伊藤さん

Gluon Scattering amplitudes and Thermodynamic Bethe Ansatz

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1 Introduction

AdS / CFT 対応 + Integrability

スペクトル, Wilson-loop, gluon scattering, form factors

Perturbative : n-pt L-loop MHV amplitudes

$$A_n^{(L)} = A_n^{tree}\, e^{M_n^{(L)}} \quad \text{(planar limit)}$$

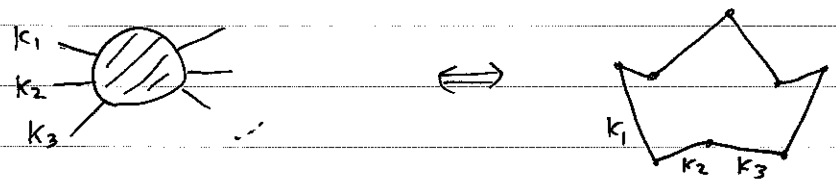
loop calculation (Anastasiou - Bern - Dixon - Kosower
Bern - Dixon - Smirnov)

(2-loop) = (1-loop)^2 … recursive

$\Rightarrow$ BDS ansatz $\quad M_n^{(L)} = M_n^{BDS\,(L)} + \underbrace{R_n^{(L)}}_{\text{Remainder function}}$

・Dual superconformal symmetry $\Longrightarrow$ $M_n^{BDS\,(L)}$, $R_n^{(L)}$: 4次元 momentum の cross-ratio の関数

・Dual to light-like polygonal Wilson loops



6-pt 2-loop numerical (Bern …, Drummond …)

BDS公式からのずれの存在が確かめられた

analytical (Del Duca - Duhr - Smirnov) 17 pages の式 (Polylog)

Goncharov - Spradin Vergu - Volovich

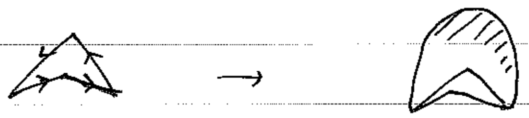
反復積 + Mixed Motive , 2行

2n-pt 2-loop momentum configuration AdS_3

Khoze - Heslop '公式' (予想) , OPE (Alday - Gaiotto - Maldacena
- Sever - Vieira)

Remainder-function at strong coupling

4-pt Alday-~~Gaiotto~~ Maldacena (Wilson-loop $\leftrightarrow$ Minimal surface)



8pt (AdS_3) AM Remainder関数の積分公式

6pt (AdS_5) Alday-Gaiotto-Maldacena TBA方程式

2n-pt (AdS_5) AMSV Y-system, T-system

(AdS_3) AMSV, Hatsuda-KI-Sakai-Satoh

Analyticな公式 CFTの perturbation

mass scale $l \to 0$ (uv limit) unpert. CFT $\leftrightarrow$ regular polygon

$l \to \infty$ (IR limit) free massive theory $\leftrightarrow$ collinear limit

6 pt (AdS_5) … CFTは $SU(2)$ parafermion, level 4

2n-pt (AdS_3) $SU(n-2)_2 / U(1)^{n-3}$

(8pt, 10pt $\longrightarrow$ 2n .. work in progress)

2. Minimal surface in AdS_3

$(z, \bar{z})$ worldsheet, $S \propto \sqrt{\lambda} \int d^2z \left[\vec{Y}_z \cdot \vec{Y}_{\bar{z}} + \mu (\vec{Y}^2 + 1) \right]$

$\vec{Y} = (Y_{-1}, Y_0, Y_1, Y_2) \in \mathbb{R}^{2,2}$

$-Y_{-1}^2 - Y_0^2 + Y_1^2 + Y_2^2 = -1$

<u>eom</u> $\vec{Y}_{z\bar{z}} - (\vec{Y}_z \cdot \vec{Y}_{\bar{z}}) \vec{Y} = 0$

<u>Virasoro cond</u> $\vec{Y}_z^2 = \vec{Y}_{\bar{z}}^2 = 0$

$\Rightarrow$ moving frameの発展方程式

$\Rightarrow$ 可積分条件 flat connection (Hitchin eq)

