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Hidden symmetry of supergravity black holes

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Ref. **Yasui and TH, Prog. Theor. Phys. Suppl. 189 (2011) 126-164.**

Motivation of this work

We want to solve the higher-dimensional gravitational theories and explicitly construct the solutions that have physically and mathematically interesting properties.

But, it is difficult to solve them in general.

Investigating known solutions, we consider a generalization of them.

We focus on hidden symmetries of black holes.

Exact solutions – vacuum black holes with S^n horizon topology

vacuum Einstein's Eq.

$$Ric(g) = \lambda g$$

Four dimensions	mass,	NUT,	rotation,	λ
Schwarzschild (1916)	○			
Kerr (1963)	○		○	
Carter (1968)	○	○	○	○

Higher dimensions	mass,	NUTs,	rotations,	λ
Tangherlini (1916)	○			
Myers-Perry (1986)	○		[(D-1)/2]	
Gibbons-Lu-Page-Pope (2004)	○		[(D-1)/2]	○
5-dim. Hawking, et al. (1998)				
Chen-Lu-Pope (2006)	○	[D/2-1]	[(D-1)/2]	○

Rotating black hole solution

In 1963, Kerr discovered a solution describing rotating black holes in a vacuum.

$$ds^2 = -\frac{\Delta}{\Sigma}\left(dt - a\sin^2\theta d\phi\right)^2 + \frac{\sin^2\theta}{\Sigma}\left(a\,dt - (r^2 + a^2)d\phi\right)^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2$$

where

$$\Delta = r^2 - 2Mr + a^2 \ , \quad \Sigma = r^2 + a^2\cos^2\theta$$

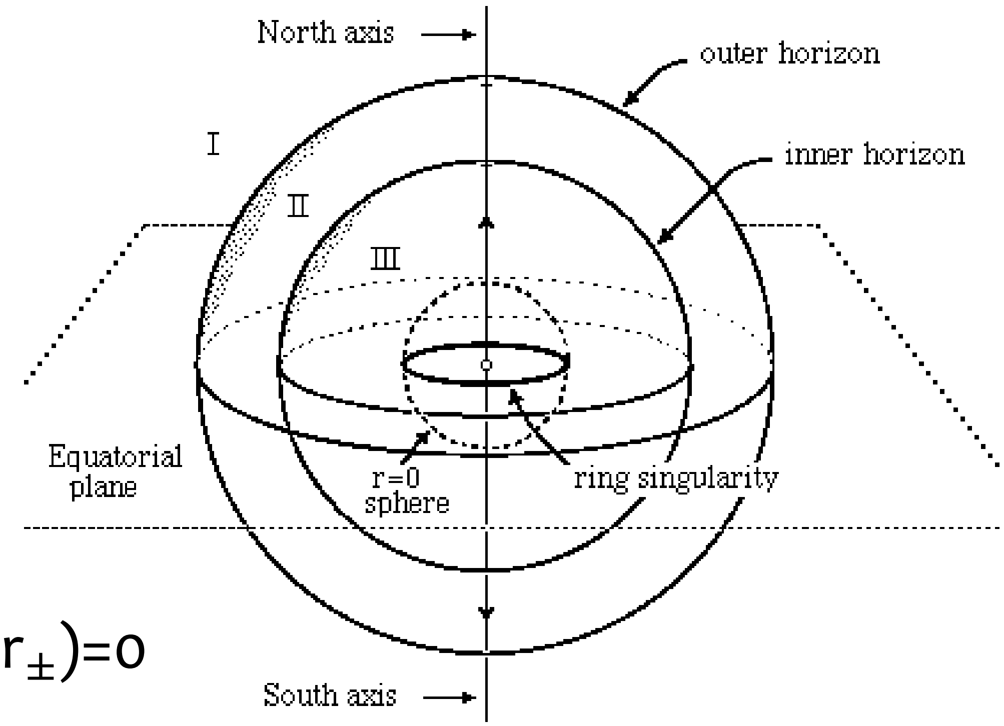
Geometry of Kerr spacetime

Kerr's metric

$$ds^2 = -\frac{\Delta}{\Sigma}\left(dt - a\sin^2\theta d\phi\right)^2 + \frac{\sin^2\theta}{\Sigma}\left(a\,dt - (r^2+a^2)d\phi\right)^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2$$

where $\Delta = r^2 - 2Mr + a^2\ ,\quad \Sigma = r^2 + a^2\cos^2\theta$

- Two parameters
  - mass M
  - angular momentum J=Ma
- Two isometries
  - time translation ∂/∂t
  - axial symmetry ∂/∂Φ
- Ring singularity at Σ=0, i.e., r=0, θ=π/2
- Two horizons at $r=r_\pm$ s.t. $\triangle(r_\pm)=0$



Geodesics in the Kerr spacetime

In 1968, Cater demonstrated that
the Hamilton-Jacobi equation for geodesics

$$\partial_\lambda S + g^{ab}\,\partial_a S\,\partial_b S = 0$$

for the Kerr's metric can be separated for a solution

$$S = -\kappa_0 \lambda - Et + L\phi + R(r) + \Theta(\theta)$$

and then the functions R(r) and Θ(θ) follow

$$(R')^2 - \frac{W_r^2}{\Delta^2} - \frac{V_r}{\Delta} = 0\ , \quad (\Theta')^2 + \frac{W_\theta^2}{\sin^2\theta} - V_\theta = 0$$

where

$$W_r = -E(r^2 + a^2) + aL\ , \ W_\theta = -aE\sin^2\theta + L$$

$$V_r = \kappa + \kappa_0 r^2\ , \qquad V_\theta = -\kappa + \kappa_0 a^2 \cos^2\theta$$

Scalar fields in the Kerr spacetime

He also demonstrated that the massive Klein-Gordon equation

$$(\nabla^2 - m^2)\Phi = 0$$

for the Kerr's metric can be separated for a solution

$$\Phi = e^{-i\omega t + in\phi} R(r)\Theta(\theta)$$

and then the functions R(r) and Θ(θ) follow

$$\frac{1}{R}\frac{d}{dr}\left(\triangle \frac{dR}{dr}\right) + \frac{U_r^2}{\triangle} - m^2 r^2 - \kappa = 0 \ ,$$

$$\frac{1}{\Theta}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) - \frac{U_\theta^2}{\sin^2\theta} - m^2 a^2 \cos^2\theta - \kappa = 0$$

where

$$U_r = an - \omega(r^2 + a^2) \ , \quad U_\theta = n - a\omega \sin^2\theta$$

Separation of variables in various equations for the Kerr's metric

- Hamilton-Jacobi equation for geodesics

$$\partial_\lambda S + g^{ab}\,\partial_a S\,\partial_b S = 0$$

- Klein-Gordon equation

$$(\nabla^2 - m^2)\Phi = 0$$

Carter (1968)

- Maxwell equation

$$\nabla_\mu F^{\mu\nu} = 0$$

- Linearized Einstein's equation

$$\delta G_{\mu\nu} = 0$$

Teukolsky (1972)

- Neutrino equation

$$\gamma^\mu(\partial_\mu + \Gamma_\mu)\psi = 0$$

Teukolsky (1973), Unruh (1973)

- Dirac equation

$$(\gamma^\mu\nabla_\mu + m)\Psi = 0$$

Chandrasekhar (1976), Page (1976)

Hidden symmetries

In order to give an account of such integrabilities and separabilities, a generalization of Killing symmetry has been studied since 1970s.

vector	**Killing vector**	**conformal Killing vector**
symmetric	**Killing-Stackel (KS)** Stackel (1895)	**conformal Killing-Stackel (CKS)**
anti-symmetric	**Killing-Yano (KY)** Yano (1952)	**conformal Killing-Yano (CKY)** Tachibana (1969), Kashiwada (1968)

Plan of this talk

0. Introduction

1. Review

– Hidden symmetry of Kerr black holes –

2. On spacetimes admitting CKY symmetry

3. A generalization of CKY symmetry

4. Summary & Outlook

1. Introduction
– Hidden symmetry of Kerr black holes –

Complete integrable system – Liouville integrability –

Geodesic equation

$$\frac{dx^\mu}{d\tau} = \frac{\partial H}{\partial p_\mu}, \quad \frac{dp_\mu}{d\tau} = -\frac{\partial H}{\partial x^\mu}, \qquad \text{for} \quad H = \frac{1}{2} g^{ab} p_a p_b \ .$$

F(*x*, *p*) *:* a constant of motion ⇔

$$0 = \frac{dF}{d\tau} = \frac{\partial F}{\partial x^\mu}\frac{\partial H}{\partial p_\mu} - \frac{\partial F}{\partial p_\mu}\frac{\partial H}{\partial x^\mu} =: \{F, H\}_P$$

Poisson's bracket

Liouville integrability means that there exists a maximal set of Poisson commuting invariants.

$$\{\alpha_i, \alpha_j\}_P = 0 \ , \quad i, j = 1, \ldots, D$$

Constants of motion and Killing tensors

$$H = \frac{1}{2} g^{ab} p_a p_b$$

Assume $C_K = K^{a_1 \ldots a_n} p_{a_1} \ldots p_{a_n}$

$$\{H, C_K\}_P = 0$$

$$\Leftrightarrow \quad \underline{\nabla^{(a_1} K^{a_2 \cdots a_{n+1})}} p_{a_1} p_{a_2} \cdots p_{a_{n+1}} = 0$$

= 0 ; Killing equation

Def. *Killing-Stackel tensor (KS) is a rank-n symmetric tensor **K** obeying the Killing equation*

Stackel (1895)

$$\nabla_{(a} K_{b_1 \cdots b_n)} = 0$$

Hamilton-Jacobi approach

For a D-dimensional manifold (M^D, g), a local coordinate system x^a is called a separable coordinate system if a Hamilton-Jacobi equation in these coordinates

$$H(x^a, p_a) = \kappa_0, \quad p_a = \frac{\partial S}{\partial x^a}$$

where κ_0 is a constant, is completely integrable by (additive) separation of variables, i.e.,

$$S = S_1(x^1, c) + S_2(x^2, c) + \cdots + S_D(x^D, c)$$

where $S_a(x^a, c)$ depends only on the corresponding coordinate x^a and includes D constants $c=(c_1, \ldots , c_D)$.

δr-Separability structure

Theor. *A D-dimensional manifold (M^D, g) admits separability of H-J equation for geodesics if and only if*

1. *There exist r indep. commuting Killing vectors* $\mathbf{X}_{(i)}$ *:*

$$[X_{(i)}, X_{(j)}] = 0,$$

2. *There exist D-r indep. rank-2 Killing tensors* $\mathbf{K}_{(\mu)}$*, which satisfy*

$$[K_{(\mu)}, K_{(\nu)}] = 0, \quad [X_{(i)}, K_{(\mu)}] = 0,$$

3. *The Killing tensors* $\mathbf{K}_{(\mu)}$ *have in common D-r eigenvectors* $\mathbf{X}_{(\mu)}$ *s.t.*

$$[X_{(\mu)}, X_{(\nu)}] = 0, \quad [X_{(i)}, X_{(\mu)}], \quad g(X_{(i)}, X_{(\mu)}) = 0.$$

Benenti-Francaviglia (1979)

Comments:

- Some examples which are not separable but integrable are known.

cf.) Gibbons-TH-Kubiznak-Warnick (2011)

Hidden symmetry of Kerr spacetime I

Kerr spacetime admits a rank-2 irreducible Killing tensor . Walker-Penrose (1970)

$$K_{ab} = K_{(ab)} \ , \quad \nabla_{(c}K_{ab)} = 0$$

Comments:

- Kerr spacetime has 4 independent and mutually commuting constants of geodesic motion, which are corresponding to 2 Killing vectors and 2 rank-2 Killing tensors.

$$(\partial_t)^a : \quad E = (\partial_t)^a p_a \qquad g_{ab} : \quad \kappa_0 = g^{ab} p_a p_b$$

$$(\partial_\phi)^a : \quad L = (\partial_\phi)^a p_a \qquad K_{ab} : \quad \kappa = K^{ab} p_a p_b$$

- One also finds that this Killing tensor admits the δ2-separability structure of the H-J equation for geodesics.

