{r^2 + \omega^2 p^2}{Q} dr^2 + \frac{r^2 + \omega^2 p^2}{P} dp^2 \right\} \]

where \( Q = -(\alpha^2 k + \lambda/3) r^4 - \frac{2\alpha n}{\omega} r^3 + \epsilon r^2 - 2mr + \omega^2 k + e^2 + g^2 \)
\( P = -[\alpha^2 (\omega^2 k + e^2 + g^2) + \omega^2 \lambda/3] p^4 + 2\alpha mp^3 - \epsilon p^2 + \frac{2n}{\omega} p + k \)

Set \( \alpha = 0 \) and \( \omega = 1 \). Then we recover the Carter’s family.
<table>
<thead>
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<td>$\bigcirc$</td>
<td>$\bigcirc$</td>
<td>$\bigcirc$</td>
</tr>
</tbody>
</table>

The most general known solution

= higher-dimensional Kerr-NUT-(A)dS
D-dimensional Kerr-NUT-(A)dS metric

\[ D = 2n + \varepsilon \ (\varepsilon = 0 \text{ or } 1) \]

\[ ds^2 = \sum_{\mu=1}^{n} \frac{dx_{\mu}^2}{Q_{\mu}} + \sum_{\mu=1}^{n} Q_{\mu} \left[ \sum_{k=0}^{n-1} A_{\mu}^{(k)} d\psi_{k} \right]^2 + \varepsilon \frac{c}{A(n)} \left[ \sum_{k=0}^{n} A^{(k)} d\psi_{k} \right]^2 \]

where

\[ Q_{\mu} = \frac{X_{\mu}}{U_{\mu}}, \quad U_{\mu} = \prod_{\nu=1}^{n} (x_{\mu}^2 - x_{\nu}^2), \quad X_{\mu} = X_{\mu}(x_{\mu}) , \]

\[ A_{\mu}^{(k)} = \sum_{1 \leq \nu_1 < \nu_2 < \cdots < \nu_k \leq n} x_{\nu_1}^2 x_{\nu_2}^2 \cdots x_{\nu_k}^2, \quad A^{(k)} = \sum_{1 \leq \nu_1 < \nu_2 < \cdots < \nu_k \leq n} x_{\nu_1}^2 x_{\nu_2}^2 \cdots x_{\nu_k}^2 , \]

\[ A_{\mu}^{(0)} = A^{(0)} = 1, \quad c = \text{const.} . \]

\[ D=2n \quad X_{\mu} = \sum_{k=0}^{n} c_{2k} x_{\mu}^{2k} + b_{\mu} x_{\mu} \quad D=2n+1 \quad X_{\mu} = \sum_{k=1}^{n} c_{2k} x_{\mu}^{2k} + b_{\mu} + \frac{(-1)^n c}{x_{\mu}^2} \]

This metric satisfies Einstein Eq.

\[ R_{ab} = -(D - 1) c_n g_{ab} \]
Four-dimensional Kerr-NUT-(A)dS metric

\[ ds^2 = \frac{x^2 - y^2}{X} dx^2 + \frac{y^2 - x^2}{Y} dy^2 \]
\[ + \frac{X}{x^2 - y^2} (d\psi_0 + y^2 d\psi_1)^2 + \frac{Y}{y^2 - x^2} (d\psi_0 + x^2 d\psi_1)^2 \]

where

\[ X = c x^4 + x^2 - a^2 - 2 M x \]
\[ Y = c y^4 + y^2 - a^2 - 2 L y \]
Five-dimensional Kerr-NUT-(A)dS metric

\[ ds^2 = \frac{x^2 - y^2}{X} dx^2 + \frac{y^2 - x^2}{Y} dy^2 \]
\[ + \frac{X}{x^2 - y^2} (d\psi_0 + y^2 d\psi_1)^2 + \frac{Y}{y^2 - x^2} (d\psi_0 + x^2 d\psi_1)^2 \]
\[ + \frac{c}{x^2 y^2} (d\psi_0 + (x^2 + y^2) d\psi_1 + x^2 y^2 d\psi_2)^2 \]

where

\[ X = c_4 x^4 + c_2 x^2 + c_0 + b_1 + \frac{c}{x^2} , \]
\[ Y = c_4 y^4 + c_2 y^2 + c_0 + b_2 + \frac{c}{y^2} \]
Six-dimensional Kerr-NUT-(A)dS metric

\[ ds^2 = \frac{(x^2 - y^2)(x^2 - z^2)}{X} dx^2 + \frac{(y^2 - x^2)(y^2 - z^2)}{Y} dy^2 + \frac{(z^2 - x^2)(z^2 - y^2)}{Z} dz^2 \]

\[ + \frac{X}{(x^2 - y^2)(x^2 - z^2)} (d\psi_0 + (y^2 + z^2) d\psi_1 + y^2 z^2 d\psi_2)^2 \]

\[ + \frac{Y}{(y^2 - x^2)(y^2 - z^2)} (d\psi_0 + (z^2 + x^2) d\psi_1 + z^2 x^2 d\psi_2)^2 \]

\[ + \frac{Z}{(z^2 - x^2)(z^2 - y^2)} (d\psi_0 + (x^2 + y^2) d\psi_1 + x^2 y^2 d\psi_2)^2 \]

where

\[ X = c_6 x^6 + c_4 x^4 + c_2 x^2 + c_0 + b_1 x , \]

\[ Y = c_6 y^6 + c_4 y^4 + c_2 y^2 + c_0 + b_2 y , \]

\[ Z = c_6 z^6 + c_4 z^4 + c_2 z^2 + c_0 + b_3 z \]
Seven-dimensional Kerr-NUT-(A)dS metric

\[ ds^2 = \frac{(x^2 - y^2)(x^2 - z^2)}{X} dx^2 + \frac{(y^2 - x^2)(y^2 - z^2)}{Y} dy^2 + \frac{(z^2 - x^2)(z^2 - y^2)}{Z} dz^2 \]

\[ + \frac{X}{(x^2 - y^2)(x^2 - z^2)} (d\psi_0 + (y^2 + z^2)d\psi_1 + y^2z^2d\psi_2)^2 \]
\[ + \frac{Y}{(y^2 - x^2)(y^2 - z^2)} (d\psi_0 + (z^2 + x^2)d\psi_1 + z^2x^2d\psi_2)^2 \]
\[ + \frac{Z}{(z^2 - x^2)(z^2 - y^2)} (d\psi_0 + (x^2 + y^2)d\psi_1 + x^2y^2d\psi_2)^2 \]
\[ + \frac{c}{x^2y^2z^2} (d\psi_0 + (x^2 + y^2 + z^2)d\psi_1 + (x^2y^2 + y^2z^2 + x^2z^2)d\psi_2 + x^2y^2z^2d\psi_3)^2 \]