$\Rightarrow$ linear problem

de Vega - Sanchez

$$\text{frame}\begin{cases}\vec{q}_1 = \vec{Y} \\ \vec{q}_2 = \vec{Y}_z \cdot e^{-\alpha} \\ \vec{q}_3 = \vec{Y}_{\bar z} \cdot e^{-\alpha} \\ \vec{q}_4 = \vec{N}\end{cases}$$

$$e^{2\alpha} \equiv \frac{1}{2}\vec{Y}_z\cdot\vec{Y}_{\bar z} \qquad \alpha(z,\bar z)$$

$$N_i \equiv \epsilon_{ijkl} Y^j \partial_z Y^k \partial_{\bar z} Y^l \cdot \frac{1}{2} e^{2\alpha}$$

$$\Longrightarrow \quad \vec{q}_1^{\,2} = -1, \quad \vec{q}_4^{\,2} = 1, \quad \vec{q}_2^{\,2} = \vec{q}_3^{\,2} = 0$$

$$\partial_z \begin{pmatrix} \vec{q}_1 \\ \vdots \\ \vec{q}_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & e^{\alpha} & 0 \\ 2e^{\alpha} & -\alpha_z & 0 & 0 \\ 0 & 0 & \alpha_z & 2pe^{-\alpha} \\ 0 & -pe^{-\alpha} & 0 & 0 \end{pmatrix} \begin{pmatrix} \vec{q}_1 \\ \vdots \\ \vec{q}_4 \end{pmatrix}$$

$$\partial_{\bar z} \begin{pmatrix} \; \end{pmatrix} = \begin{pmatrix} \text{略} \end{pmatrix} \begin{pmatrix} \; \end{pmatrix} \qquad P \equiv -\frac{1}{2}\vec{N}\cdot\vec{Y}_{zz}$$

$$\mathbb{R}^{2,2} \qquad SO(2,2) \to SL(2\mathbb{R}) \times SL(2\mathbb{R})$$

$$Y_m \to Y_{a\dot a} \qquad (\vec{q}_1 \pm \vec{q}_4)^2 = \vec{q}_2^{\,2} = \vec{q}_3^{\,2} = 0$$

$$\vec{q}_i^{\,2} = 0 \Longrightarrow q_i \Leftrightarrow q_{\alpha\dot\alpha}$$

$$W_{\alpha\dot\alpha,a\dot a} = \begin{pmatrix} W_{11a\dot a} & W_{12a\dot a} \\ W_{21a\dot a} & W_{22a\dot a} \end{pmatrix} = \begin{pmatrix} (q_1+q_4)_{a\dot a} & q_{2a\dot a} \\ q_{3a\dot a} & (q_1-q_4)_{a\dot a} \end{pmatrix}$$

$$\partial_z W_{\alpha\dot\alpha,a\dot a} = -(B_z^L)_\alpha{}^\beta W_{\beta\dot\alpha,a\dot a} - (B_z^R)_{\dot\alpha}{}^{\dot\beta} W_{\alpha\dot\beta,a\dot a}$$

$$B_z^L = \begin{pmatrix} \frac{1}{2}\alpha_z & -e^{\alpha} \\ -pe^{\alpha} & -\frac{1}{2}\alpha_z \end{pmatrix} \qquad B_z^R = \begin{pmatrix} -\frac{1}{2}\alpha_z & e^{-\alpha}p \\ -e^{\alpha} & \frac{1}{2}\alpha_z \end{pmatrix}$$

両立条件: $\partial_z B_{\bar z}^L - \partial_{\bar z} B_z^L + [B_z^L, B_{\bar z}^L] = 0$, $L \leftrightarrow R$

$\Longrightarrow$ Generalized Sinh-Gordon 方程式

$$\partial_z\partial_{\bar z}\alpha - e^{2\alpha} + |p(z)|^2 e^{-2\alpha} = 0$$

$$\partial_{\bar z} p = \partial_z \bar p = 0$$

$\Longrightarrow$ auxiliary linear problem : $\partial_z \psi_\alpha^L + (B_z^L)_\alpha{}^\beta \psi_\beta^L = 0$

$$\partial_{\bar z} \psi_\alpha^L + (B_{\bar z}^L)_\alpha{}^\beta \psi_\beta^L = 0 \quad , \quad (L \leftrightarrow R)$$

解 $\psi_{\alpha,a}^L$ $(a=1,2)$, 内積 $\langle \psi_a^L, \psi_b^L \rangle = \varepsilon^{\alpha\beta}\psi_{\alpha a}^L \psi_{\beta b}^L$

$$W_{\alpha\dot\alpha,a\dot a} = \Psi^L_{\alpha a}\Psi^R_{\dot\alpha\dot a}\ , \quad Y_{a\dot a} = \Psi^L_{\alpha a} M_1^{\alpha\dot\beta}\Psi^R_{\dot\beta\dot a}$$