Hidden symmetry of Kerr spacetime II

The Killing tensor ***K*** can be written as the square of a rank-2 Killing-Yano tensor ***f***. Penrose-Floyd (1973)

$$^{\exists} f \quad \text{s.t.} \ K_{ab} = f_a{}^c f_{bc}, \quad f_{ba} = -f_{ab}, \quad \nabla_{(a} f_{b)c} = 0$$

↑ **rank-2 KY equation**

<u>Comments:</u>

- Killing-Yano tensor (KY) is a rank-*p* anti-symmetric tensor ***f*** obeying $\nabla_{(a} f_{b_1)b_2 \cdots b_p} = 0$. Yano (1952)

- Having a Killing-Yano tensor, one can always construct the corresponding Killing tensor. On the other hand, not every Killing tensor can be decomposed in terms of a Killing-Yano tensor. Collinson (1976), Stephani (1978)

Hidden symmetry of Kerr spacetime III

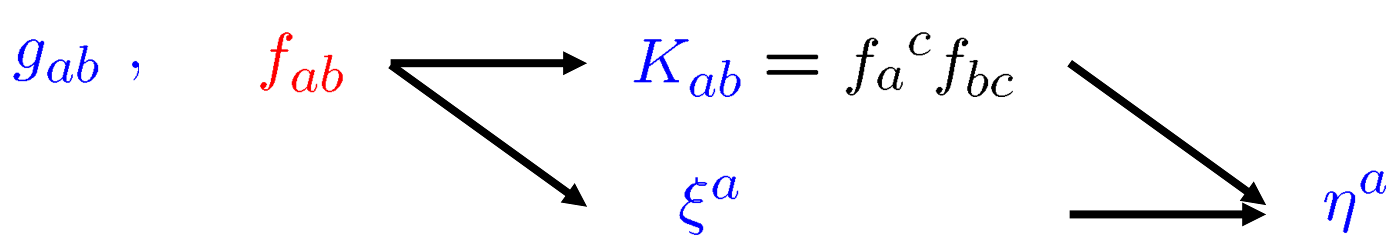
Moreover, the Killing-Yano tensor ***f*** generates two Killing vectors.

Hughston-Sommers (1973)

$$\xi^a \equiv (\partial_t)^a = \frac{1}{3}\nabla_b(*f)^{ba}$$

$$\eta^a \equiv -a^2(\partial_t)^a - a(\partial_\phi)^a = K^a{}_b\xi^b$$

In the end, all the symmetries necessary for complete integrability and separability of the H-J equation for geodesics can be generated by a single rank-2 Killing-Yano tensor.



Hidden symmetry of Kerr spacetime IV

The Killing-Yano tensor is derived from a 1-form potential ***b***,

$$\boldsymbol{f} = *d\boldsymbol{b}$$

Carter (1987)

Comments:

- Obviously, $\boldsymbol{h} = *\boldsymbol{f}$ is closed 2-form.

- One finds that ***h*** is a conformal Killing-Yano tensor (CKY) of rank-2, i.e., it follows

$$\nabla_a h_{bc} + \nabla_b h_{ac} = 2g_{ab}\xi_c - g_{ac}\xi_b - g_{bc}\xi_a$$

where

$$\xi^a = \frac{1}{3}\nabla_b h^{ba}$$

Tachibana (1969)

Hidden symmetry of Kerr spacetime V

Kerr's metric

$$ds^2 = -e^0e^0 + e^1e^1 + e^2e^2 + e^3e^3$$

where $e^0 = \sqrt{\frac{\Delta}{\Sigma}}\Big(dt - a\sin^2\theta d\phi\Big),\ e^2 = \sqrt{\frac{\sin^2\theta}{\Sigma}}\Big(a\,dt - (r^2 + a^2)d\phi\Big)$

$e^1 = \sqrt{\frac{\Sigma}{\Delta}}dr, \qquad e^3 = \sqrt{\Sigma}d\theta$

- KY 2-form

$$f = a\cos\theta\, e^0 \wedge e^1 + r\, e^2 \wedge e^3$$

- CCKY 2-form

$$h = r\, e^0 \wedge e^1 + a\cos\theta\, e^2 \wedge e^3$$

- rank-2 Killing tensor

$$K = a^2\cos^2\theta(e^0e^0 - e^1e^1) + r^2(e^2e^2 + e^3e^3)$$

Symmetry operators

$\mathcal{O}$: a sym. op. for ***D***
⇔ $[\mathcal{O}, D] = 0$ for a diff. op. ***D***

Klein-Gordon equation

For the scalar Laplacian □,

$$\hat{\eta}^{(j)} = \eta^{(j)a}\nabla_a, \;\; \hat{K}^{(j)} = \nabla_a K^{(j)ab}\nabla_b,$$

are symmetry operators, i.e.,

$$[\hat{\eta}^{(j)}, \Box] = [\hat{K}^{(j)}, \Box] = 0$$

Carter (1977)

Dirac equation

For the Dirac operator ***D***, the operator

$$\hat{f} \equiv i\gamma_5\gamma^a\left(f_a{}^b\nabla_b - \frac{1}{6}\gamma^b\gamma^c\nabla_c f_{ab}\right)$$

is symmetry operator whenever ***f*** is a Killing-Yano tensor.

Carter-McLenaghan (1979)

Separability structures for Kerr black hole

Algebraic type of curvature is type-D.

Geodesic motion is completely integrable.

Carter (1968)

Hamilton-Jacobi equation is separable.

Klein-Gordon equation is separable.

Carter (1968)

K-G symmetry operators exist.

Carter (1977)

Dirac equation is separable.

Chandrasekhar (1976)

Dirac symmetry operators exist.

Carter-McLenaghan (1979)

A closed CKY 2-form exists.

Carter (1987)

Carter's metric

$$\xi^a = (\partial_\tau)^a$$
$$\eta^a = (\partial_\sigma)^a$$

Kerr's metric

$$ds^2 = -\frac{\Delta}{\Sigma}\left(dt - a\sin^2\theta d\phi\right)^2 + \frac{\sin^2\theta}{\Sigma}\left(a\,dt - (r^2+a^2)d\phi\right)^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2$$

where $\Delta = r^2 - 2Mr + a^2$, $\Sigma = r^2 + a^2\cos^2\theta$

coord. trasf. $p = a\cos\theta, \quad \tau = t - a\phi, \quad \sigma = -\frac{\phi}{a}$

$$ds^2 = -\frac{Q}{r^2+p^2}(d\tau - p^2 d\sigma)^2 + \frac{P}{r^2+p^2}(d\tau + r^2 d\sigma)^2 + \frac{r^2+p^2}{Q}dr^2 + \frac{r^2+p^2}{P}dp^2$$

(Boyer's coordinates)

where $Q = r^2 - 2Mr + a^2, \quad P = -p^2 + a^2$

Carter (1968)

The "off-shell" metric with *Q* and *P* replaced by arbirary functions *Q*(*r*) and *P*(*p*) is said to be of Carter's class.

Spacetimes admitting a Killing-Yano tensor

<u>**Theor.**</u> *Let (M^4, g) be a vacuum type-D space-time. The following conditions are equivalent:*

1. *(M^4, g) is without acceleration.*
2. *(M^4, g) is one of Carter's class.*
3. *(M^4, g) admits a δ2-separability structure.*
4. *(M^4, g) admits a Killing-Yano tensor.*

Demianski-Francaviglia (1980)

<u>**Theor.**</u> *A spacetime (M^4, g) admits a rank-2 Killing-Yano tensor if and only if the metric is of Carter's class, i.e.,*

$$ds^2 = -\frac{Q(r)}{r^2+p^2}(d\tau - p^2 d\sigma)^2 + \frac{P(p)}{r^2+p^2}(d\tau + r^2 d\sigma)^2 + \frac{r^2+p^2}{Q(r)}dr^2 + \frac{r^2+p^2}{P(p)}dp^2$$

Dietz-Rudiger (1982), Taxiarchis (1985)

Carter's metric in Einstein-Maxwell theory

The Carter's metric

$$ds^2 = -\frac{Q(r)}{r^2+p^2}(d\tau - p^2 d\sigma)^2 + \frac{P(p)}{r^2+p^2}(d\tau + r^2 d\sigma)^2 + \frac{r^2+p^2}{Q(r)}dr^2 + \frac{r^2+p^2}{P(p)}dp^2$$

obeys the Einstein-Maxwell equations when provided that the functions take the form

$$Q = -\frac{\lambda}{3}r^4 + \epsilon r^2 - 2mr + k + e^2 + g^2$$

$$P = -\frac{\lambda}{3}p^4 - \epsilon p^2 + 2np + k$$

and the vector potential reads

$$A = -\frac{1}{r^2+p^2}[er(d\tau - p^2\, d\sigma) + gp(d\tau + r^2\, d\sigma)]$$

This metric has six independent parameters.

Plebanski-Demianski metric

The important family of type D in four dimensions can be represented by the seven-parameter metric.

Plebanski-Demianski (1976)

$$ds^2 = \frac{1}{(1-pr)^2}\Big\{ -\frac{Q}{r^2+p^2}(d\tau - p^2 d\sigma)^2 + \frac{P}{r^2+p^2}(d\tau + r^2 d\sigma)^2 + \frac{r^2+p^2}{Q}dr^2 + \frac{r^2+p^2}{P}dp^2 \Big\}$$

This metric obeys the Einstein-Maxwell equations provided that the functions take the form

$$Q = -(k+\lambda/3)r^4 - 2nr^3 + \epsilon r^2 - 2mr + k + e^2 + g^2$$
$$P = -(k+e^2+g^2+\lambda/3)p^4 + 2mp^3 - \epsilon p^2 + 2np + k$$

and the vector potential reads

$$A = -\frac{1}{r^2+p^2}[er(d\tau - p^2\, d\sigma) + gp(d\tau + r^2\, d\sigma)]$$

Relationship b/w Carter's metric and P-D metric

Plebanski-Demianski metric

$$ds^2 = \frac{1}{(1-pr)^2}\left\{ -\frac{Q}{r^2+p^2}(d\tau - p^2 d\sigma)^2 + \frac{P}{r^2+p^2}(d\tau + r^2 d\sigma)^2 + \frac{r^2+p^2}{Q}dr^2 + \frac{r^2+p^2}{P}dp^2 \right\}$$

where $Q = -(k+\lambda/3)r^4 - 2nr^3 + \epsilon r^2 - 2mr + k + e^2 + g^2$

$P = -(k+e^2+g^2+\lambda/3)p^4 + 2mp^3 - \epsilon p^2 + 2np + k$

rescale $p \to \sqrt{\alpha\omega}p,\ r \to \sqrt{\frac{\alpha}{\omega}}r,\ \tau \to \sqrt{\frac{\omega}{\alpha}}\tau,\ \sigma \to \sqrt{\frac{\omega}{\alpha^3}}\sigma$

relabel $m \to \left(\frac{\alpha}{\omega}\right)^{3/2}m,\ n \to \left(\frac{\alpha}{\omega}\right)^{3/2}n,\ e \to \frac{\alpha}{\omega}e,\ g \to \frac{\alpha}{\omega}g,\ \epsilon \to \frac{\alpha}{\omega}\epsilon,\ k \to \alpha^2 k$

$$ds^2 = \frac{1}{(1-\alpha pr)^2}\left\{ -\frac{Q}{r^2+\omega^2p^2}(d\tau - \omega^2p^2 d\sigma)^2 + \frac{P}{r^2+\omega^2p^2}(\omega d\tau + r^2 d\sigma)^2 + \frac{r^2+\omega^2p^2}{Q}dr^2 + \frac{r^2+\omega^2p^2}{P}dp^2 \right\}$$

where $Q = -(\alpha^2 k+\lambda/3)r^4 - \frac{2\alpha n}{\omega}r^3 + \epsilon r^2 - 2mr + \omega^2 k + e^2 + g^2$

$P = -[\alpha^2(\omega^2 k+e^2+g^2)+\omega^2\lambda/3]p^4 + 2\alpha mp^3 - \epsilon p^2 + \frac{2n}{\omega}p + k$

Set α=0 and ω=1. Then we recover the Carter's family.