where

\[ X = c_6x^6 + c_4x^4 + c_2x^2 + c_0 + b_1 - \frac{c}{x^2} , \]
\[ Y = c_6y^6 + c_4y^4 + c_2y^2 + c_0 + b_2 - \frac{c}{y^2} , \]
\[ Z = c_6z^6 + c_4z^4 + c_2z^2 + c_0 + b_3 - \frac{c}{z^2} \]
D-dimensional Kerr-NUT-(A)dS metric

\[ D = 2n + \varepsilon \quad (\varepsilon = 0 \text{ or } 1) \]

\[
\sum_{\mu=1}^{n} \frac{dx_{\mu}^{2}}{Q_{\mu}} + \sum_{\mu=1}^{n} Q_{\mu} \left[ \sum_{k=0}^{n-1} A_{\mu}^{(k)} d\psi_{k} \right]^{2} + \varepsilon \frac{c}{A(n)} \left[ \sum_{k=0}^{n} A^{(k)} d\psi_{k} \right]^{2} 
\]

where

\[ Q_{\mu} = \frac{X_{\mu}}{U_{\mu}}, \quad U_{\mu} = \prod_{\nu=1}^{n} \left( x_{\mu}^{2} - x_{\nu}^{2} \right), \quad X_{\mu} = X_{\mu}(x_{\mu}) \, , \]

\[ A_{\mu}^{(k)} = \sum_{1 \leq \nu_{1} < \nu_{2} < \ldots < \nu_{k} \leq n, \nu_{i} \neq \mu} x_{\nu_{1}}^{2} x_{\nu_{2}}^{2} \ldots x_{\nu_{k}}^{2}, \quad A^{(k)} = \sum_{1 \leq \nu_{1} < \nu_{2} < \ldots < \nu_{k} \leq n} x_{\nu_{1}}^{2} x_{\nu_{2}}^{2} \ldots x_{\nu_{k}}^{2}, \]

\[ A_{\mu}^{(0)} = A^{(0)} = 1, \quad c = \text{const.} \, . \]

D=2n  \quad X_{\mu} = \sum_{k=0}^{n} c_{2k} x_{\mu}^{2k} + b_{\mu} x_{\mu} \quad D=2n+1  \quad X_{\mu} = \sum_{k=1}^{n} c_{2k} x_{\mu}^{2k} + b_{\mu} + \frac{(-1)^{n} c}{x_{\mu}^{2}}

This metric satisfies Einstein Eq.

\[ R_{ab} = -(D - 1) c_{n} g_{ab} \]
How about higher dimensions? – Higher dim. Kerr-NUT-(A)dS –

A closed CKY 2-form exists. \[\text{Kubiznak-Frolov (2007)}\]

Geodesic motion is completely integrable. \[\text{Page-Kubiznak-Vasudevan-Krtous (2007)}\]

Algebraic type of curvature is type-D. \[\text{Hamamoto-TH-Oota-Yasui (2007)}\]

Hamilton-Jacobi equation is separable. \[\text{Kubiznak-Frolov (2007)}\]

Klein-Gordon equation is separable. \[\text{Frolov-Krtous-Kubiznak (2007)}\]

K-G symmetry operators exist. \[\text{Sergyeyev, Krtous (2008)}\]

Dirac equation is separable. \[\text{Oota-Yasui (2008)}\]

Dirac symmetry operators exist. \[\text{Benn-Charlton (1996), Wu (2009)}\]
Hidden symmetries

There exist two “natural” (symmetric and anti-symmetric) generalizations of (conformal) Killing vector.

<table>
<thead>
<tr>
<th>vector</th>
<th>Killing vector</th>
<th>conformal Killing vector</th>
</tr>
</thead>
<tbody>
<tr>
<td>symmetric</td>
<td>Killing-Stackel (KS)</td>
<td>conformal Killing-Stackel (CKS)</td>
</tr>
<tr>
<td></td>
<td>Stackel (1895)</td>
<td></td>
</tr>
<tr>
<td>anti-symmetric</td>
<td>Killing-Yano (KY)</td>
<td>conformal Killing-Yano (CKY)</td>
</tr>
<tr>
<td></td>
<td>Yano (1952)</td>
<td>Tachibana (1969), Kashiwada (1968)</td>
</tr>
</tbody>
</table>
**Def.** Killing-Stackel tensor (KS) is a rank-$p$ symmetric tensor $K$ obeying

$$\nabla_{(a} K_{b_1 \ldots b_p)} = 0$$  \hspace{1cm} \text{Stackel (1895)}

**Def.** Killing-Yano tensor (KY) is a rank-$p$ anti-symmetric tensor $f$ obeying

$$\nabla_{(a} f_{b_1} b_2 \ldots b_p) = 0$$  \hspace{1cm} \text{Yano (1952)}
Prop. When $f$ is a rank-n Killing-Yano (KY) tensor, then rank-2 symmetric tensor $K$ defined by
\[ K_{ab} = f_a \ldots f_b \ldots \]
is a Killing-Stackel (KS) tensor.

Prop. Let $K$ be a rank-n Killing-Stackel tensor field and $\gamma$ be a geodesic with tangent $p$. Then
\[ K^{abc} \ldots p_a p_b p_c \ldots \]
is constant along $\gamma$. 
**Def.** Conformal Killing-Yano tensor (CKY) is a rank-\(p\) anti-symmetric tensor \(k\) obeying

\[
\nabla (a_k b)_{c_1...c_{p-1}} = g_{ab} \xi_{c_1...c_{p-1}} + \sum_{i=1}^{p-1} (-1)^i g_{c_i(a} \xi_{b)c_1...\hat{c}_i...c_{p-1}}
\]

where \(\xi_{c_1...c_{p-1}} = \frac{1}{D - p + 1} \nabla^a k_{ac_1...c_{p-1}}\)

Tachibana (1969), Kashiwada (1968)

**Prop.** Let \(k\) be a CKY \(p\)-form for a metric \(g\). Then, \(\sim k = \Omega^{p+1} k\) is a CKY \(p\)-form for the metric \(\sim g = \Omega^2 g\).
Subclasses of CKY tensors

**Def.** Equivalently, CKY is a $p$-form $k$ obeying

$$
\nabla_X k = \frac{1}{p-1} X \downarrow dk - \frac{1}{D-p+1} X^* \wedge \delta k
$$

for an arbitrary vector $X$.