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spectral parameter ζ

$$B_z(\zeta) = \begin{pmatrix} \frac{1}{2}\alpha_z & -\frac{1}{\zeta}e^{\alpha} \\ -\frac{1}{\zeta}e^{-\alpha}p & -\frac{1}{2}\alpha_z \end{pmatrix} \qquad B_{\bar z}(\zeta) = \begin{pmatrix} -\frac{1}{2}\alpha_{\bar z} & -\zeta e^{-\alpha}p \\ -\zeta e^{\alpha} & \frac{1}{2}\alpha_{\bar z} \end{pmatrix}$$

$$\partial_{\bar z}B_z - \partial_z B_{\bar z} + [B_z, B_{\bar z}] = 0$$

$$B_z = A_z + \frac{1}{\zeta}\Phi_z\ , \quad B_{\bar z} = A_{\bar z} + \zeta\Phi_{\bar z}$$

Hitchin system ~~problem~~ $\quad D_{\bar z}\phi_z = D_z\phi_{\bar z} = 0 \qquad B^L_z = B_z(1)$

$$F_{z\bar z} + [\phi_z, \phi_{\bar z}] = 0 \qquad B^R_{\bar z} = U B_z(i) U^{-1}$$

$$U = \begin{pmatrix} 0 & e^{+i\pi/4} \\ e^{3i\pi/4} & 0 \end{pmatrix}$$

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後半の目標 Remainder function (2n-pt, n:odd)

$$R_{2n} = \frac{7\pi}{12}(n-2) + \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} d\theta \sum_{s=1}^{n-3} |m_s|\cosh\theta \log(1+\tilde Y_s(\theta))}_{\text{TBA free energy}}$$

$$- \frac{1}{4}\sum_{k=1}^{\frac{n-3}{2}}\sum_{i=k}^{\frac{n-3}{2}} (-1)^{i+k}(m_{2i}\bar m_{2k-1} + \bar m_{2i} m_{2k-1}) + \frac{1}{4}\sum_{i,j=1}^{n} \log\frac{C^+_{i,j}}{C^+_{i,j+1}} \log\frac{C^-_{i-1,j}}{C^-_{i,j}}$$

$$C^+_{ij} = T^{[i+j]}_{|i-j|-1}$$

$$C^-_{ij} = T^{[i+j+1]}_{|i-j|-1} \qquad n\text{が even の時も似た式}$$

$$(\partial_z + B_z(\zeta))\,\psi(z,\bar z,\zeta) = 0$$

$$(\partial_{\bar z} + B_{\bar z}(\zeta))\,\psi(z,\bar z,\zeta) = 0 \qquad P(z):\ (n-2)\text{次多項式} \quad (2n\text{-ptのとき})$$

$$dw = \sqrt{P(z)}\,dz \quad , \ y^2 = P(z)$$

変数変換 $\quad (\partial_w + \hat B_w)\hat\psi = 0$

$$(\partial_{\bar w} + \hat B_{\bar w})\hat\psi = 0$$

$$\alpha \to \alpha - \tfrac{1}{4}\ln P\bar P = \hat\alpha \qquad \partial_w\partial_{\bar w}\hat\alpha - e^{2\hat\alpha} + e^{-2\hat\alpha} = 0$$

4点 $\quad n=2 \quad w=z$

b.c. $\quad \hat\alpha = 0 \quad (w\to\infty) \qquad B_w = \frac{1}{\zeta}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} \qquad B_{\bar w} = \zeta\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$

$$\hat\eta_+ = \begin{pmatrix} e^{\frac{w}{\zeta}+\bar w\zeta} \\ 0 \end{pmatrix} \qquad \hat\eta_- = \begin{pmatrix} 0 \\ e^{-\frac{w}{\zeta}-\bar w\zeta} \end{pmatrix}$$

$\psi^L_a = c^+_a \hat\eta^L_+ + c^-_a \hat\eta^L_- \qquad$ similarly for R

$$Y_{a\dot a} = \frac{1}{\sqrt2}\begin{pmatrix} e^{u} & e^{-v} \\ -e^{v} & e^{-u}\end{pmatrix} \qquad u = w+\bar w + \frac{w-\bar w}{i}, \quad v = -(w+\bar w) + \frac{w-\bar w}{i}$$

$$Y_{-1}^2 - Y_2^2 = Y_0^2 - Y_1^2$$

embedding 座標 $\longrightarrow$ Poincaré 座標

$$Y_{-1} + Y_2 = \frac{1}{r}, \quad x^\pm = \frac{Y_1 \pm Y_0}{Y_{-1}+Y_2}$$

$$\frac{1}{r} = \frac{1}{\sqrt2}e^{u}, \quad x^+ = e^{-v-u} \qquad x^+x^- = -\frac{r^2}{2}$$

$$x^- = -e^{v-u}$$

$$P(z) = z^{n-2} + \cdots$$

$$\frac{x^\pm_{ij}\,x^\pm_{kl}}{x^\pm_{ik}\,x^\pm_{jl}} = \frac{\langle s_i, s_j\rangle\,\langle s_k, s_l\rangle}{\langle s_i, s_k\rangle\,\langle s_j, s_l\rangle} \qquad (\zeta = 1, \zeta = i)$$