TABLE I

Known exact solutions of the Einstein and Einstein–Maxwell Equations of type D

		$m+in, a+ib, e+ig, \lambda$			
$m+in, a, e+ig, \lambda$ Plebanski [3] 1975		$m+in, a+ib, e+ig$ Kinnersley [2] 1975			$m+in, b, e+ig, \lambda$
$m+in, a, e\lambda$ Carter [11] 1968 $m+in, a, \lambda$ Frolov [22] 1973		$m+in, a, e+ig$ Demianski, Newman [12] 1966	$m+in, a+ib, \lambda$ Carter [11] 1968 $m+in, a+ib$ Kinnersley [13] 1969	$m+in, b, e, \lambda$ Carter [11] 1968 $m+in, b, e$ Levi-Civita [4] 1918 Newman, Tamburino [5] 1961 Robinson, Trautman [6] 1962 Ehlers, Kundt [5] 1962	
m, a, λ Demianski [14] 1973	m, a, e Newman *et. al.* [10] 1965 Perjes [25] 1969 Ernst [26] 1968	$m+in, a$ Demianski [15] 1966 Kramer, Neugebauer [24] 1968 Robinson, J. Robinson Zund [27] 1969	$m+in, e$ Brill [23] 1964		$m+in, \lambda$ 1972 Demianski [16] Frolov [22] 1973
m, λ Kottler [17] 1918	m, e Reisner, Nordstrom [18] 1916	m, a Kerr [9] 1963	$m+in$ Newman, Tamburino, Unti [19] 1963 Taub [21] 1951	e, b Bertotti [28] 1959 Robinson [29] 1959	
λ de Sitter [30] 1917			m Schwarzschild [20] 1916		

TABLE I in Plebanski, Demianski, Annal. Phys. **98** (1976) 98-127

2. On Spacetimes admitting conformal Killing-Yano (CKY) symmetry

Exact solutions – vacuum black holes with S^n horizon topology

vacuum Einstein's Eq.

$$Ric(g) = \lambda g$$

Four dimensions	mass,	NUT,	rotation,	λ
Schwarzschild (1916)	○			
Kerr (1963)	○		○	
Carter (1968)	○	○	○	○

Higher dimensions	mass,	NUT,	rotation,	λ
Tangherlini (1916)	○			
Myers-Perry (1986)	○		○	
Gibbons-Lu-Page-Pope (2004)	○		○	○
5-dim. Hawking, et al. (1998)				
Chen-Lu-Pope (2006)	○	○	○	○

↑ **The most general known solution**
= higher-dimensional Kerr-NUT-(A)dS

D-dimensional Kerr-NUT-(A)dS metric

$D = 2n + \varepsilon \quad (\varepsilon = 0 \text{ or } 1)$

$$ds^2 = \sum_{\mu=1}^{n} \frac{dx_\mu^2}{Q_\mu} + \sum_{\mu=1}^{n} Q_\mu \left[\sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k\right]^2 + \varepsilon \frac{c}{A^{(n)}} \left[\sum_{k=0}^{n} A^{(k)} d\psi_k\right]^2$$

where

Chen-Lu-Pope (2006)

$$Q_\mu = \frac{X_\mu}{U_\mu}\ , \quad U_\mu = \prod_{\substack{\nu=1 \\ \nu\neq\mu}}^{n} (x_\mu^2 - x_\nu^2)\ , \quad X_\mu = X_\mu(x_\mu)\ ,$$

$$A_\mu^{(k)} = \sum_{\substack{1\le\nu_1<\nu_2<\cdots<\nu_k\le n \\ \nu_i\neq\mu}} x_{\nu_1}^2 x_{\nu_2}^2 \cdots x_{\nu_k}^2\ , \quad A^{(k)} = \sum_{1\le\nu_1<\nu_2<\cdots<\nu_k\le n} x_{\nu_1}^2 x_{\nu_2}^2 \cdots x_{\nu_k}^2\ ,$$

$$A_\mu^{(0)} = A^{(0)} = 1\ , \quad c = \text{const.}\ .$$

$D=2n$ $\quad X_\mu = \sum_{k=0}^{n} c_{2k} x_\mu^{2k} + b_\mu x_\mu$ $\qquad$ $D=2n+1$ $\quad X_\mu = \sum_{k=1}^{n} c_{2k} x_\mu^{2k} + b_\mu + \frac{(-1)^n c}{x_\mu^2}$

This metric satisfies Einstein Eq.

$$R_{ab} = -(D-1) c_n\, g_{ab}$$

Four-dimensional Kerr-NUT-(A)dS metric

$$
\begin{aligned}
ds^2 &= \frac{x^2-y^2}{X}dx^2 + \frac{y^2-x^2}{Y}dy^2 \\
&+ \frac{X}{x^2-y^2}(d\psi_0 + y^2 d\psi_1)^2 + \frac{Y}{y^2-x^2}(d\psi_0 + x^2 d\psi_1)^2
\end{aligned}
$$

where

$$
X = cx^4 + x^2 - a^2 - 2Mx\ , \qquad Y = cy^4 + y^2 - a^2 - 2Ly
$$

Five-dimensional Kerr-NUT-(A)dS metric

$$
\begin{aligned}
ds^2 \;=\;& \frac{x^2-y^2}{X}dx^2+\frac{y^2-x^2}{Y}dy^2 \\
&+\frac{X}{x^2-y^2}(d\psi_0+y^2d\psi_1)^2+\frac{Y}{y^2-x^2}(d\psi_0+x^2d\psi_1)^2 \\
&+\frac{c}{x^2y^2}(d\psi_0+(x^2+y^2)d\psi_1+x^2y^2d\psi_2)^2
\end{aligned}
$$

where

$$
\begin{aligned}
X &= c_4x^4+c_2x^2+c_0+b_1+\frac{c}{x^2}\ , \\
Y &= c_4y^4+c_2y^2+c_0+b_2+\frac{c}{y^2}
\end{aligned}
$$

Six-dimensional Kerr-NUT-(A)dS metric

$$
\begin{aligned}
ds^2 \; = \; & \frac{(x^2-y^2)(x^2-z^2)}{X}dx^2 + \frac{(y^2-x^2)(y^2-z^2)}{Y}dy^2 + \frac{(z^2-x^2)(z^2-y^2)}{Z}dz^2 \\
& + \frac{X}{(x^2-y^2)(x^2-z^2)}(d\psi_0 + (y^2+z^2)d\psi_1 + y^2z^2d\psi_2)^2 \\
& + \frac{Y}{(y^2-x^2)(y^2-z^2)}(d\psi_0 + (z^2+x^2)d\psi_1 + z^2x^2d\psi_2)^2 \\
& + \frac{Z}{(z^2-x^2)(z^2-y^2)}(d\psi_0 + (x^2+y^2)d\psi_1 + x^2y^2d\psi_2)^2
\end{aligned}
$$

where

$$
\begin{aligned}
X &= c_6x^6 + c_4x^4 + c_2x^2 + c_0 + b_1x \;, \\
Y &= c_6y^6 + c_4y^4 + c_2y^2 + c_0 + b_2y \;, \\
Z &= c_6z^6 +_c 4z^4 + c_2z^2 + c_0 + b_3z
\end{aligned}
$$

Seven-dimensional Kerr-NUT-(A)dS metric

$$
\begin{aligned}
ds^2 \;=\; & \frac{(x^2-y^2)(x^2-z^2)}{X}dx^2 + \frac{(y^2-x^2)(y^2-z^2)}{Y}dy^2 + \frac{(z^2-x^2)(z^2-y^2)}{Z}dz^2 \\
& + \frac{X}{(x^2-y^2)(x^2-z^2)}(d\psi_0 + (y^2+z^2)d\psi_1 + y^2z^2 d\psi_2)^2 \\
& + \frac{Y}{(y^2-x^2)(y^2-z^2)}(d\psi_0 + (z^2+x^2)d\psi_1 + z^2x^2 d\psi_2)^2 \\
& + \frac{Z}{(z^2-x^2)(z^2-y^2)}(d\psi_0 + (x^2+y^2)d\psi_1 + x^2y^2 d\psi_2)^2 \\
& + \frac{c}{x^2y^2z^2}(d\psi_0 + (x^2+y^2+z^2)d\psi_1 + (x^2y^2+y^2z^2+x^2z^2)d\psi_2 + x^2y^2z^2 d\psi_3)^2
\end{aligned}
$$

where

$$
\begin{aligned}
X &= c_6x^6 + c_4x^4 + c_2x^2 + c_0 + b_1 - \frac{c}{x^2}\,, \\
Y &= c_6y^6 + c_4y^4 + c_2y^2 + c_0 + b_2 - \frac{c}{y^2}\,, \\
Z &= c_6z^6 +_c 4z^4 + c_2z^2 + c_0 + b_3 - \frac{c}{z^2}
\end{aligned}
$$

D-dimensional Kerr-NUT-(A)dS metric

$D = 2n + \varepsilon \quad (\varepsilon = 0 \text{ or } 1)$

$$ds^2 = \sum_{\mu=1}^{n} \frac{dx_\mu^2}{Q_\mu} + \sum_{\mu=1}^{n} Q_\mu \left[\sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k \right]^2 + \varepsilon \frac{c}{A^{(n)}} \left[\sum_{k=0}^{n} A^{(k)} d\psi_k \right]^2$$

where

Chen-Lu-Pope (2006)

$$Q_\mu = \frac{X_\mu}{U_\mu} , \quad U_\mu = \prod_{\substack{\nu=1 \\ \nu\neq\mu}}^{n} (x_\mu^2 - x_\nu^2) , \quad X_\mu = X_\mu(x_\mu) ,$$

$$A_\mu^{(k)} = \sum_{\substack{1\leq\nu_1<\nu_2<\cdots<\nu_k\leq n \\ \nu_i\neq\mu}} x_{\nu_1}^2 x_{\nu_2}^2 \cdots x_{\nu_k}^2 , \quad A^{(k)} = \sum_{1\leq\nu_1<\nu_2<\cdots<\nu_k\leq n} x_{\nu_1}^2 x_{\nu_2}^2 \cdots x_{\nu_k}^2 ,$$

$$A_\mu^{(0)} = A^{(0)} = 1 , \quad c = \text{const.} .$$

D=2*n* $\quad X_\mu = \sum_{k=0}^{n} c_{2k} x_\mu^{2k} + b_\mu x_\mu$ $\qquad$ *D*=2*n*+1 $\quad X_\mu = \sum_{k=1}^{n} c_{2k} x_\mu^{2k} + b_\mu + \frac{(-1)^n c}{x_\mu^2}$

This metric satisfies Einstein Eq.

$$R_{ab} = -(D-1)c_n \, g_{ab}$$

How about higher dimensions?

– Higher dim. Kerr-NUT-(A)dS –

A closed CKY 2-form exists. → Kubiznak-Frolov (2007)

Geodesic motion is completely integrable.
Page-Kubiznak-Vasudevan-Krtous (2007)

Algebraic type of curvature is type-D.
Hamamoto-TH-Oota-Yasui (2007)

Hamilton-Jacobi equation is separable.
Klein-Gordon equation is separable.
Frolov-Krtous-Kubiznak (2007)

K-G symmetry operators exist.
Sergyeyev, Krtous (2008)

Dirac equation is separable.
Oota-Yasui (2008)

Dirac symmetry operators exist.
Benn-Charlton (1996), Wu (2009)

Hidden symmetries

There exist two "natural" (symmetric and anti-symmetric) generalizations of (conformal) Killing vector.

vector	Killing vector	conformal Killing vector
symmetric	Killing-Stackel (KS) Stackel (1895)	conformal Killing-Stackel (CKS)
anti-symmetric	Killing-Yano (KY) Yano (1952)	conformal Killing-Yano (CKY) Tachibana (1969), Kashiwada (1968)

Generalizations of Killing vector

<u>Def.</u> *Killing-Stackel tensor (KS) is a rank-p symmetric tensor* ***K*** *obeying*

$$\nabla_{(a}K_{b_1\cdots b_p)} = 0$$

Stackel (1895)

<u>Def.</u> *Killing-Yano tensor (KY) is a rank-p anti-symmetric tensor* ***f*** *obeying*

$$\nabla_{(a}f_{b_1)b_2\cdots b_p} = 0$$

Yano (1952)

Properties of KY tensors and KS tensors

<u>Prop.</u> *When* ***f*** *is a rank-n Killing-Yano (KY) tensor, then rank-2 symmetric tensor* **K** *defined by*

$$K_{ab} = f_{a\cdots} f_{b}{}^{\cdots}$$

is a Killing-Stackel (KS) tensor.

<u>Prop.</u> *Let* **K** *be a rank-n Killing-Stackel tensor field and γ be a geodesic with tangent* ***p****. Then*

$$K^{abc\cdots} p_a p_b p_c \cdots$$

is constant along γ.