$d h = 0$ ; $h$ is a closed CKY

$\delta f = 0$ ; $f$ is a KY

$\nabla_X (d\psi) = c X^* \wedge \psi$

; $\psi$ is a special KY

Tachibana-Yu (1970)
Prop. The Hodge star $\star$ maps CKY $p$-forms into CKY $(D-p)$-forms. In particular, the Hodge star of a closed CKY $p$-form is a KY $(D-p)$-form and vice versa.

Prop. When $h_1$ and $h_2$ is a closed CKY $p$-form and $q$-form, respectively, then $h_3 = h_1 \wedge h_2$ is a closed CKY $(p+q)$-form.
Basic properties of hidden symmetries

CKY

CCKY

* * *

KY

Λ

rank-2 KS

rank-2 CKS
Tower of hidden symmetries

\[
\begin{align*}
CCKY_{(2)} & \\
h & \\
CCKY_{(0)} & \xrightarrow{h(0)} \quad CCKY_{(2)} & \xrightarrow{h(1)} \quad CCKY_{(4)} & \xrightarrow{\ldots} & \quad CCKY_{(2n-2)} & \xrightarrow{h(n-1)} \\
volume & \xrightarrow{\omega} \quad form & \xrightarrow{KY_{(D-2)}} & \quad KY_{(D-4)} & \xrightarrow{\ldots} & \quad KY_{(D-2n+1)} & \xrightarrow{f(n-1)} \\
metric & \xrightarrow{g} \quad metric & \xrightarrow{KS_{(2)}} & \quad KS_{(2)} & \xrightarrow{\ldots} & \quad KS_{(2)} & \xrightarrow{K(n-1)}
\end{align*}
\]
Geodesic integrability in higher dimensions

- **closed CKY 2-form** $h$
  - $h^{(j)} = h \wedge \ldots \wedge h$
  - *nontrivial*

- **closed CKY 2j-form** $f^{(j)} = *h^{(j)}$

- **KY (D-2j)-form** $f^{(j)} = f_a^{(j)} f_b^{(j)}$

- **Killing vector** $\xi_a = \nabla^b h_{ba}$

- **Killing vector** $\eta_a^{(j)} = K^{(j)}_{ab} \xi^b$

- **const. of motion** $C_j = K^{(j)}_{ab} p^a p^b$

- **const. of motion** $F_j = \eta_a^{(j)} p^a$

<table>
<thead>
<tr>
<th>dimension</th>
<th># Killing vector</th>
<th># KS tensor</th>
</tr>
</thead>
<tbody>
<tr>
<td>even (D=2n)</td>
<td>$n$</td>
<td>$n$</td>
</tr>
<tr>
<td>odd (D=2n+1)</td>
<td>$n+1$</td>
<td>$n$</td>
</tr>
</tbody>
</table>

\[
\{C_i, C_j\}_P = 0 \quad \{F_i, F_j\}_P = 0 \quad \{C_i, F_j\}_P = 0
\]


One further finds that such a spacetime admits $\delta(n+\varepsilon)$-separability structure, that is, separability of H-J equation for geodesics.

TH-Oota-Yasui (2007)
Theor. Suppose a Riemannian manifold \((M^D, g)\) admits a non-degenerate closed CKY 2-form \(h\). Then the metric takes the form

\[
g = \sum_{\mu=1}^{n} \frac{d x_{\mu}^2}{Q_\mu} + \sum_{\mu=1}^{n} Q_\mu \left[ \sum_{k=0}^{n-1} A_{\mu}^{(k)} d\psi_k \right]^2 + \varepsilon S \left[ \sum_{k=0}^{n} A^{(k)} d\psi_k \right]^2,
\]

where

\[
Q_\mu = \frac{X_\mu}{U_\mu}, \quad U_\mu = \prod_{\nu=1, \nu \neq \mu}^{n} (x_{\mu}^2 - x_{\nu}^2), \quad X_\mu = X_\mu(x_\mu), \quad S = \frac{c}{A(n)}, \quad A^{(0)} = A^{(0)} = 1,
\]

\[
A_{\mu}^{(k)} = \sum_{1 \leq \nu_1 < \nu_2 < \cdots < \nu_k \leq n} x_{\nu_1}^2 x_{\nu_2}^2 \cdots x_{\nu_k}^2, \quad A^{(k)} = \sum_{1 \leq \nu_1 < \nu_2 < \cdots < \nu_k \leq n} x_{\nu_1}^2 x_{\nu_2}^2 \cdots x_{\nu_k}^2.
\]

Einstein metrics with a non-degenerate CKY 2-form

when
\[
X_\mu = \sum_{k=0}^{n} c_k x_\mu^{2k} + b_\mu x_\mu
\]  \quad \text{in 2n dimension}

\[
X_\mu = \sum_{k=1}^{n} c_k x_\mu^{2k} + b_\mu + \frac{(-1)^n c}{x_\mu^2}
\]  \quad \text{in 2n+1 dimension ,}

This metric satisfies Einstein Eq.

\[
R_{ab} = -(D - 1) c_n g_{ab}
\]

Then, the metric coincides with that of Kerr-NUT-(A)dS metric. In this mean, only vacuum spacetime admitting a non-degenerate CKY 2-form is the Kerr-NUT-(A)dS spacetime.
In the case of degenerate CCKY tensors

It is convenient to see the eigenvalues of a rank-2 closed CKY by $Q^a_b = -h^a_{ch}c^b_b$.

$$V^{-1}(Q^a_b)V = \left\{ x_1^2, x_1^2, \ldots, x_n^2, x_n^2, \xi_1^2, \ldots, \xi_1^2, \ldots, \xi_N^2, \ldots, \xi_N^2, 0, \ldots, 0 \right\}$$

The D-dim. generalized Kerr-NUT-(A)dS offshell metric is

$$g = \sum_{\mu=1}^{n} \frac{dx_{\mu}^2}{P_{\mu}} + \sum_{\mu=1}^{n} P_{\mu} \left[ \sum_{k=0}^{n-1} A^{(k)}_{\mu} \theta_k \right]^2 + \sum_{j=1}^{N} \prod_{\mu=1}^{n} (x_{\mu}^2 - \xi_j^2) g^{(j)} + \left( \prod_{\mu} x_{\mu}^2 \right) g^{(0)}$$

