Instanton partition function (with a general surface operator) and Whittaker vector of W algebra

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Instanton partition function

The instanton expansion of the partition function of $\mathcal{N}=2$ SUSY Yang-Mills theory in four dimensions;

$$Z_{\mathrm{inst}}(a,\Lambda)=1+\sum_{k=1}^{\infty}\Lambda^k n_k(a), \qquad n_k(a)=\int_{\mathcal{M}_k} e^{-S_{\mathrm{top}}+[Q_B,V]} .$$

The integral over the moduli space $\mathcal{M}_k$ can be computed by the localization theorem with the "Ω-background" (ϵ_1,ϵ_2), that is a kind of regularization parameters.

We obtain Nekrasov's partition function

$$Z_{\mathrm{Nek}}(a,\epsilon_{1,2};\Lambda)=\exp\left(-\frac{1}{\epsilon_1\epsilon_2}\mathcal{F}_{SW}(a;\Lambda)+\cdots\right).$$

U(1) partition function by a coherent state

For U(1) theory with $\hbar = \epsilon_1 = -\epsilon_2$;

$$
\begin{aligned}
Z_{U(1)}(\hbar;\Lambda) &= \sum_{k=0}^{\infty}\left[\sum_{|\lambda|=k}\frac{1}{\prod_{\square\in\lambda}(\hbar h(\square))^2}\right]\Lambda^{2k} = \exp\left(\frac{\Lambda^2}{\hbar^2}\right) \\
&= \langle\Psi|\Psi\rangle,
\end{aligned}
$$

where the fixed points are labeled by a single Young diagram $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d)$ and $h(\square)$ denotes the hook length.

We can find a coherent state (a Whittaker vector) $|\Psi\rangle = \exp\left(\frac{\Lambda}{\hbar}J_{-1}\right)|0\rangle$ in the Verma module of the Heisenberg algebra $[J_n, J_m] = \delta_{n+m,0}$, or the Fock space of a free boson.

The coherent state for $Z_{U(1)}$

By the boson-fermion correspondence $J_n = \sum_{m \in \mathbb{Z}+\frac{1}{2}} :\psi^*_{n-m}\psi_m:$,

$$J_{-1}^k|0\rangle = \sum_{|\lambda|=k} \frac{k!}{\prod_{\square\in\lambda} h(\square)} |\lambda\rangle_F = \sum_{|\lambda|=k} \dim R_\lambda \cdot |\lambda\rangle_F \ .$$

Hence,

$$|\Psi\rangle = \exp\left(\frac{\Lambda}{\hbar}J_{-1}\right)|0\rangle = \sum_{k=0}^{\infty} \frac{\Lambda^k}{\hbar^k} \sum_{|\lambda|=k} \frac{1}{\prod_{\square\in\lambda} h(\square)} |\lambda\rangle_F .$$

We can easily compute the norm of $|\Psi\rangle$;

$$\langle\Psi|\Psi\rangle = \langle 0|e^{\frac{\Lambda^2[J_1,J_{-1}]}{\hbar^2}}|0\rangle = \exp\left(\frac{\Lambda^2}{\hbar^2}\right) .$$

Whittaker vecotrs for instanton partition function

The coherent state $|\Psi\rangle$ is an eigenstate of J_1;

$$J_1|\Psi\rangle = \frac{\Lambda}{\hbar}|\Psi\rangle.$$

In general a Whittaker vector is characterized as a simultaneous eigenvector of annihilation operators.

$$\chi : \mathfrak{n}_- \to \mathbb{C}, \qquad a|W\rangle = \chi(a)|W\rangle, \quad (a \in \mathfrak{n}_-)$$

$\chi \equiv 0$ for a highest weight vector.

We will propose a similar definition for $|\Psi\rangle$ with $Z_{\mathrm{inst}}^{\vec{n}}(a_\alpha, \epsilon_{1,2}; \vec{q}) = \langle\Psi|\Psi\rangle$ in $SU(N)$ theory with a (general) surface operator.

Definition of the surface operator

The surface operator is a half-BPS object in SUSY Yang-Mills theory. [Gukov-Witten, hep-th/0612073]

A surface operator at $w = 0$, filling the z-plane induces a singular behavior of $SU(N)$ gauge field A_μ on $\mathbb{R}^4 \simeq \mathbb{C}^2 \ni (z, w)$;

$$A_\mu dx^\mu \sim \mathrm{diag}(\alpha_1, \alpha_2, \cdots, \alpha_N)\sqrt{-1}d\theta, \qquad r \to 0$$

(r, θ) : the polar coordinates of the w-plane

$\vec{\alpha} := \mathrm{diag}(\alpha_1, \alpha_2, \cdots, \alpha_N)$ is a monodromy around $w = 0$.