Conformal Killing-Yano tensor

Def. *Conformal Killing-Yano tensor (CKY) is a rank-p anti-symmetric tensor **k** obeying*

$$\nabla_{(a}k_{b)c_1\ldots c_{p-1}} = g_{ab}\xi_{c_1\ldots c_{p-1}} + \sum_{i=1}^{p-1}(-1)^i g_{c_i(a}\xi_{b)c_1\ldots\hat{c}_i\ldots c_{p-1}}$$

where $\xi_{c_1\ldots c_{p-1}} = \frac{1}{D-p+1}\nabla^a k_{ac_1\ldots c_{p-1}}$

Tachibana (1969), Kashiwada (1968)

Prop. *Let **k** be a CKY p-form for a metric **g**. Then, ~**k** = Ω^{p+1} **k** is a CKY p-form for the metric ~**g** = Ω^2 **g**.*

Subclasses of CKY tensors

Def. *Equivalently,* CKY *is a p-form* ***k*** *obeying*

$$\nabla_X \boldsymbol{k} = \frac{1}{p-1} \boldsymbol{X} \lrcorner\, d\boldsymbol{k} - \frac{1}{D-p+1} \boldsymbol{X}^* \wedge \boldsymbol{\delta k}$$

for an arbitrary vector ***X.***

covariantly constant form

$d\boldsymbol{h} = 0$; ***h*** *is a* *closed CKY*

$\boldsymbol{\delta f} = 0$; ***f*** *is a* KY

$\nabla_X (d\boldsymbol{\psi}) = c\boldsymbol{X}^* \wedge \boldsymbol{\psi}$

; ***ψ*** *is a* *special KY*

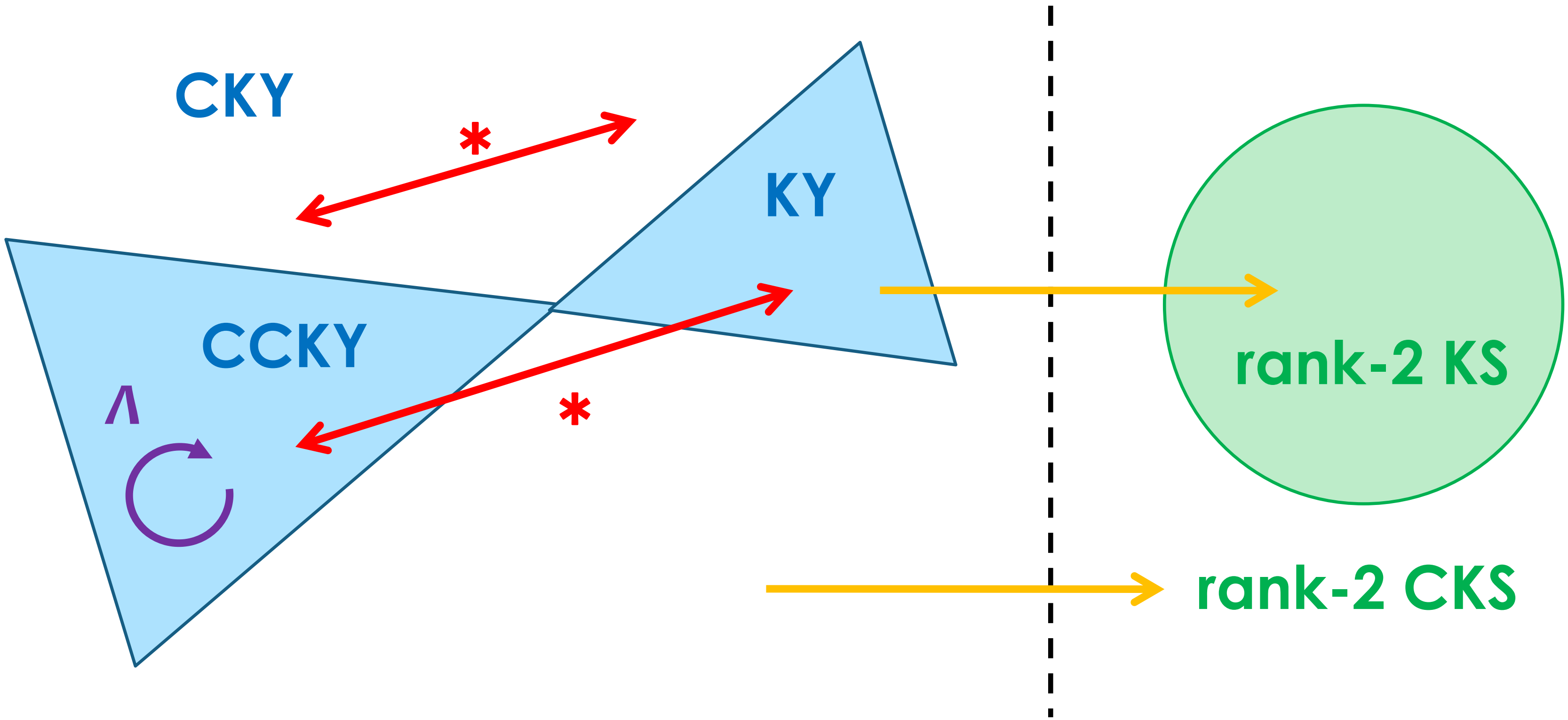
Tachibana-Yu (1970)

Basic properties of CKY tensors

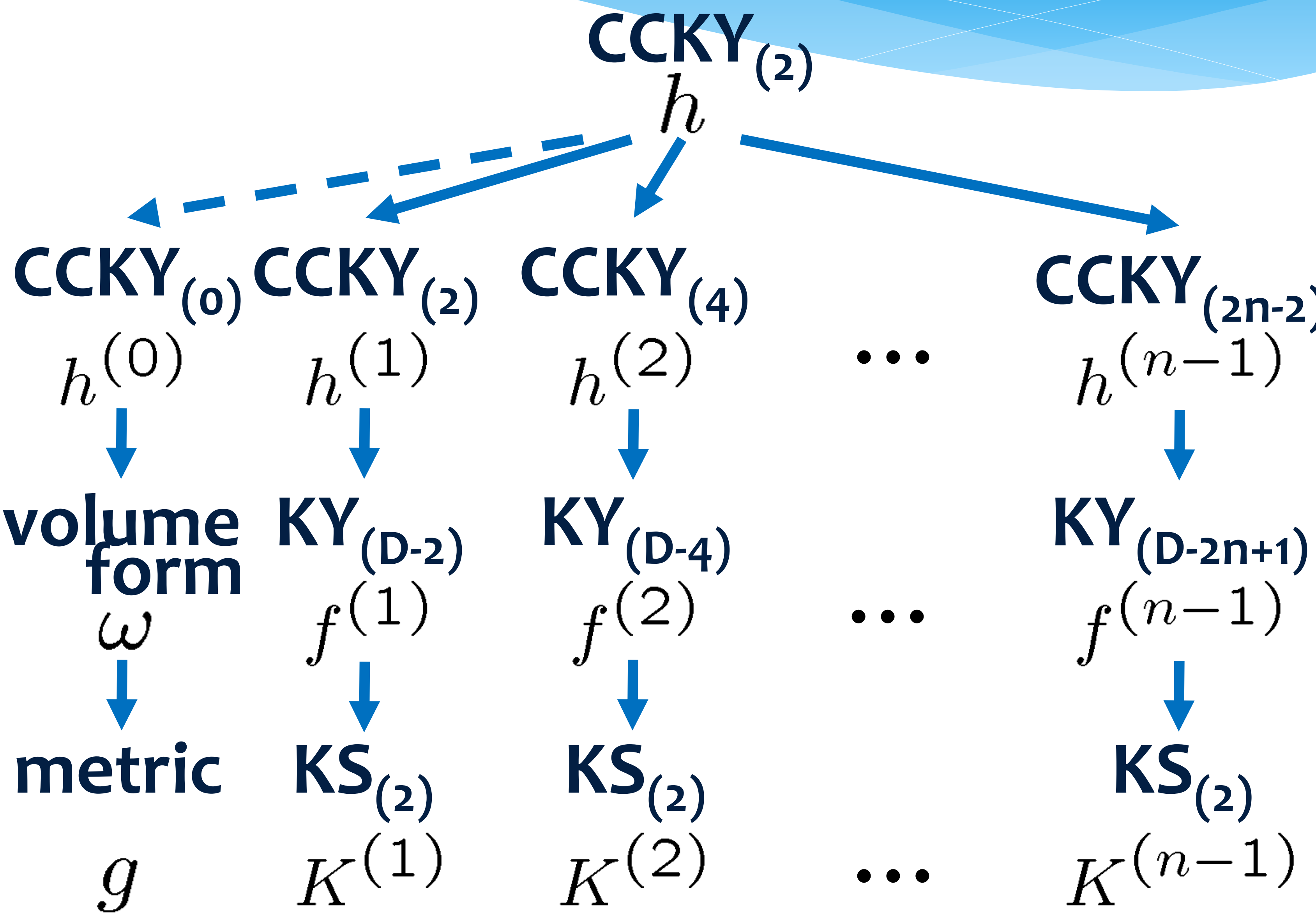
Prop. *The Hodge star $*$ maps CKY p-forms into CKY (D-p)-forms. In particular, the Hodge star of a closed CKY p-form is a KY (D-p)-form and vice versa.*

Prop. *When **h1** and **h2** is a closed CKY p-form and q-form, respectively, then **h3 = h1 ∧ h2** is a closed CKY (p+q)-form.*

Basic properties of hidden symmetries



Tower of hidden symmetries



Geodesic integrability in higher dimensions

closed CKY 2-form h

- ▸ **closed CKY 2*j*-form** $h^{(j)} = h \wedge \ldots \wedge h$ ▸ **KY (*D*-2*j*)-form** $f^{(j)} = *h^{(j)}$ ▸ **rank-2 KS tensor** $K^{(j)}_{ab} = f^{(j)}_{a\cdots} f^{(j)\cdots}_{b}$ ▸ **const. of motion** $C_j = K^{(j)}_{ab} p^a p^b$
- ▾ (from rank-2 KS tensor)
- ▸ **Killing vector** $\xi_a = \nabla^b h_{ba}$ ▸ **Killing vector** $\eta^{(j)}_a = K^{(j)}_{ab}\xi^b$ ▸ **const. of motion** $F_j = \eta^{(j)}_a p^a$

***nontrivial**

dimension	# Killing vector	# KS tensor
even (D=2n)	n	n
odd (D=2n+1)	n+1	n

$\{C_i, C_j\}_P = 0$ $\{F_i, F_j\}_P = 0$ $\{C_i, F_j\}_P = 0$

Krtous-Kubiznak-Page-Frolov (2006) TH-Oota-Yasui (2007)

One further finds that such a spacetime admits $\delta(n+\varepsilon)$-separability structure, that is, separability of H-J equation for geodesics.

TH-Oota-Yasui (2007)

Manifolds admitting a closed CKY 2-form

Theor. *Suppose a Riemannian manifold (M^D, **g**) admits a* non-degenerate *closed CKY 2-form **h**. Then the metric takes the form*

$$g = \sum_{\mu=1}^{n} \frac{dx_\mu^2}{Q_\mu} + \sum_{\mu=1}^{n} Q_\mu \left[\sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k \right]^2 + \varepsilon S \left[\sum_{k=0}^{n} A^{(k)} d\psi_k \right]^2 ,$$

where

$$Q_\mu = \frac{X_\mu}{U_\mu} , \quad U_\mu = \prod_{\substack{\nu=1 \\ \nu \neq \mu}}^{n} (x_\mu^2 - x_\nu^2) , \quad \boxed{X_\mu = X_\mu(x_\mu)} , \quad S = \frac{c}{A^{(n)}} , \quad A_\mu^{(0)} = A^{(0)} = 1 ,$$

$$A_\mu^{(k)} = \sum_{\substack{1 \leq \nu_1 < \nu_2 < \cdots < \nu_k \leq n \\ \nu_i \neq \mu}} x_{\nu_1}^2 x_{\nu_2}^2 \cdots x_{\nu_k}^2 , \quad A^{(k)} = \sum_{1 \leq \nu_1 < \nu_2 < \cdots < \nu_k \leq n} x_{\nu_1}^2 x_{\nu_2}^2 \cdots x_{\nu_k}^2 .$$

TH-Oota-Yasui (2007), Krtous-Frolov-Kubiznak (2008)

Einstein metrics with a non-degenerate CKY 2-form

when $X_\mu = \sum_{k=0}^{n} c_k x_\mu^{2k} + b_\mu x_\mu$ in 2n dimension

$X_\mu = \sum_{k=1}^{n} c_k x_\mu^{2k} + b_\mu + \frac{(-1)^n c}{x_\mu^2}$ in 2n+1 dimension ,

This metric satisfies Einstein Eq.

$$R_{ab} = -(D-1)c_n\, g_{ab}$$

Then, the metric coincides with that of Kerr-NUT-(A)dS metric. In this mean, only vacuum spacetime admitting a non-degenerate CKY 2-form is the Kerr-NUT-(A)dS spacetime.