Where $g^{(0)}$ is arbitrary K-dim metric and $g^{(j)}$ is $2m_j$-dim Kahler metric with the Kahler form $\omega^{(j)}$.

$$P_{\mu} = \frac{X_{\mu}(x_{\mu})}{x_{\mu}^K \prod_{j=1}^{N} (x_{\mu}^2 - \xi_j^2)^{m_j} \prod_{\nu=1}^{n} (x_{\mu}^2 - x_{\nu}^2)}$$

$$A^{(k)}_{\mu} = \sum_{\nu_i \neq \mu} x_{\nu_1}^2 x_{\nu_2}^2 \ldots x_{\nu_k}^2$$

$$d\theta_k + 2 \sum_{j=1}^{N} (-1)^{n-k} \xi_j^{2n-2k-1} \omega^{(j)} = 0$$

TH-Oota-Yasui (2008)

We can’t determine them any more without Einstein's Eq.
When $g^{(0)}$ is $K$-dim Einstein metric, $g^{(j)}$ is $2m_j$-dim Einstein-Kahler metric with the Kahler form $\omega^{(j)}$ and

$$X_\mu = x_\mu \int dx_\mu \, \chi(x_\mu) x^K x^2 i \prod_{i=1}^N \left( x^2_i - \xi^2_i \right)^{m_i} + d_\mu x_\mu$$

where

$$\chi(x_\mu) = \sum_{i=0}^n \alpha_i x^{2i}, \quad \alpha_0 = (-1)^{n-1} \lambda^{(0)}$$

$$\lambda^{(j)} = (-1)^{n-1} \chi(\xi^2_j)$$

This metric satisfies Einstein Eq.

$$R_{ab} = -(D - 1) \alpha_n g_{ab}$$
**Theor.** Let \((M^n, g)\) be a compact, simply connected manifold admitting a special KY. Then \(M\) is either isometric to \(S^n\) or \(M\) is a Sasakian, 3-Sasakian, nearly Kahler or weak \(G_2\)-manifold.

Semmelmann (2002)

**Example** Let \((M^{2n+1}, g, \xi, \eta)\) be a Sasakian manifold with Killing vector field \(\xi\). Then

\[
\omega_k := \xi^* \wedge (d\xi^*)^k
\]

is a rank-(2\(k+1\)) special KY for \(k = 0, \ldots, n\), which satisfies for any vector field \(X\) and any \(k\)

\[
\nabla_X (d\omega_k) = -2(k + 1)X^* \wedge \omega_k
\]
3. A generalization of CKY symmetry
Known facts:

- Existence of a rank-2 Killing tensor
  - Davis-Kunduri-Lucietti (2005)

- Existence of a GCCKY 2-form
  - Kubiznak-Kundri-Yasui (2009)

Hidden symmetry of charged BH in 5-dim. minimal SUGRA

\[ S_5 = \int R \ast 1 - \frac{1}{2} \ast F \land F + \frac{1}{3\sqrt{3}} F \land F \land A \]

- Charged rotating BH

\[
g = \frac{x^2 - y^2}{X} dx^2 + \frac{y^2 - x^2}{Y} dy^2 \\
+ \frac{X}{x^2 - y^2} [dt + y^2 d\phi]^2 + \frac{Y}{y^2 - x^2} [dt + x^2 d\phi]^2 \\
+ \frac{1}{x^2 y^2} [c\{dt + (x^2 + y^2) d\phi + x^2 y^2 d\psi\} - y^2 A(1)]^2
\]

\[ A(1) = \frac{\sqrt{3} q}{x^2 - y^2} [dt + y^2 d\phi] \]

Chong-Cvetic-Lu-Pope (2005)
**Def.** Generalized CKY is a $p$-form $k$ if a 3-form $T$ exists obeying

$$\nabla^T_X k = \frac{1}{p-1} X \lrcorner d^T k - \frac{1}{D-p+1} X^* \wedge \delta^T k$$

for an arbitrary vector $X$.

$$\nabla^T_{a b_1 \ldots b_p} := \nabla_{a b_1 \ldots b_p} - \frac{1}{2} T_{ca [b_1 k^c_{b_2 \ldots b_p}]}
$$

$$(d^T k)_{a_1 \ldots a_{p+1}} := (p+1) \nabla^T_{[a_1 k_{a_2 \ldots a_{p+1}]}]
$$

$$(\delta^T k)_{a_1 \ldots a_{p-1}} := -\nabla^T_c k^c_{a_1 \ldots a_{n-1}}
$$

Note: This connection gives $\nabla^T g = 0$. 

Subclasses of GCKY tensors

\[
\nabla^T_X k = \frac{1}{p - 1} X \perp d^T k - \frac{1}{D - p + 1} X^* \wedge \delta^T k
\]

\[d^T h = 0 \; ; \; h \text{ is a generalized closed CKY}\]

\[\delta^T f = 0 \; ; \; f \text{ is a GKY}\]
1) A GCKY 1-form is equal to a conformal Killing 1-form.

2) The Hodge star $\ast$ maps GCKY $p$-forms into GCKY $(D-p)$-forms. In particular, the Hodge star of a closed GCKY $p$-form is a GKY $(D-p)$-form and vice versa.

3) When $h_1$ and $h_2$ is a closed GCKY $p$-form and $q$-form, respectively, then $h_3 = h_1 \wedge h_2$ is a closed GCKY $(p+q)$-form.

4) When $f$ is a G(C)KY $p$-form, then rank-2 symmetric tensor $K$ defined by $K_{ab} = f_a \cdots f_b$ is a (conformal) Killing tensor.