Classification of the surface operator

Suppose $\vec{\alpha}$ has the structure with $\alpha_{(\ell)} > \alpha_{(\ell+1)}$,

$$\vec{\alpha} = (\underbrace{\alpha_{(1)}, \ldots, \alpha_{(1)}}_{n_1 \text{ times}}, \underbrace{\alpha_{(2)}, \ldots, \alpha_{(2)}}_{n_2 \text{ times}}, \ldots, \underbrace{\alpha_{(M)}, \ldots, \alpha_{(M)}}_{n_M \text{ times}}),$$

where (n_ℓ) is a composition of N, not a partition of N.

Then the gauge group $SU(N)$ is broken to the Levi subgroup $\mathbb{L}$ (=the commutant of $\vec{\alpha}$) on the surface:

$$SU(N) \supset \mathbb{L} = S[U(n_1) \times U(n_2) \times \cdots \times U(n_M)] \supset U(1)^{M-1}.$$

The surface operator is classified by $\mathbb{L}$ which depends only on the partition $[n_\ell]$ of N.

Partition function with a surface operator

Instanton partition function with a surface operator of type $\vec{n} = (n_1, n_2, \cdots, n_M)$;

$$Z_{\text{inst}}^{\vec{n}}(a_\alpha, \epsilon_{1,2}; \vec{q}) = 1 + \sum_{\vec{k}} \vec{q}^{\vec{k}} \int_{\mathcal{M}_{\vec{n},\vec{k}}} e^{-[Q_B, V]}$$

$\mathcal{M}_{\vec{n},\vec{k}}$: Instanton moduli space
$\vec{k} = (k_1, k_2, \cdots, k_M) \in (\mathbb{Z}_{\geq 0})^M$: topological numbers
$\vec{q} = (q_1, q_2, \cdots, q_M)$: parameters of topological expansion

a_α : VEV of $U(1)^{N-1}$ (the Coulomb moduli); $\sum_{\alpha=1}^{N} a_\alpha = 0$

Surface operator as a defect in six dimensional theory

In Gaiotto's construction [0904.2715], four dim. theory is identified with (twisted) compactification of the six dim. $\mathcal{N} = (2,0)$ theory on a punctured Riemann surface C.
The surface operator can be realized as either codimension *four* defects or codimension *two* defects.
[Drukker-Gaiotto-Gomis,1003.1112; Alday-Tachikawa,1005.4468]

- Codim. four defect = an $M2$ brane ending on N $M5$'s
- Codim. two defect = an $\widetilde{M5}$ brane intersecting with N $M5$'s

The second view point implies that the presence of the surface operator can be described by a $\mathbb{Z}_M$ orbifold action on $\mathbb{C}^2 \ni (z, w)$ by $(z, w) \to (z, \omega w)$ with $\omega = \exp(2\pi i/M)$.

ADHM data and ADHM conditions

ADHM data : (A, B, P, Q)

W and V : the Chan-Paton spaces of D4 and D0 branes

$$\dim W = N = \text{the number of D4 branes}$$
$$\dim V = k = \text{the number of D0 branes}$$

$A, B \in \mathrm{Hom}\,(V, V)$ from "0-0" string
$P \in \mathrm{Hom}\,(W, V),\ Q \in \mathrm{Hom}\,(V, W)$ from "4-0" and "0-4" string

ADHM equation : BPS condition for the effective theory on $D0$

$$[A, B] + PQ = 0\ , \quad \text{F-term condition}$$
$$[A, A^\dagger] + [B, B^\dagger] + PP^\dagger - Q^\dagger Q = 0\ . \quad \text{D-term condition}$$

Orbifold decomposition of the Chan-Paton space

In the presence of the surface operator we have the orbifold action $(z, w) \to (z, \omega w)$. The vector spaces V and W are decomposed according to the representation under $\mathbb{Z}_M$ action,

$$W = \bigoplus_{\ell=1}^{M} W_\ell, \qquad V = \bigoplus_{\ell=1}^{M} V_\ell, \qquad \dim W_\ell = n_\ell, \qquad \dim V_\ell = k_\ell.$$

n_ℓ : the number of D4-branes on which $\mathbb{Z}_M$ acts as ω^ℓ.

k_ℓ : the number of fractional instantons (D0-branes).

$a_1, \ldots, a_N \Longrightarrow a_{s,\ell},\ 1 \leq s \leq n_\ell$: eigenvalues of $SU(N)$ acting on $W_\ell,\ 1 \leq \ell \leq M$.