In the case of degenerate CCKY tensors

It is convenient to see the eigenvalues of a rank-2 closed CKY by $Q^a{}_b = -h^a{}_c h^c{}_b$.

$$V^{-1}(Q^a{}_b)V = \{\underbrace{x_1^2, x_1^2, \ldots, x_n^2, x_n^2}_{2n}, \underbrace{\xi_1^2, \ldots, \xi_1^2}_{2m_1}, \ldots, \underbrace{\xi_N^2, \ldots, \xi_N^2}_{2m_N}, \underbrace{0, \ldots, 0}_{K}\}$$

TH-Oota-Yasui (2008)

The *D*-dim. generalized Kerr-NUT-(A)dS offshell metric is

$$g = \sum_{\mu=1}^{n} \frac{dx_\mu^2}{P_\mu} + \sum_{\mu=1}^{n} P_\mu \left[\sum_{k=0}^{n-1} A_\mu^{(k)} \theta_k\right]^2 + \sum_{j=1}^{N} \prod_{\mu=1}^{n} (x_\mu^2 - \xi_j^2) g^{(j)} + (\prod_\mu x_\mu^2) g^{(0)}$$

Where $g^{(0)}$ is arbitrary K-dim metric and $g^{(j)}$ is $2m_j$-dim Kahler metric with the Kahler form $\omega^{(j)}$.

$$P_\mu = \frac{X_\mu(x_\mu)}{x_\mu^K \prod_{j=1}^{N} (x_\mu^2 - \xi_j^2)^{m_j} \prod_{\substack{\nu=1 \\ \nu\neq\mu}}^{n} (x_\mu^2 - x_\nu^2)}, \quad A_\mu^{(k)} = \sum_{\nu_i \neq \mu} x_{\nu_1}^2 x_{\nu_2}^2 \ldots x_{\nu_k}^2$$

$$d\theta_k + 2 \sum_{j=1}^{N} (-1)^{n-k} \xi_j^{2n-2k-1} \omega^{(j)} = 0$$

We can't determine them any more without Einstein's Eq.

Einstein metrics with a degenerate CKY 2-form

When $g^{(0)}$ is K-dim Einstein metric, $g^{(j)}$ is $2m_j$-dim Einstein-Kahler metric with the Kahler form $\omega^{(j)}$ and

$$X_\mu = x_\mu \int dx_\mu \; \chi(x_\mu) x_\mu^{K-2} \prod_{i=1}^{N} (x_\mu^2 - \xi_i^2)^{m_i} \; + \; d_\mu x_\mu$$

where

$$\chi(x_\mu) = \sum_{i=0}^{n} \alpha_i x^{2i} \; , \quad \alpha_0 = (-1)^{n-1} \lambda^{(0)}$$

$$\lambda^{(j)} = (-1)^{n-1} \chi(\xi_j^2)$$

This metric satisfies Einstein Eq.

$$R_{ab} = -(D-1)\alpha_n \, g_{ab}$$

Manifolds admitting a special KY

Theor. *Let* $(M^n, \mathbf{g})$ *be a compact, simply connected manifold admitting a special KY. Then M is either isometric to* S^n *or M is a Sasakian, 3-Sasakian, nearly Kahler or weak* G_2*-manifold.*

Semmelmann (2002)

Example Let $(M^{2n+1}, \mathbf{g}, \boldsymbol{\xi}, \boldsymbol{\eta})$ be a Sasakian manifold with Killing vector field $\boldsymbol{\xi}$. Then

$$\omega_k := \xi^* \wedge (d\xi^*)^k$$

is a rank-$(2k+1)$ special KY for $k = 0, \dots, n$, which satisfies for any vector field $\mathbf{X}$ and any k

$$\nabla_X(d\omega_k) = -2(k+1)X^* \wedge \omega_k$$

3. A generalization of CKY symmetry

Hidden symmetry of charged BH in 5-dim. minimal SUGRA

$$S_5 = \int R * 1 - \frac{1}{2} * F \wedge F + \frac{1}{3\sqrt{3}} F \wedge F \wedge A$$

- Charged rotating BH

$$\begin{aligned} g &= \frac{x^2 - y^2}{X} dx^2 + \frac{y^2 - x^2}{Y} dy^2 \\ &+ \frac{X}{x^2 - y^2}[dt + y^2 d\phi]^2 + \frac{Y}{y^2 - x^2}[dt + x^2 d\phi]^2 \\ &+ \frac{1}{x^2 y^2}[c\{dt + (x^2 + y^2)d\phi + x^2 y^2 d\psi\} - y^2 A_{(1)}]^2 \end{aligned}$$

$$A_{(1)} = \frac{\sqrt{3}q}{x^2 - y^2}[dt + y^2 d\phi]$$

Chong-Cvetic-Lu-Pope (2005)

<u>Known facts :</u>

Existence of a rank-2 Killing tensor

Davis-Kunduri-Lucietti (2005)

Existence of a GCCKY 2-form

Kubiznak-Kundri-Yasui (2009)

Generalized conformal Killing-Yano tensor

<u>Def.</u> *Generalized CKY is a p-form **k** if a 3-form **T** exists obeying*

$$\nabla^T_X \boldsymbol{k} = \frac{1}{p-1} \boldsymbol{X} \lrcorner\, \boldsymbol{d}^T \boldsymbol{k} - \frac{1}{D-p+1} \boldsymbol{X}^* \wedge \boldsymbol{\delta}^T \boldsymbol{k}$$

*for an arbitrary vector **X**.*

$$\nabla^T_a k_{b_1 \cdots b_p} := \nabla_a k_{b_1 \cdots b_p} - \frac{1}{2} T_{ca[b_1} k^c{}_{b_2 \cdots b_p]}$$

$$(d^T k)_{a_1 \cdots a_{p+1}} := (p+1) \nabla^T_{[a_1} k_{a_2 \cdots a_{p+1}]}$$

$$(\delta^T k)_{a_1 \cdots a_{p-1}} := -\nabla^T_c k^c{}_{a_1 \cdots a_{n-1}}$$

Note: This connection gives $\nabla^T g = 0$.

Subclasses of GCKY tensors

$$\nabla^T_X \boldsymbol{k} = \frac{1}{p-1} \boldsymbol{X} \lrcorner \, \boldsymbol{d}^T \boldsymbol{k} - \frac{1}{D-p+1} \boldsymbol{X}^* \wedge \boldsymbol{\delta}^T \boldsymbol{k}$$

$\boldsymbol{d}^T \boldsymbol{h} = 0$; ***h*** *is a generalized closed CKY*

$\boldsymbol{\delta}^T \boldsymbol{f} = 0$; ***f*** *is a* GKY

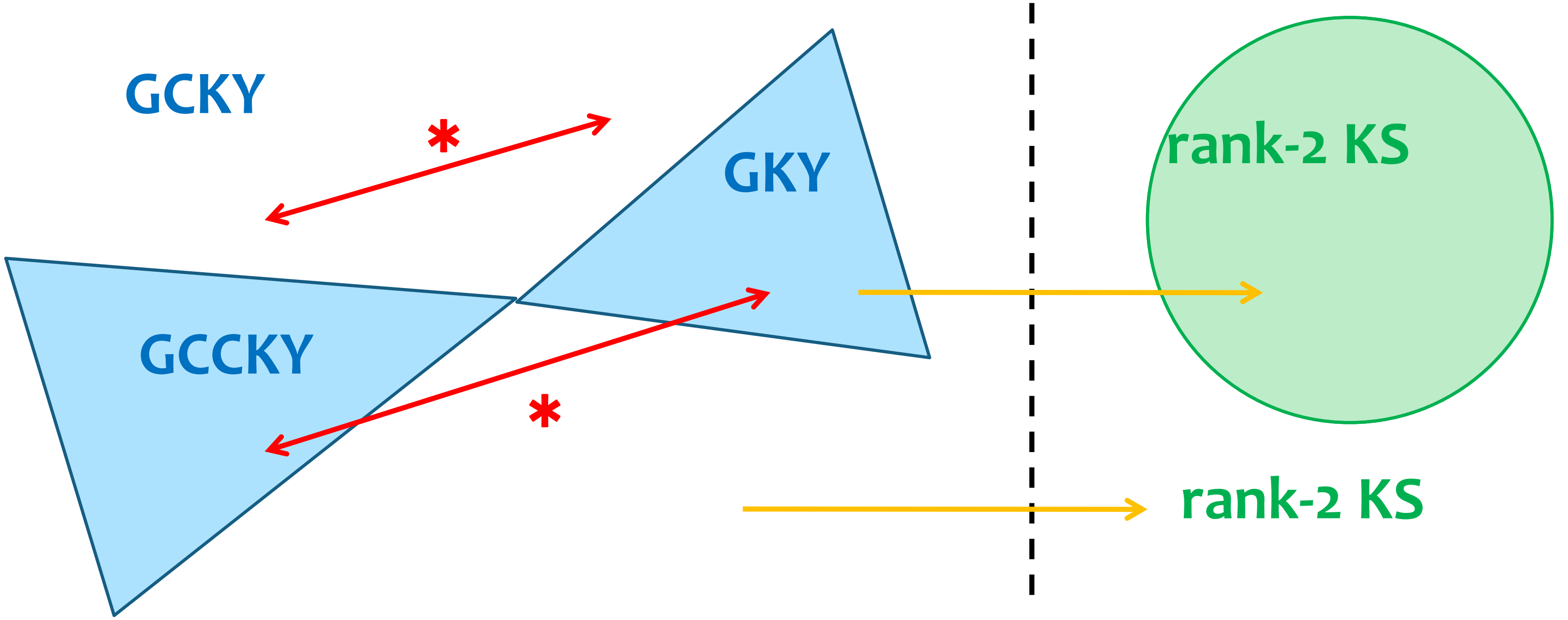
Basic Properties of GCKY symmetry

1) A GCKY 1-form is equal to a conformal Killing 1-form.