**Basic Properties of GCKY symmetry**
Tower of hidden symmetries

- **GCCKY\(_{(2)}\)**
  - **GCCKY\(_{(0)}\)**
  - **GCCKY\(_{(2)}\)**
  - **GCCKY\(_{(4)}\)**
  - **GCCKY\(_{(2n-2)}\)**

- **GKY\(_{(D-2)}\)**
  - **GKY\(_{(D-4)}\)**
  - **GKY\(_{(D-2n+1)}\)**

- **KS\(_{(2)}\)**
  - **KS\(_{(2)}\)**
  - **KS\(_{(2)}\)**
  - **KS\(_{(2)}\)**

- **volume form** \(\omega\)
- **metric** \(g\)
- **KS**
  - **KS**
  - **KS**
  - **KS**
- Constants of motion generated from a GCCKY 2-form are in involution, i.e., $\{C_i, C_j\}_P = 0$

- one doesn’t have Killing vectors.
Th. Let \( \omega \) be a generalized conformal Killing-Yano (GCKY) \( p \)-form obeying
\[
\nabla^T_X \omega - \frac{1}{p+1} X \wedge d^T \omega + \frac{1}{n-p+1} X^b \wedge \delta^T \omega = 0.
\]
Then the operator
\[
L_\omega = e^a \omega \nabla^T_{e_a} + \frac{p}{p+1} d^T \omega - \frac{n-p}{n-p+1} \delta^T \omega + \frac{1}{2} T_\omega
\]
satisfies
\[
D L_\omega = \omega D^2 + \frac{(-1)^p}{p+1} d^T \omega D + \frac{(-1)^p}{n-p+1} \delta^T \omega D - A.
\]
In the case $A$ vanishes, $L_\omega$ is an symmetry operator for massless Dirac equation, i.e.,

$$\mathcal{D}L_\omega - L_\omega \mathcal{D} = 0 \quad \text{(on-shell)}$$

Anomaly

The last term $A = A_{(p+2)} + A_{(p-2)}$ is written explicitly as

$$A_{(p+2)} = \frac{d(d^T \omega)}{p+1} - \frac{T \wedge \delta^T \omega}{n-p+1} - \frac{1}{2} d^T \wedge \omega$$

$$A_{(p-2)} = \frac{\delta(\delta^T \omega)}{n-p+1} - \frac{1}{6(p+1)} T \wedge d^T \omega + \frac{1}{12} d^T \wedge \omega.$$
Col. Let \( \omega \) be a generalized Killing-Yano (GKY) \( p \)-form such that an anomaly \( A \) vanishes. Then there exists an operator \( K_\omega \) such that
\[
\mathcal{D} K_\omega + (-1)^p K_\omega \mathcal{D} = 0 \quad (\text{off-shell})
\]

Col. Let \( \omega \) be a generalized closed conformal Killing-Yano (GCCKY) \( p \)-form such that an anomaly \( A \) vanishes. Then there exists an operator \( M_\omega \) such that
\[
\mathcal{D} M_\omega - (-1)^p M_\omega \mathcal{D} = 0 \quad (\text{off-shell})
\]
\[
d^T \omega = 0
\]
The symmetry operators in terms of gamma matrices

\[ L_\omega = \left[ \omega^{a}_{b_1...b_{p-1}} \gamma^{b_1...b_p}_{b_1...b_p} + \frac{1}{p(p+1)} \omega_{b_1...b_p} \gamma^{ab_1...b_p}_{ab_1...b_p} \right] \nabla a \]

\[ + \frac{1}{(p+1)^2} (d\omega)_{b_1...b_{p+1}} \gamma^{b_1...b_{p+1}}_{b_1...b_{p+1}} - \frac{n-p}{n-p+1} (\delta\omega)_{b_1...b_{p-1}} \gamma^{b_1...b_p}_{b_1...b_p} - \frac{1}{24} T_{b_1b_2b_3} \omega_{b_4...b_{p+3}} \gamma^{b_1...b_{p+3}}_{b_1...b_{p+3}} + \frac{3-p}{8(p+1)} T^a_{b_1} \omega_{b_2b_3...b_{p+1}} \gamma^{b_1...b_{p+1}}_{b_1...b_{p+1}} \]

\[ + \frac{(n-p-3)(p-1)}{8(n-p+1)} T^{ab}_{b_1} \omega_{ab2...b_{p-1}} \gamma^{b_1...b_{p-1}}_{b_1...b_{p-1}} + \frac{(p-1)(p-2)}{24} T^{abc} \omega_{abc1...b_{p-3}} \gamma^{b_1...b_{p-3}}_{b_1...b_{p-3}}. \]

\[ K_\omega = \omega^{a}_{b_1...b_{p-1}} \gamma^{b_1...b_p}_{b_1...b_p} \nabla a \]

\[ + \frac{1}{2(p+1)^2} (d\omega)_{b_1...b_{p+1}} \gamma^{b_1...b_{p+1}}_{b_1...b_{p+1}} + \frac{1-p}{8(p+1)} T^a_{b_1b_2} \omega_{ab3...b_{p+1}} \gamma^{b_1...b_{p+1}}_{b_1...b_{p+1}} - \frac{p-1}{4} T^{ab}_{b_1} \omega_{ab2...b_{p-1}} \gamma^{b_1...b_{p-1}}_{b_1...b_{p-1}} + \frac{(p-1)(p-2)}{24} T^{abc} \omega_{abc1...b_{p-3}} \gamma^{b_1...b_{p-3}}_{b_1...b_{p-3}}. \]

\[ M_\omega = \omega_{b_1...b_p} \gamma^{ab_1...b_p}_{ab_1...b_p} \nabla a \]

\[ - \frac{p(n-p)}{2(n-p+1)} (\delta\omega)_{b_1...b_{p-1}} \gamma^{b_1...b_{p-1}}_{b_1...b_{p-1}} - \frac{1}{24} T_{b_1b_2b_3} \omega_{b_4...b_{p+3}} \gamma^{b_1...b_{p+3}}_{b_1...b_{p+3}} \]

\[ + \frac{p}{4} T^a_{b_1b_2} \omega_{ab3...b_{p+1}} \gamma^{b_1...b_{p+1}}_{b_1...b_{p+1}} + \frac{p(n-p-1)}{8(n-p+1)} T^{ab}_{b_1} \omega_{ab2...b_{p-1}} \gamma^{b_1...b_{p-1}}_{b_1...b_{p-1}}. \]
Hidden symmetry of CCLP black hole

- GCCKY 2-form

Kubiznak-Kunduri-Yasui (2009)

\[ h = x_1 e^1 \wedge \bar{e}^1 + x_2 e^2 \wedge \bar{e}^2 \quad \text{with} \quad T = \frac{1}{\sqrt{3}} * F \]

It was shown that this 2-form produces a rank-2 Killing tensor discovered by Davis-Kunduri-Lucietti.

- Separation of variables

H-J, K-G and Dirac equations are separable.