T-system, Y-system

$T_s(\theta)$, $Y_s(\theta)$ $\quad (s=1, \cdots n-3)$

$\zeta = e^{\theta}$ $\quad T_s^+ T_s^- = 1 + T_{s-1} T_{s+1}$, $\quad Y_s^+ Y_s^- = (1+Y_{s+1})(1+Y_{s-1})$

$f^{\pm}(\theta) = f(\theta \pm \frac{i\pi}{2})$

odd $n \longrightarrow T_0 = T_{n-2} = 1, \quad Y_0 = Y_{n-2} = 0$

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2-loop Heslop - Khoze

$$R_{2n}^{2\text{-loop}} = -\frac{1}{2} \sum_S \log(u_{i_1 i_5}) \log(u_{i_2 i_6}) \log(u_{i_3 i_7}) \log(u_{i_4 i_8}) - \frac{\pi^4}{36}(n-2)$$

$S = \{1 \le i_1 \le \cdots i_8 \le 2n\}$

$i_k - i_{k-1} = \text{odd}$

$$u_{ij} = \frac{x_{i\,j+1}^2 \, x_{i+1\,j}^2}{x_{ij}^2 \, x_{i+1\,j+1}^2}$$

Rescaled Remainder function

$$\overline{R_{2n}} = \frac{R_{2n} - R_{2n,reg}}{R_{2n,reg} - (n-2) R_{6,reg}}$$

large mass $\overline{R_{2n}} \to -1$

small mass $\overline{R_{2n}} \to 0$

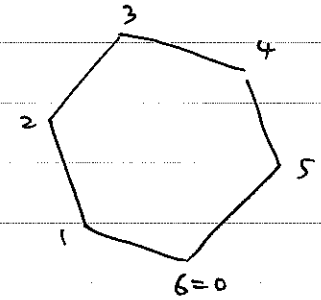
$\overline{R_8}^{strong} / \overline{R_8}^{2\text{-loop}} \longrightarrow 1.025[illegible]7$ (@ small mass)

$\overline{R_{10}}^{strong} / \overline{R_{10}}^{2\text{-loop}} \longrightarrow 0.984137$

14 $\longrightarrow 0.946321$

$2n \quad (n \to \infty) \longrightarrow 0.9048\cdots$

$$R_{2n} \sim a_0 n + a_1 + \frac{a_1}{n} + m^{4(1-\Delta)} \left(\frac{b_1}{n} + \frac{b_2}{n^2} + \cdots \right)$$



$$\langle S_{-k-1}, S_{k+1}\rangle(\zeta) = T_{2k+1}(\zeta)$$

$$\langle S_{-k-1}, S_k^+\rangle(\zeta) = T_{2k}(\zeta)$$

$$T_s^+ T_s^- \cancel{= 1 + T_{s-1} T_{s+1}}$$

wronskian $\rightarrow$ T- function

cross-ratio $\rightarrow$ Y- function

$$\log Y_{2k} \sim Z_{2k}/\zeta \qquad \log Y_{2k+1} \sim Z_{2k+1}/i\zeta$$

$$Z_{2k} = -\int_{\gamma_{2k}} \sqrt{p}\, dz, \quad Z_{2k+1} = -\int_{\gamma_{2k+1}} \sqrt{p}\, dz$$

$\Rightarrow$ TBA eq

$$\text{Area} = \int d^2z\, e^{2\alpha} = \int d^2z\, \mathrm{tr}(\Phi_z \Phi_{\bar z})$$

$$\log Y_s \sim -\frac{1}{\zeta}\int_{\gamma_s} \sqrt{p}\, dz + \zeta \int_{\gamma_s} \phi$$

$$\underbrace{\sim \text{Free energy} + \int \sqrt{p\bar p}\, d^2z}_{R^{\text{strong}}_{2n}} + \underbrace{\text{divergent} + \text{cross}}_{M^{BDS} + \Delta A_{BDS}}$$

$$\int \sqrt{p\bar p}\, d^2z \downarrow m^2$$