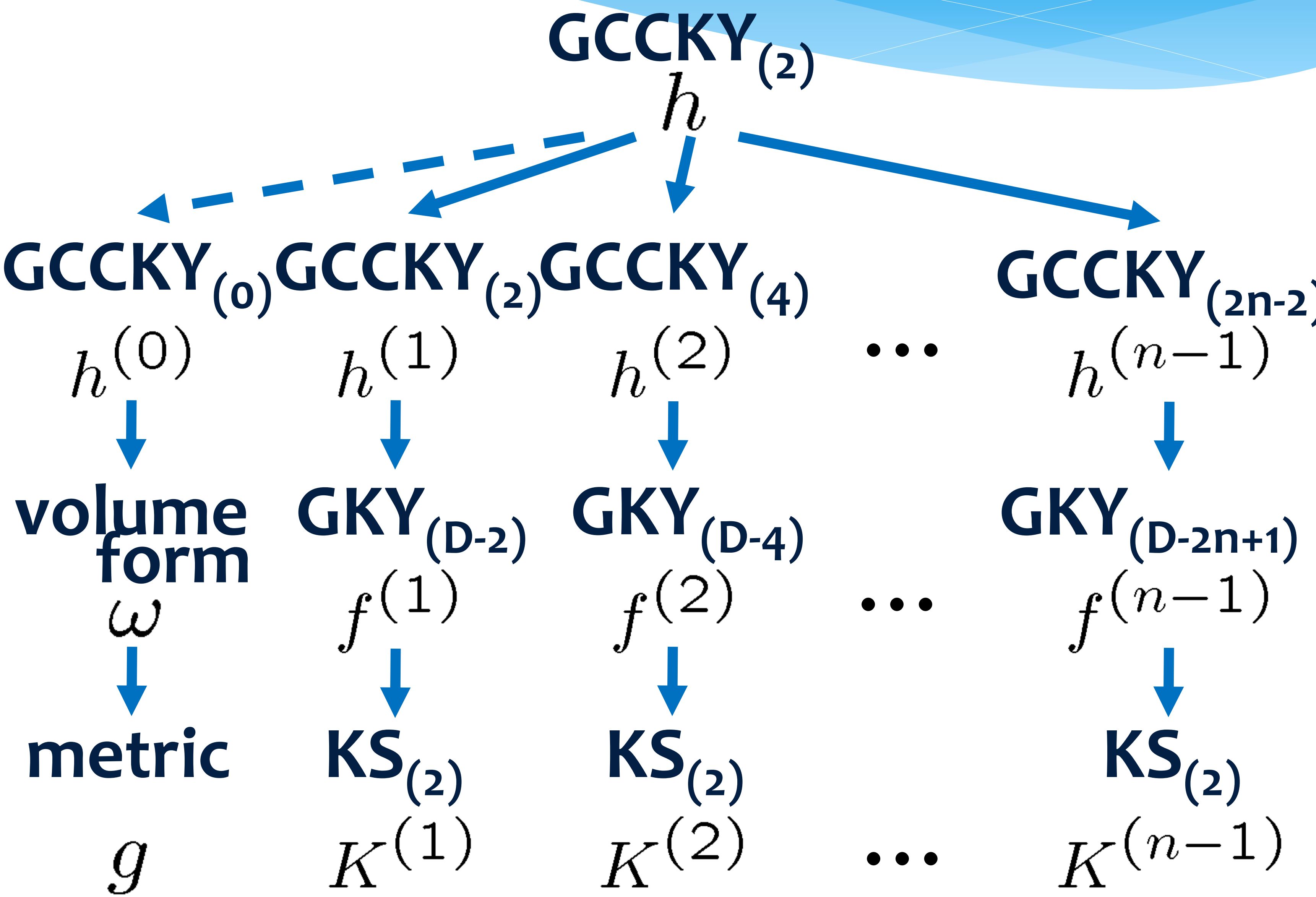
2) The Hodge star ∗ maps GCKY *p*-forms into GCKY (*D-p*)-forms.
In particular, the Hodge star of a closed GCKY *p*-form is a GKY (*D-p*)-form and vice versa.

3) When ***h1*** and ***h2*** is a closed GCKY *p*-form and *q*-form, respectively, then ***h3 = h1 ∧ h2*** is a closed GCKY (*p+q*)-form.

4) When f is a G(C)KY *p*-form, then rank-2 symmetric tensor ***K*** defined by $K_{ab}=f_{a\ldots}f_b{}^{\ldots}$ is a (conformal) Killing tensor.



Tower of hidden symmetries



Geodesic integrability

GCCKY 2-form h

- **GCCKY 2j-form** $h^{(j)} = h \wedge \ldots \wedge h$ ▶ **GKY (D-2j)-form** $f^{(j)} = *h^{(j)}$ ▶ **rank-2 Killing tensor** $K^{(j)}_{ab} = f^{(j)}_{a\cdots} f^{(j)\cdots}_{b}$ ▶ **const. of motion** $C_j = K^{(j)}_{ab} p^a p^b$
- ~~Killing vector $\xi_a = \nabla^b h_{ba}$ ▶ Killing vector $\eta^{(j)}_a = K^{(j)}_{ab} \xi^b$ ▶ const. of motion $F_j = \eta^{(j)}_a p^a$~~

comments:

- Constants of motion generated from a GCCKY 2-form are in involution, i.e., $\{C_i, C_j\}_P = 0$

- one doesn't have Killing vectors.

Dirac symmetry operator

Benn-Charlton, Class.Quant.Grav.14 (1997)
TH-Kubiznak-Warnick-Yasui, arXiv:1002.3616

<u>Th.</u> Let ω be a generalized conformal Killing-Yano (GCKY) p-form obeying

$$\nabla^T_X \omega - \frac{1}{p+1} X \lrcorner\, d^T \omega + \frac{1}{n-p+1} X^\flat \wedge \delta^T \omega = 0\,.$$

Then the operator

$$L_\omega = e^a \omega \nabla^T_{e_a} + \frac{p}{p+1} d^T \omega - \frac{n-p}{n-p+1} \delta^T \omega + \frac{1}{2} T \omega$$

satisfies

$$\mathcal{D} L_\omega = \omega \mathcal{D}^2 + \frac{(-1)^p}{p+1} d^T \omega \mathcal{D} + \frac{(-1)^p}{n-p+1} \delta^T \omega \mathcal{D} - A\,.$$

Massless Dirac symmetry operators

In the case A vanishes, $\mathbf{L}_\omega$ is an symmetry operator for massless Dirac equation, i.e.,

$$\mathcal{D}L_\omega - L_\omega\mathcal{D} = 0 \ . \qquad \text{(on-shell)}$$

$$\mathcal{D}\psi = 0$$

Anomaly

The last term A = $A_{(p+2)}$ + $A_{(p-2)}$ is written explicitly as

$$A_{(p+2)} = \frac{d(d^T\omega)}{p+1} - \frac{T\wedge\delta^T\omega}{n-p+1} - \frac{1}{2}dT\underset{1}{\wedge}\omega$$

$$A_{(p-2)} = \frac{\delta(\delta^T\omega)}{n-p+1} - \frac{1}{6(p+1)}T\underset{3}{\wedge}d^T\omega + \frac{1}{12}dT\underset{3}{\wedge}\omega\,.$$

Massive Dirac symmetry operators

$$(\mathcal{D} + m)\psi = 0$$

<u>Col.</u> Let ω be a generalized Killing-Yano (GKY) p-form such that an anomaly A vanishes. Then there exists an operator $\mathbf{K}_\omega$ such that

$$\delta^T \omega = 0$$

$$\mathcal{D} K_\omega + (-1)^p K_\omega \mathcal{D} = 0 \ . \quad \text{(off-shell)}$$

<u>Col.</u> Let ω be a generalized closed conformal Killing-Yano (GCCKY) p-form such that an anomaly A vanishes. Then there exists an operator M_ω such that

$$\mathcal{D} M_\omega - (-1)^p M_\omega \mathcal{D} = 0 \ . \quad \text{(off-shell)} \quad d^T \omega = 0$$

The symmetry operators in terms of gamma matrices

$$L_\omega = \left[\omega^a{}_{b_1\dots b_{p-1}}\gamma^{b_1\dots b_{p-1}} + \frac{1}{p(p+1)}\omega_{b_1\dots b_p}\gamma^{ab_1\dots b_p}\right]\nabla_a$$
$$+\frac{1}{(p+1)^2}(d\omega)_{b_1\dots b_{p+1}}\gamma^{b_1\dots b_{p+1}} - \frac{n-p}{n-p+1}(\delta\omega)_{b_1\dots b_{p-1}}\gamma^{b_1\dots b_{p-1}}$$
$$-\frac{1}{24}T_{b_1b_2b_3}\omega_{b_4\dots b_{p+3}}\gamma^{b_1\dots b_{p+3}} + \frac{3-p}{8(p+1)}T^a{}_{b_1b_2}\omega_{ab_3\dots b_{p+1}}\gamma^{b_1\dots b_{p+1}}$$
$$+\frac{(n-p-3)(p-1)}{8(n-p+1)}T^{ab}{}_{b_1}\omega_{abb_2\dots b_{p-1}}\gamma^{b_1\dots b_{p-1}} + \frac{(p-1)(p-2)}{24}T^{abc}\omega_{abcb_1\dots b_{p-3}}\gamma^{b_1\dots b_{p-3}}.$$

$$K_\omega = \omega^a{}_{b_1\dots b_{p-1}}\gamma^{b_1\dots b_{p-1}}\nabla_a$$
$$+\frac{1}{2(p+1)^2}(d\omega)_{b_1\dots b_{p+1}}\gamma^{b_1\dots b_{p+1}} + \frac{1-p}{8(p+1)}T^a{}_{b_1b_2}\omega_{ab_3\dots b_{p+1}}\gamma^{b_1\dots b_{p+1}}$$
$$-\frac{p-1}{4}T^{ab}{}_{b_1}\omega_{abb_2\dots b_{p-1}}\gamma^{b_1\dots b_{p-1}} + \frac{(p-1)(p-2)}{24}T^{abc}\omega_{abcb_1\dots b_{p-3}}\gamma^{b_1\dots b_{p-3}}.$$

$$M_\omega = \omega_{b_1\dots b_p}\gamma^{ab_1\dots b_p}\nabla_a$$
$$-\frac{p(n-p)}{2(n-p+1)}(\delta\omega)_{b_1\dots b_{p-1}}\gamma^{b_1\dots b_{p-1}} - \frac{1}{24}T_{b_1b_2b_3}\omega_{b_4\dots b_{p+3}}\gamma^{b_1\dots b_{p+3}}$$
$$+\frac{p}{4}T^a{}_{b_1b_2}\omega_{ab_3\dots b_{p+1}}\gamma^{b_1\dots b_{p+1}} + \frac{p(p-1)(n-p-1)}{8(n-p+1)}T^{ab}{}_{b_1}\omega_{abb_2\dots b_{p-1}}\gamma^{b_1\dots b_{p-1}}.$$

Hidden symmetry of CCLP black hole

- GCCKY 2-form

Kubiznak-Kunduri-Yasui (2009)

$$h = x_1\, e^1 \wedge e^{\hat{1}} + x_2\, e^2 \wedge e^{\hat{2}} \quad \text{with} \quad T = \frac{1}{\sqrt{3}} * F$$

It was shown that this 2-form produces a rank-2 Killing tensor discovered by Davis-Kunduri-Lucietti.

- Separation of variables

H-J, K-G and Dirac equations are separable.

Davis-Kunduri-Lucietti (2005), Wu (2009)

4-dim. heterotic SUGRA

We consider the following theory

$$\mathcal{L}_4 = e^{-\varphi}(R * 1 + *d\varphi \wedge d\varphi - \frac{1}{4} * F_{(2)} \wedge F_{(2)} - \frac{1}{2} * H_{(3)} \wedge H_{(3)})$$

where

$$F_{(2)} = dA_{(1)} \ , \quad H_{(3)} = dB_{(2)} - \frac{1}{4} A_{(1)} \wedge dA_{(1)}$$

This action gives an bosonic part of the low-energy effective action of heterotic string theory.

Kerr-Sen black holes

$$
\begin{aligned}
ds^2 &= e^{\Phi}\{-\frac{\Delta}{\rho_b^2}(dt - a\sin^2\theta d\varphi)^2 \\
&\qquad + \frac{\sin^2\theta}{\rho_b^2}[adt - (r^2 + 2br + a^2)d\varphi]^2 + \frac{\rho_b^2}{\Delta}dr^2 + \rho_b^2 d\theta^2\} \ , \\
H &= -\frac{2ba}{\rho_b^4} dt \wedge d\varphi \wedge [(r^2 - a^2\cos^2\theta)\sin^2\theta dr - r\Delta \sin 2\theta d\theta] \ , \\
A &= -\frac{Qr}{\rho_b^2}(dt - a\sin^2\theta d\varphi) \ , \\
\Phi &= 2\ln\left(\frac{\rho}{\rho_b}\right)
\end{aligned}
$$

Sen (1992)

where

$$
\rho^2 = r^2 + a^2\cos^2\theta \ , \quad \rho_b^2 = \rho^2 + 2br \ , \quad \Delta = r^2 - 2(M-b)r + a^2 \, .
$$

Hidden symmetry of Kerr-Sen black holes

Known facts :

Algebraic properties of curvature

Burinskii (1995)

Separability of the Hamilton-Jacobi equation

Blaga-Blaga (2001)

Separability of the Klein-Gordon equation

Wu-Cai (2003)

Existence of a rank-2 Killing tensor (string frame)

Hioki-Miyamoto (2008)

Questions :

Separability of the Dirac equation ?

Why does such a separation occur?

D-dimensional heterotic SUGRA

We consider the 'naïve' generalization of heterotic supergravity

$$\mathcal{L}_D = e^{\varphi\sqrt{(D-2)/2}}\{R*1 - \frac{1}{2}*d\varphi\wedge d\varphi - *F_{(2)}\wedge F_{(2)} \\ -\frac{1}{2}*H_{(3)}\wedge H_{(3)}\}$$

where

$$F_{(2)} = dA_{(1)}\ , \quad H_{(3)} = dB_{(2)} - A_{(1)}\wedge dA_{(1)}\ .$$

This kind of action gives a bosonic part of supergravity such as heterotic supergravity compactified on a torus in each dimension.