Davis-Kunduri-Lucietti (2005), Wu (2009)
We consider the following theory

\[ \mathcal{L}_4 = e^{-\varphi} (R \star 1 + *d\varphi \wedge d\varphi - \frac{1}{4} * F(2) \wedge F(2) - \frac{1}{2} * H(3) \wedge H(3)) \]

where

\[ F(2) = dA(1) , \quad H(3) = dB(2) - \frac{1}{4} A(1) \wedge dA(1) \]

This action gives an bosonic part of the low-energy effective action of heterotic string theory.
Kerr-Sen black holes

\[
\begin{align*}
 ds^2 &= e^\Phi \left\{ -\frac{\Delta}{\rho^2_b} (dt - a \sin^2 \theta d\varphi)^2 \\
 &\quad + \frac{\sin^2 \theta}{\rho^2_b} [adt - (r^2 + 2br + a^2) d\varphi]^2 + \frac{\rho^2_b}{\Delta} dr^2 + \rho^2_b d\theta^2 \right\}, \\
 H &= -\frac{2ba}{\rho^4_b} dt \wedge d\varphi \wedge [(r^2 - a^2 \cos^2 \theta) \sin^2 \theta dr - r \Delta \sin 2\theta d\theta], \\
 A &= -\frac{Qr}{\rho^2_b} (dt - a \sin^2 \theta d\varphi), \\
 \Phi &= 2 \ln \left( \frac{\rho}{\rho_b} \right)
\end{align*}
\]

where

\[
\begin{align*}
 \rho^2 &= r^2 + a^2 \cos^2 \theta, \quad \rho^2_b = \rho^2 + 2br, \quad \Delta = r^2 - 2(M - b)r + a^2.
\end{align*}
\]
Hidden symmetry of Kerr-Sen black holes

Known facts:

Algebraic properties of curvature

Separability of the Hamilton-Jacobi equation

Separability of the Klein-Gordon equation

Existence of a rank-2 Killing tensor (string frame)

Questions:

Separability of the Dirac equation?

Why does such a separation occur?
We consider the ‘naïve’ generalization of heterotic supergravity

\[ \mathcal{L}_D = e^\varphi \sqrt{\frac{(D-2)}{2}} \{ R \ast 1 - \frac{1}{2} \ast d\varphi \wedge d\varphi - \ast F(2) \wedge F(2) - \frac{1}{2} \ast H(3) \wedge H(3) \} \]

where

\[ F(2) = dA(1) , \quad H(3) = dB(2) - A(1) \wedge dA(1) . \]

This kind of action gives a bosonic part of supergravity such as heterotic supergravity compactified on a torus in each dimension.
Higher-dimensional Kerr-Sen black holes

\[ g_D = \sum_{\mu=1}^{n} \frac{dx_\mu^2}{Q_\mu} + \sum_{\mu=1}^{n} Q_\mu \left( A_\mu - \sum_{\nu=1}^{n} \frac{2N_\nu s^2}{HU_\nu} A_\nu \right)^2 + \varepsilon S \left( A - \sum_{\nu=1}^{n} \frac{2N_\nu s^2}{HU_\nu} A_\nu \right)^2 \]

\[ \phi = \sqrt{\frac{2}{D-2}} \ln H , \quad A_{(1)} = \sum_{\mu=1}^{n} \frac{2N_\mu s c}{HU_\mu} A_\mu , \]

\[ B_{(2)} = \left( \sum_{k=0}^{n-1} (-1)^k c_{n-k-1} d\psi_k + \varepsilon \tilde{c} d\psi_n \right) \wedge \left( \sum_{\nu=1}^{n} \frac{2N_\nu s^2}{HU_\nu} A_\nu \right) \]

where \( A_\mu = \sum_{k=0}^{n-1} A^{(k)}_\mu d\psi_k \), \( A = \sum_{k=0}^{n} A^{(k)} d\psi_k \), \( H = 1 + \sum_{\mu=1}^{n} \frac{2N_\mu s^2}{U_\mu} \), \( N_\mu = m_\mu x_\mu^{1-\varepsilon} \),

\[ Q_\mu = \frac{X_\mu}{U_\mu} , \quad U_\mu = \prod_{\nu=1}^{n} (x_\mu^2 - x_\nu^2) , \quad X_\mu = \sum_{k=0}^{n-1} c_k x_\mu^{2k} + 2N_\mu + \varepsilon \frac{(-1)^{n\tilde{c}}}{x_\mu^2} \], \( c_{n-1} = -1 \),

\[ A^{(k)}_\mu = \sum_{1 \leq \nu_1 < \cdots < \nu_k \leq n} x_{\nu_1}^{2} \cdots x_{\nu_k}^{2} , \quad A^{(k)} = \sum_{1 \leq \nu_1 < \cdots < \nu_k \leq n} x_{\nu_1}^{2} \cdots x_{\nu_k}^{2} , \quad A^{(0)} = A^{(0)} = 1 \),

\[ S = \frac{\tilde{c}}{A^{(n)}} , \quad \tilde{c} = \text{const.} , \quad s = \sinh \delta , \quad c = \cosh \delta . \]
Known facts:

Hamilton-Jacobi equation is separable.
Rank-2 Killing tensors exist.

\[ K^{(j)} = \sum_{\mu=1}^{n} A^{(j)}_{\mu} (e^\mu e^{\mu} + e^{\tilde{\mu}} e^{\tilde{\mu}}) + \varepsilon A^{(j)} e^0 e^0 \]

Questions:

Does the separation of the K-G equation occurs?
How about the Dirac equation?
If separable, where does such a structure come from?
- **GCCKY 2-form**

\[ h = \sum_{\mu=1}^{n} x_{\mu} e^{\mu} \wedge e^{\bar{\mu}} \quad \text{with} \quad T = H \]

TH-Kubiznak-Warnick-Yasui (2010)

- **Separation of variables**


<table>
<thead>
<tr>
<th>frame</th>
<th>Einstein</th>
<th>string</th>
</tr>
</thead>
<tbody>
<tr>
<td>H-J</td>
<td>separable</td>
<td>separable</td>
</tr>
<tr>
<td>K-G</td>
<td>separable</td>
<td>×</td>
</tr>
<tr>
<td>Dirac*</td>
<td>×</td>
<td>separable</td>
</tr>
</tbody>
</table>

- **Symmetry operators**

For the torsion \( T=H \), one can produce the symmetry operators for the Laplacian and the modified Dirac operator \( D^{T/3} \).
4. Summary & Outlook
We have studied properties of spacetimes admitting a conformal Killing-Yano symmetry and its generalization. Especially, a rank-2 CCKY and GCCKY 2-form.

If the torsion is absent, we have shown that such symmetry characterizes vacuum black hole solutions with spherical horizon topology.

If the torsion is present, we have shown that such symmetry are seen in the solutions of supergravities such as 5-dim. minimal SUGRA and heterotic supergravity.
Exact solutions of 5-dim. $U(1)^3$ SUGRA

\[ a = b \]

- Galt’ssov-Sherbluk (2008)

\[ g^2 = 0 \]

- Mei-Pope (2007)

\[ \delta_1 = \delta_2 \]

- Chong-Cvetic-Lu-Pope (2005)
- Chong-Cvetic-Lu-Pope (2005)
- Chong-Cvetic-Lu-Pope (2005)
- Chow (2007)

\[ \delta_3 = 0 \]


minimal SUGRA

\[ \delta_1 = \delta_2 = \delta_3 \]


susy limit

- Susy limit
Manifolds with special holonomy

Type-IIB supergravity on $\text{AdS}_5 \times X^5$

$\mathcal{N} = 1$ SCFT correspondence

(Examples of Sasaki-Einstein) $S^5 \quad T^{1,1}$

It is known that Sasaki-Einstein and Calabi-Yau metrics are derived from vacuum rotating BH by taking a limit.

$vacuum \text{ rotating BH} \quad \rightarrow \quad \begin{cases} even & \text{Calabi-Yau} \\ odd & \text{Sasaki-Einstein} \end{cases}

\text{Ex})

5-dim. Kerr-(A)dS $\quad \rightarrow \quad$ Sasaki-Einstein $\quad Y^{p,q} \quad L^{a,b,c}

6-dim. Kerr-NUT-(A)dS $\quad \rightarrow \quad$ resolved Calabi-Yau cone
Even Dimensions

- Space admitting a CCKY 2-form
  - vacuum rotating BH
  - TH, Oota, Yasui (2008)
  - Krtous, Frolov, Kubiznak (2008)

- Space admitting a GCCKY 2-form
  - charged rotating BH
  - HKWY (2010)

≠

- Kahler manifold admitting a Hamiltonian 2-form
  - Calabi-Yau manifold

∩

∩

- KT manifold admitting a Hamiltonian 2-form with torsion
  - Calabi-Yau with torsion
  - HKWY (2010)
Spacetimes admitting a GCCKY 2-form

• assumption

\[ g = g_{ab}(z) dz^a dz^b \quad : \text{D-dim metric} \]

\[ h = \frac{1}{2} h_{ab}(z) dz^a \wedge dz^b \quad : \text{GCCKY 2-form} \]

i.e. \[ \nabla^T_a h_{bc} = g_{ab} \xi_c - g_{ac} \xi_b \quad \xi_a = \frac{1}{D-1} \nabla^T_b h_{ba} \]
• introduce canonical basis

Orthonormal frame \( \{ e^a \} = \{ e^\mu, e^{\hat{\mu}} \} \)

s.t. \( g = \sum_{\mu=1}^{n} (e^\mu e^\mu + e^{\hat{\mu}} e^{\hat{\mu}}), \quad h = \sum_{\mu=1}^{n} x_\mu e^\mu \wedge e^{\hat{\mu}} \)

non-degenerate: \( x_\mu \neq x_\nu \)

• the form of \( \xi \)

\[
\xi = \sum_{\mu=1}^{n} \sqrt{Q_\mu} e^{\hat{\mu}} \quad \text{where} \quad Q_\mu \text{ is an arbitrary fn.}
\]
\[ [e_\mu, e_\nu] = -\frac{x_\nu \sqrt{Q_\nu}}{x_\mu^2 - x_\nu^2} e_\mu - \frac{x_\mu \sqrt{Q_\mu}}{x_\mu^2 - x_\nu^2} e_\nu \quad (\mu \neq \nu) , \]

\[ [e_\mu, e_{\tilde{\mu}}] = K_\mu e_\mu + L_\mu e_{\tilde{\mu}} + \sum_{\rho \neq \mu} \frac{2x_\mu \sqrt{Q_\rho}}{x_\mu^2 - x_\rho^2} e_{\tilde{\rho}} , \]

\[ [e_\mu, e_{\tilde{\nu}}] = -\frac{x_\mu \sqrt{Q_\mu}}{x_\mu^2 - x_\tilde{\nu}^2} e_{\tilde{\nu}} \quad (\mu \neq \nu) , \]

\[ [e_{\tilde{\mu}}, e_{\tilde{\nu}}] = 0 \quad (\mu \neq \nu) , \]

\[ K_\mu = \frac{\kappa_{\mu \mu}}{\sqrt{Q_\mu}}, \quad L_\mu = -\frac{1}{\sqrt{Q_\mu}} \left( \sum_{\rho \neq \mu} \frac{x_\mu Q_\rho}{x_\mu^2 - x_\rho^2} - \kappa_{\tilde{\mu} \mu} \right), \quad M_{\mu \nu} = \frac{2x_\mu \sqrt{Q_\nu}}{x_\mu^2 - x_\nu^2} - T_{\mu \tilde{\nu} \tilde{\nu}} \]
\[
[[e_A, e_B], e_C] + [[e_B, e_C], e_A] + [[e_C, e_A], e_B] = 0
\]

\[
M_{\mu\nu}K_{\nu} = 0 ,
\]

\[
e_\nu(K_{\mu}) = \frac{x_\nu \sqrt{Q_{\nu}}}{x_\mu^2 - x_\nu^2}K_{\mu} , \quad e_{\bar{\nu}}(K_{\mu}) = 0 ,
\]

\[
e_\nu(L_{\mu}) = \frac{x_\nu \sqrt{Q_{\nu}}}{x_\mu^2 - x_\nu^2}L_{\mu} - M_{\mu\nu}M_{\nu\mu} - \frac{2x_\mu x_\nu \sqrt{Q_{\mu}} \sqrt{Q_{\nu}}}{(x_\mu^2 - x_\nu^2)^2} , \quad e_{\bar{\nu}}(L_{\mu}) = 0 ,
\]

\[
e_\nu(M_{\mu\nu}) = \left(\frac{2x_\nu \sqrt{Q_{\nu}}}{x_\mu^2 - x_\nu^2} - L_{\nu}\right)M_{\mu\nu} , \quad e_{\bar{\nu}}(M_{\mu\nu}) = 0 ,
\]

\[
e_\nu(M_{\mu\rho}) = \left(\frac{2x_\nu \sqrt{Q_{\nu}}}{x_\mu^2 - x_\nu^2} + \frac{x_\nu \sqrt{Q_{\nu}}}{x_\nu^2 - x_\rho^2}\right)M_{\mu\rho} - M_{\mu\nu}M_{\nu\rho} , \quad e_{\bar{\nu}}(M_{\mu\rho}) = 0 .
\]

\[
T_{\mu\nu\rho} = 0 \quad (\mu, \nu, \rho: \text{different}) .
\]
the only components $T_{\mu\tilde{\mu}\tilde{\nu}}$ are non-vanishing.