Higher-dimensional Kerr-Sen black holes

$$g_D = \sum_{\mu=1}^{n} \frac{dx_\mu^2}{Q_\mu} + \sum_{\mu=1}^{n} Q_\mu (\mathcal{A}_\mu - \sum_{\nu=1}^{n} \frac{2N_\nu s^2}{HU_\nu} \mathcal{A}_\nu)^2 + \varepsilon S (\mathcal{A} - \sum_{\nu=1}^{n} \frac{2N_\nu s^2}{HU_\nu} \mathcal{A}_\nu)^2$$

$$\phi = \sqrt{\frac{2}{D-2}} \ln H \ , \quad A_{(1)} = \sum_{\mu=1}^{n} \frac{2N_\mu sc}{HU_\mu} \mathcal{A}_\mu \ ,$$

$$B_{(2)} = (\sum_{k=0}^{n-1} (-1)^k c_{n-k-1} d\psi_k + \varepsilon\, \tilde{c}\, d\psi_n) \wedge (\sum_{\nu=1}^{n} \frac{2N_\nu s^2}{HU_\nu} \mathcal{A}_\nu)$$

where

$$\mathcal{A}_\mu = \sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k \ , \quad \mathcal{A} = \sum_{k=0}^{n} A^{(k)} d\psi_k \ , \quad H = 1 + \sum_{\mu=1}^{n} \frac{2N_\mu s^2}{U_\mu} \ , \quad N_\mu = m_\mu x_\mu^{1-\varepsilon} \ ,$$

$$Q_\mu = \frac{X_\mu}{U_\mu} \ , \quad U_\mu = \prod_{\substack{\nu=1 \\ \nu\neq\mu}}^{n} (x_\mu^2 - x_\nu^2) \ , \quad X_\mu = \sum_{k=0}^{n-1} c_k x_\mu^{2k} + 2N_\mu + \varepsilon \frac{(-1)^n \tilde{c}}{x_\mu^2} \ , \quad c_{n-1} = -1 \ ,$$

$$A_\mu^{(k)} = \sum_{\substack{1\leq\nu_1<\cdots<\nu_k\leq n \\ \nu_i\neq\mu}} x_{\nu_1}^2 \cdots x_{\nu_k}^2 \ , \quad A^{(k)} = \sum_{1\leq\nu_1<\cdots<\nu_k\leq n} x_{\nu_1}^2 \cdots x_{\nu_k}^2 \ , \quad A_\mu^{(0)} = A^{(0)} = 1 \ ,$$

$$S = \frac{\tilde{c}}{A^{(n)}} \ , \quad \tilde{c} = \text{const.} \ , \quad s = \sinh\delta \ , \quad c = \cosh\delta \ .$$

Cvetic-Youm (1996), Chow (2008)

Hidden symmetry of Kerr-Sen black holes

Known facts : Chow (2008)

Hamilton-Jacobi equation is separable.

Rank-2 Killing tensors exist.

$$K^{(j)} = \sum_{\mu=1}^{n} A_{\mu}^{(j)}(e^{\mu}e^{\mu} + e^{\hat{\mu}}e^{\hat{\mu}}) + \varepsilon A^{(j)}e^{0}e^{0}$$

Questions :

Does the separation of the K-G equation occurs?

How about the Dirac equation?

If separable, where does such a structure come from?

Hidden symmetry of Kerr-Sen black holes

- GCCKY 2-form

$$h = \sum_{\mu=1}^{n} x_\mu \, e^\mu \wedge e^{\hat{\mu}} \quad \text{with} \quad T = H$$

TH-Kubiznak-Warnick-Yasui (2010)

- Separation of variables

Okai (1994), Blaga, et al. (2001), Wu-Cai (2003), Hioki-Miyamoto(2008)
Chow (2008), HKWY (2010)

	frame	
	Einstein	string
H-J	separable	separable
K-G	separable	×
Dirac*	×	separable

- Symmetry operators

TH-Kubiznak-Warnick-Yasui (2010)

For the torsion ***T=H***, one can produce the symmetry operators for the Laplacian and the modified Dirac operator $D^{T/3}$.

4. Summary & Outlook

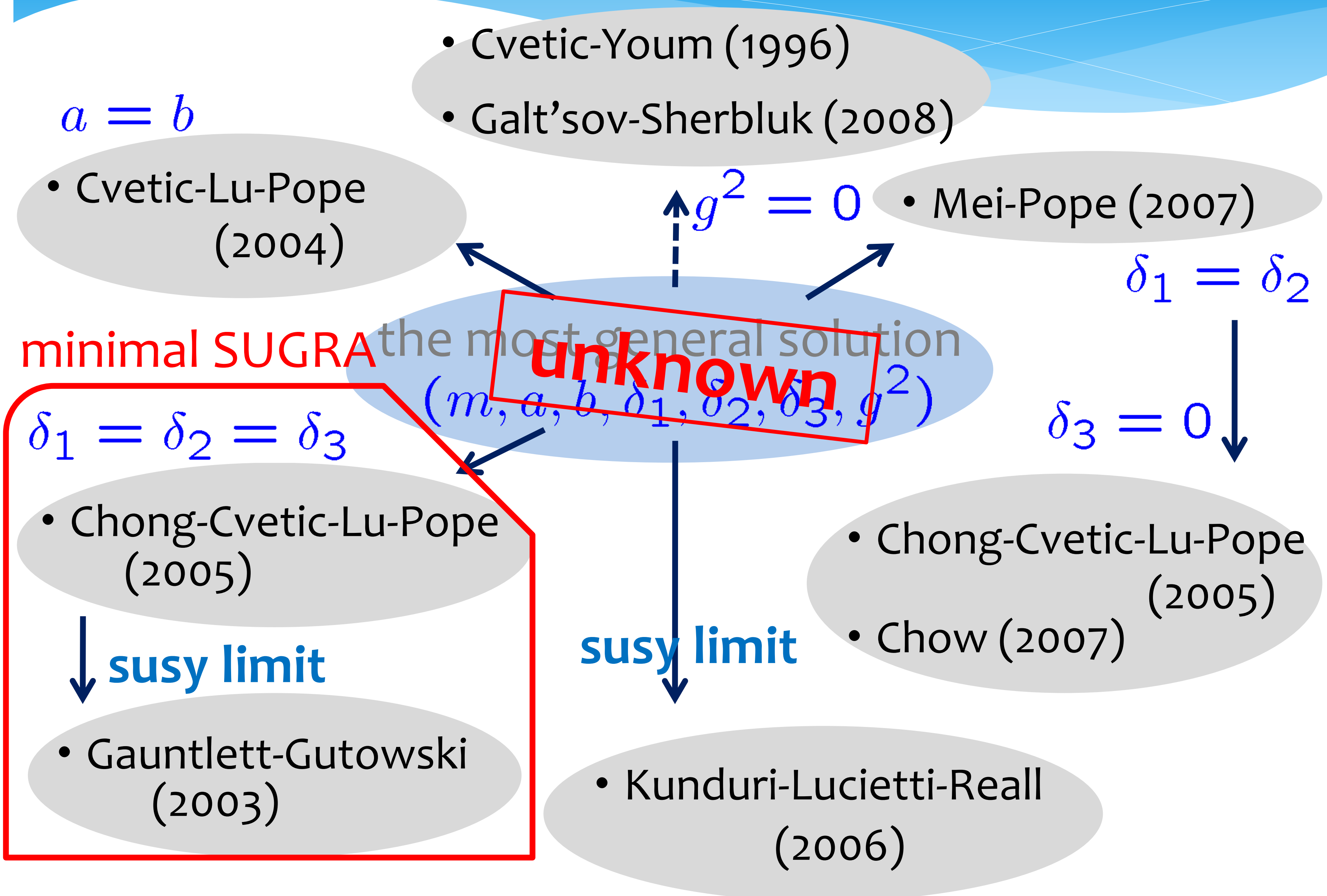
Summary

We have studied properties of spacetimes admitting a conformal Killing-Yano symmetry and its generalization. Especially, a rank-2 CCKY and GCCKY 2-form.

If the torsion is absent, we have shown that such symmetry characterizes vacuum black hole solutions with spherical horizon topology.

If the torsion is persent, we have shown that such symmetry are seen in the solutions of supergravities such as 5-dim. minimal SUGRA and heterotic supergravity.

Exact solutions of 5-dim. $U(1)^3$ SUGRA



Manifolds with special holonomy

Type-IIB supergravity
on $AdS_5 \times X^5$ $\longleftrightarrow$ correspondence $\mathcal{N}=1$ SCFT

(Examples of Sasaki-Einstein) S^5 $T^{1,1}$

It is known that Sasaki-Einstein and Calabi-Yau metrics are derived from vacuum rotating BH by taking a limit.

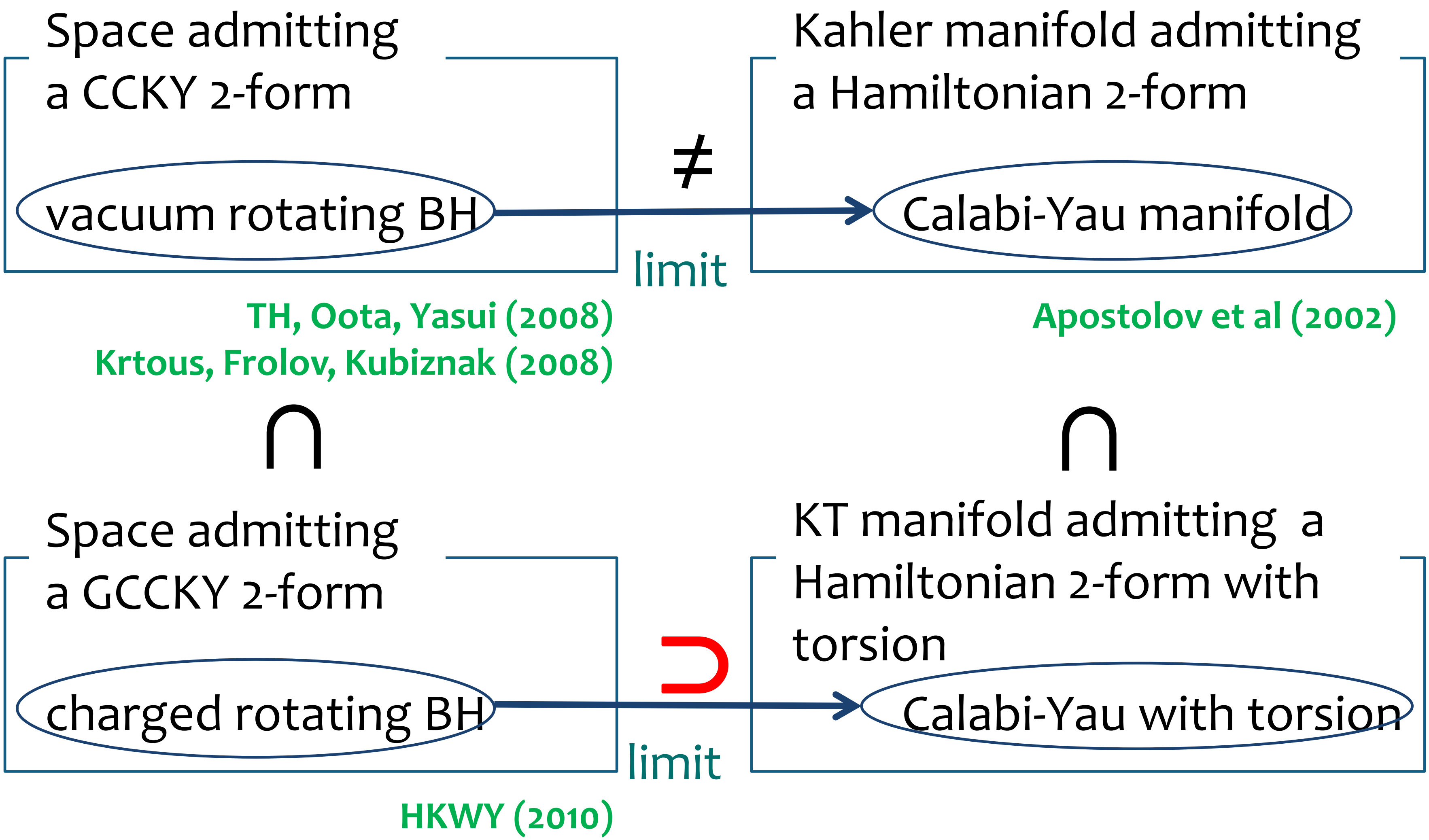
vacuum rotating BH $\xrightarrow{\text{limit}}$ <u>even</u> Calabi-Yau
<u>odd</u> Sasaki-Einstein