\[ T = T_{\mu\tilde{\mu}\tilde{\nu}} e^\mu \wedge e^{\tilde{\mu}} \wedge e^{\tilde{\nu}} \]

local multi-Hermitian structure

for each $\epsilon = (\epsilon_1, \cdots, \epsilon_n)$ with $\epsilon_i = \pm 1$

\[ \exists \quad J_\epsilon(e_\mu) = -\epsilon_\mu e^{\tilde{\mu}} \quad J_\epsilon(e^{\tilde{\mu}}) = \epsilon_\mu e_\mu \]

s.t.

\[ N_\epsilon(X, Y) \equiv [J_\epsilon X, J_\epsilon Y] - [X, Y] \]

\[ - J_\epsilon [X, J_\epsilon Y] - J_\epsilon [J_\epsilon X, Y] = 0 \]

(1) For each $\epsilon$, $J_\epsilon$ is complex structure.

(2) $g$ is Hermitian: $g(X, Y) = g(X, J_\epsilon Y)$
• Bismut torsion

\[ B = -d\Omega(JX, JY, JZ) \]

where \( \Omega(X, Y) = g(X, JY) \)

s.t. \( \nabla^B g = 0, \ \nabla^B J = 0, \ \nabla^B \Omega = 0 \)

\( (M, g, J, \Omega, B) \) is called Kahler with torsion (KT) manifold. When \( B=0 \), then it becomes Kahler manifold.

• relationship b/w the torsion \( T \) of GCCKY 2-form and the Bismut torsion \( B \)

\[ T = \sum_{\mu \neq \nu} \frac{2\epsilon_{\mu} \sqrt{Q_{\nu}}}{\epsilon_{\mu} x_{\mu} + \epsilon_{\nu} x_{\nu}} e^\mu \wedge \overset{\nu}{e} \wedge \overset{\mu}{e} + B . \]
• 3 types of solutions: $\mathbf{K}_{\mu} = 0, \mathbf{M}_{\mu \nu} = 0, \text{Mixed}$

(These doesn’t exist when $T=0$.)

(1) $\mathbf{K}_{\mu} = 0$ type: special solution

$$L_{\mu} = -\partial_{\mu} \sqrt{Q_{\mu}} + \partial_{\mu} \left( \ln H - \sum_{\nu=1}^{n} \ln f_{\nu} \right) \sqrt{Q_{\mu}},$$

$$M_{\mu \nu} = \frac{f_{\nu}}{f_{\mu}} \left( \frac{2x_{\mu}}{x_{\mu}^2 - x_{\nu}^2} + \partial_{\mu} \ln H \right) \sqrt{Q_{\nu}},$$

where $H = 1 + \sum_{\mu=1}^{n} \frac{N_{\mu}}{U_{\mu}}, \partial_{\nu} N_{\mu} = 0$ and $f_{\mu} = f_{\mu}(x_{\mu})$

$$T = \sum_{\mu \neq \nu} \left\{ \frac{2x_{\mu}}{x_{\mu}^2 - x_{\nu}^2} \left( 1 - \frac{f_{\nu}}{f_{\mu}} \right) - \frac{f_{\nu} \left( \partial_{\mu} \ln H \right)}{f_{\mu}} \right\} \sqrt{Q_{\nu}} e^{\mu} \wedge e^{\mu} \wedge e^{\nu}.$$
(2) \( M_{\mu \nu} = 0 \): we have general solution

\[
K_\mu = -\frac{U_\mu}{2}, \quad L_\mu = \sqrt{Q_\mu} \left( \sum_{\rho \neq \mu} \frac{x_\mu}{x_\mu^2 - x_\rho^2} + h_\mu \right),
\]

where \( \partial_\nu h_\mu = e_\nu(h_\mu) = 0 \) for \( \mu \neq \nu \)

\[
T = \sum_{\mu \neq \nu} \frac{2x_\mu \sqrt{Q_\nu}}{x_\mu^2 - x_\nu^2} e_\mu \wedge e_\nu \wedge e_\nu.
\]

(3) Mixed: for simplicity, in 4 dimensions

\[
K_1 = -\frac{1}{2}(x_1^2 - x_2^2), \quad K_2 = 0,
\]

\[
L_1 = \left( \frac{x_1}{x_1^2 - x_2^2} + h_1 \right)\sqrt{Q_1}, \quad L_2 = \left( \frac{x_2}{x_2^2 - x_1^2} + h_2 \right)\sqrt{Q_2},
\]

\[
M_{12} = \exp \left( \int h_2 dx_2 \right) \times (x_1^2 - x_2^2)^{-3/2} + f_1(x_1), \quad M_{21} = 0.
\]
• construction of metrics

(1) \( K_\mu = 0 \):
\[
e^\mu = \frac{dx_\mu}{\sqrt{Q_\mu}}, \quad \tilde{e}^\mu = \sqrt{R_\mu(\mathcal{A}_\mu - A)}
\]

where \( Q_\mu = \frac{X_\mu(x_\mu)}{U_\mu} \), \( R_\mu = \frac{Y_\mu(x_\mu)}{U_\mu} \), \( Y_\mu = f_\mu(x_\mu)^2 \).

\[
\mathcal{A}_\mu = \sum_{k=0}^{n-1} A^{(k)}_\mu \psi^k \quad \text{and} \quad dA = \sum_{\mu=1}^{n} \frac{\partial_\mu \ln H}{f_\mu} e^\mu \wedge \tilde{e}^\mu.
\]

(This includes Kerr-Sen black holes)

(2) \( M_{\mu\nu} = 0 \):
\[
e^\mu = \frac{dx_\mu}{\sqrt{Q_\mu}}, \quad \tilde{e}^\mu = \frac{dy_\mu}{\sqrt{R_\mu}}
\]

where \( Q_\mu = \frac{X_\mu(x_\mu, y_\mu)}{U_\mu} \), \( R_\mu = \frac{Y_\mu(x_\mu, y_\mu)}{U_\mu} \).

(3) Mixed: not yet