Ex)

5-dim. Kerr-(A)dS $\longrightarrow$ Sasaki-Einstein $Y^{p,q}$ $L^{a,b,c}$

6-dim. Kerr-NUT-(A)dS $\longrightarrow$ resolved Calabi-Yau cone

Even Dimensions



Spacetimes admitting a GCCKY 2-form

- assumption

$$g = g_{ab}(z)dz^a dz^b$$: *D*-dim metric

$$h = \frac{1}{2}h_{ab}(z)dz^a \wedge dz^b$$: GCCKY 2-form

i.e. $\nabla^T_a h_{bc} = g_{ab}\xi_c - g_{ac}\xi_b \quad \xi_a = \frac{1}{D-1}\nabla^{Tb}h_{ba}$

- introduce canonical basis

Orthonoramal frame $\{e^a\} = \{e^\mu, e^{\hat{\mu}}\}$

s.t. $g = \sum_{\mu=1}^{n} (e^\mu e^\mu + e^{\hat{\mu}} e^{\hat{\mu}}),\ h = \sum_{\mu=1}^{n} x_\mu e^\mu \wedge e^{\hat{\mu}}$

non-degenerate: $x_\mu \neq x_\nu$

- the form of ξ

$$\xi = \sum_{\mu=1}^{n} \sqrt{Q_\mu}\, e^{\hat{\mu}}$$

where Q_μ is an arbitrary fn.

$$[\boldsymbol{e}_\mu, \boldsymbol{e}_\nu] = -\frac{x_\nu\sqrt{Q_\nu}}{x_\mu^2 - x_\nu^2}\boldsymbol{e}_\mu - \frac{x_\mu\sqrt{Q_\mu}}{x_\mu^2 - x_\nu^2}\boldsymbol{e}_\nu \quad (\mu \neq \nu) \ ,$$

$$[\boldsymbol{e}_\mu, \boldsymbol{e}_{\widehat{\mu}}] = K_\mu \boldsymbol{e}_\mu + L_\mu \boldsymbol{e}_{\widehat{\mu}} + \sum_{\rho\neq\mu} \frac{2x_\mu\sqrt{Q_\rho}}{x_\mu^2 - x_\rho^2}\boldsymbol{e}_{\widehat{\rho}} \ ,$$

$$[\boldsymbol{e}_\mu, \boldsymbol{e}_{\widehat{\nu}}] = -\frac{x_\mu\sqrt{Q_\mu}}{x_\mu^2 - x_\nu^2}\boldsymbol{e}_{\widehat{\nu}} \quad (\mu \neq \nu) \ ,$$

$$[\boldsymbol{e}_{\widehat{\mu}}, \boldsymbol{e}_{\widehat{\nu}}] = 0 \quad (\mu \neq \nu) \ ,$$

$$K_\mu = \frac{\kappa_{\mu\mu\mu}}{\sqrt{Q_\mu}}, \ \ L_\mu = -\frac{1}{\sqrt{Q_\mu}}\left(\sum_{\rho\neq\mu}\frac{x_\mu Q_\rho}{x_\mu^2 - x_\rho^2} - \kappa_{\widehat{\mu}\mu}\right), \ \ M_{\mu\nu} = \frac{2x_\mu\sqrt{Q_\nu}}{x_\mu^2 - x_\nu^2} - T_{\mu\widehat{\mu}\widehat{\nu}}$$

$$[[\boldsymbol{e}_A, \boldsymbol{e}_B], \boldsymbol{e}_C] + [[\boldsymbol{e}_B, \boldsymbol{e}_C], \boldsymbol{e}_A] + [[\boldsymbol{e}_C, \boldsymbol{e}_A], \boldsymbol{e}_B] = 0$$

$$M_{\mu\nu} K_\nu = 0 \ ,$$

$$e_\nu(K_\mu) = \frac{x_\nu \sqrt{Q_\nu}}{x_\mu^2 - x_\nu^2} K_\mu \ , \quad e_{\hat{\nu}}(K_\mu) = 0 \ ,$$

$$e_\nu(L_\mu) = \frac{x_\nu \sqrt{Q_\nu}}{x_\mu^2 - x_\nu^2} L_\mu - M_{\mu\nu} M_{\nu\mu} - \frac{2 x_\mu x_\nu \sqrt{Q_\mu} \sqrt{Q_\nu}}{(x_\mu^2 - x_\nu^2)^2} \ , \quad e_{\hat{\nu}}(L_\mu) = 0 \ ,$$

$$e_\nu(M_{\mu\nu}) = \left(\frac{2 x_\nu \sqrt{Q_\nu}}{x_\mu^2 - x_\nu^2} - L_\nu \right) M_{\mu\nu} \ , \quad e_{\hat{\nu}}(M_{\mu\nu}) = 0 \ ,$$

$$e_\nu(M_{\mu\rho}) = \left(\frac{2 x_\nu \sqrt{Q_\nu}}{x_\mu^2 - x_\nu^2} + \frac{x_\nu \sqrt{Q_\nu}}{x_\nu^2 - x_\rho^2} \right) M_{\mu\rho} - M_{\mu\nu} M_{\nu\rho} \ , \quad e_{\hat{\nu}}(M_{\mu\rho}) = 0 \ .$$

$$T_{\hat{\mu}\hat{\nu}\hat{\rho}} = 0 \quad (\mu, \nu, \rho\text{: different}) \ .$$

- the only components $T_{\mu\hat{\mu}\hat{\nu}}$ are non-vanishing.

$$\boldsymbol{T} = T_{\mu\hat{\mu}\hat{\nu}}\, \boldsymbol{e}^{\mu} \wedge \boldsymbol{e}^{\hat{\mu}} \wedge \boldsymbol{e}^{\hat{\nu}}$$

- local multi-Hermitian structure

for each $\epsilon = (\epsilon_1, \cdots, \epsilon_n)$ with $\epsilon_i = \pm 1$

$$\exists \quad J_\epsilon(e_\mu) = -\epsilon_\mu e_{\hat{\mu}} \ , \quad J_\epsilon(e_{\hat{\mu}}) = \epsilon_\mu e_\mu$$

s.t. $$N_\epsilon(X, Y) \equiv [J_\epsilon X,\ J_\epsilon Y] - [X,\ Y] - J_\epsilon[X,\ J_\epsilon Y] - J_\epsilon[J_\epsilon X,\ Y] = 0$$

(1) For each ε, Jε is complex structure.

(2) g is Hermitian : $g(X, Y) = g(X, J_\epsilon Y)$

- Bismut torsion

$$\boldsymbol{B} = -d\Omega(JX, JY, JZ)$$

where $\Omega(X,Y) = g(X, JY)$

s.t. $\nabla^B g = 0, \quad \nabla^B J = 0, \quad \nabla^B \Omega = 0$

∗ (M, g, J, Ω, B) is called Kahler with torsion (KT) manifond. When B=0, then it becomes Kahler manifold.

- relationship b/w the torsion ***T*** of GCCKY 2-form and the Bismut torsion ***B***

$$\boldsymbol{T} = \sum_{\mu \neq \nu} \frac{2\epsilon_\mu \sqrt{Q_\nu}}{\epsilon_\mu x_\mu + \epsilon_\nu x_\nu} \boldsymbol{e}^\mu \wedge \boldsymbol{e}^{\hat{\mu}} \wedge \boldsymbol{e}^{\hat{\nu}} + \boldsymbol{B} \ .$$

- 3 types of solutions: **Kμ＝0、Mμν＝0、**Mixed

(These doesn't exist when T=0。)

(1) **Kμ＝0** type: special solution

$$L_\mu = -\partial_\mu\sqrt{Q_\mu} + \partial_\mu\Big(\ln H - \sum_{\nu=1}^{n}\ln f_\nu\Big)\sqrt{Q_\mu}\ ,$$

$$M_{\mu\nu} = \frac{f_\nu}{f_\mu}\left(\frac{2x_\mu}{x_\mu^2 - x_\nu^2} + \partial_\mu \ln H\right)\sqrt{Q_\nu}\ ,$$

where $H = 1 + \sum_{\mu=1}^{n}\frac{N_\mu}{U_\mu},\ \partial_\nu N_\mu = 0$ and $f_\mu = f_\mu(x_\mu)$

$$\boldsymbol{T} = \sum_{\mu\neq\nu}\left\{\frac{2x_\mu}{x_\mu^2 - x_\nu^2}\Big(1 - \frac{f_\nu}{f_\mu}\Big) - \frac{f_\nu\big(\partial_\mu \ln H\big)}{f_\mu}\right\}\sqrt{Q_\nu}\,\boldsymbol{e}^\mu \wedge \boldsymbol{e}^{\hat\mu} \wedge \boldsymbol{e}^{\hat\nu}\ .$$

(2) **Mμν=0**: we have general solution

$$K_\mu = -\frac{U_\mu}{2} \ , \qquad L_\mu = \sqrt{Q_\mu}\left(\sum_{\rho\neq\mu}\frac{x_\mu}{x_\mu^2 - x_\rho^2} + h_\mu\right) \ ,$$

where $\partial_\nu h_\mu = e_{\hat{\nu}}(h_\mu) = 0$ for $\mu \neq \nu$

$$\boldsymbol{T} = \sum_{\mu\neq\nu}\frac{2x_\mu\sqrt{Q_\nu}}{x_\mu^2 - x_\nu^2}\boldsymbol{e}^\mu \wedge \boldsymbol{e}^{\hat{\mu}} \wedge \boldsymbol{e}^{\hat{\nu}} \ .$$

(3) Mixed: for simplicity, in 4 dimensions

$$K_1 = -\frac{1}{2}(x_1^2 - x_2^2) \ , \qquad K_2 = 0 \ ,$$

$$L_1 = \left(\frac{x_1}{x_1^2 - x_2^2} + h_1\right)\sqrt{Q_1} \ , \qquad L_2 = \left(\frac{x_2}{x_2^2 - x_1^2} + h_2\right)\sqrt{Q_2} \ ,$$

$$M_{12} = \exp\left(\int h_2 dx_2\right) \times (x_1^2 - x_2^2)^{-3/2} + f_1(x_1) \ , \qquad M_{21} = 0 \ .$$

- construction of metrics

(1) **Kμ=0:** $\boldsymbol{e}^{\mu} = \frac{\boldsymbol{d}x_{\mu}}{\sqrt{Q_{\mu}}}$, $\boldsymbol{e}^{\hat{\mu}} = \sqrt{R_{\mu}}(\boldsymbol{\mathcal{A}}_{\mu} - \boldsymbol{A})$

where $Q_{\mu} = \frac{X_{\mu}(x_{\mu})}{U_{\mu}}, \ R_{\mu} = \frac{Y_{\mu}(x_{\mu})}{U_{\mu}}, \ \frac{Y_{\mu}}{X_{\mu}} = f_{\mu}(x_{\mu})^{2}$.

$$\boldsymbol{\mathcal{A}}_{\mu} = \sum_{k=0}^{n-1} A_{\mu}^{(k)} d\psi_{k} \ . \quad d\boldsymbol{A} = \sum_{\mu=1}^{n} \frac{\partial_{\mu} \ln H}{f_{\mu}} \boldsymbol{e}^{\mu} \wedge \boldsymbol{e}^{\hat{\mu}} \ .$$

(This includes Kerr-Sen black holes)

(2) **Mμν=0:** $\boldsymbol{e}^{\mu} = \frac{\boldsymbol{d}x_{\mu}}{\sqrt{Q_{\mu}}}$, $\boldsymbol{e}^{\hat{\mu}} = \frac{\boldsymbol{d}y_{\mu}}{\sqrt{R_{\mu}}}$

where $Q_{\mu} = \frac{X_{\mu}(x_{\mu}, y_{\mu})}{U_{\mu}}, \ R_{\mu} = \frac{Y_{\mu}(x_{\mu}, y_{\mu})}{U_{\mu}}$.

(3) Mixed: not yet