Orbifold quotient of ADHM data

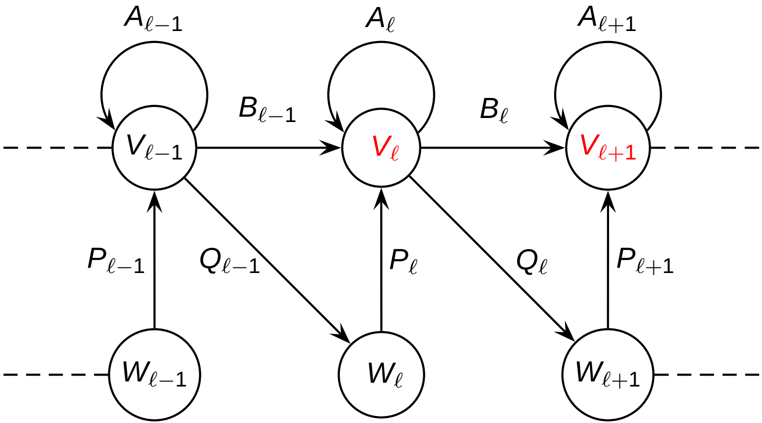
The toric action of $\mathbb{C}^2$ on the ADHM data is
$(A, B, P, Q) \to (e^{\epsilon_1} \cdot A, e^{\epsilon_2} \cdot B, P, e^{\epsilon_1+\epsilon_2} \cdot Q)$.

$\Longrightarrow$ Under the $\mathbb{Z}_M$ orbifold action $(z, w) \to (z, \omega w)$,
$A_\ell \in \mathrm{Hom}\,(V_\ell, V_\ell)$, $B_\ell \in \mathrm{Hom}\,(V_\ell, V_{\ell+1})$,
$P_\ell \in \mathrm{Hom}\,(W_\ell, V_\ell)$ and $Q_\ell \in \mathrm{Hom}\,(V_\ell, W_{\ell+1})$ survive.

$$A_{\ell+1}B_\ell - B_\ell A_\ell + P_{\ell+1}Q_\ell = 0\,, \quad \text{F-term condition}$$

We obtain the chain-saw quiver. [Finkelberg-Rybnikov, 1009.0676]
(Compare it with the (Kronheimer-Nakajima) quiver for ADHM on ALE by another $\mathbb{Z}_M$ orbifold action $(z, w) \to (\omega z, \omega^{-1} w)$)

A part of $\mathbb{Z}_M$ chain-saw quiver



Formula of the character

$\mathbb{Z}_M$ invariant part of the character at a fixed point $\vec{Y}$ is

$$\chi_{\vec{Y}} = \sum_{\ell=1}^{M} \Big[-(1-e^{\epsilon_1}) V_\ell^* V_\ell + (1-e^{\epsilon_1}) e^{\frac{\epsilon_2}{M}} V_{\ell-1}^* V_\ell + W_\ell^* V_\ell + e^{\epsilon_1 + \frac{\epsilon_2}{M}} V_{\ell-1}^* W_\ell \Big]$$

where we identified the vector spaces and their characters,

$$V_\ell = \sum_{\tilde{\ell}=1}^{M} \sum_{s=1}^{n_{\ell-\tilde{\ell}+1}} e^{\lfloor \frac{\ell-\tilde{\ell}}{M} \rfloor \epsilon_2 - \frac{\ell}{M}\epsilon_2 + a_{s,\ell-\tilde{\ell}+1}} \sum_{(i,\, j\cdot M+\tilde{\ell}) \in Y^{s,\ell-\tilde{\ell}+1}} e^{(1-i)\epsilon_1 - j\epsilon_2},$$

$$W_\ell = \sum_{s=1}^{n_\ell} e^{-\frac{\ell}{M}\epsilon_2 + a_{s,\ell}}$$

Expansion of this formula reproduces the conjecture made by Wyllard [1012.1355].

Partition function from the character

The character $\chi_{\vec{Y}}$ at a fixed point $\vec{Y}$ after expansion,

$$\chi_{\vec{Y}} = \sum_{i=1}^{\dim(\vec{Y})} e^{w_i(\vec{Y})}, \qquad \dim(\vec{Y}) \equiv \sum_{\ell=1}^{M} (n_\ell + n_{\ell+1}) k_\ell(\vec{Y}),$$

where w_i are integral linear combinations of $a_{s,l}$ and $\epsilon_{1,2}$.

Localization $Z \sim \int d\sigma \frac{\Delta_F(\sigma)}{\Delta_B(\sigma)} e^{-S_{top}(\sigma)}$;

$$Z^{\vec{n}}_{\text{inst}}(a_{s,\ell}, \epsilon_{1,2}; \vec{q}) = \sum_{\vec{Y}} \frac{1}{\prod_{i=1}^{\dim(\vec{Y})} w_i(\vec{Y})} \prod_{\ell=1}^{M} q_\ell^{k_\ell(\vec{Y})}.$$

It turns out that the partition function $Z^{\vec{n}}_{\text{inst}}(a_{s,\ell}, \epsilon_{1,2}; \vec{q})$ depends on the ordering of $\vec{n} = (n_1, \ldots, n_M)$.

W-algebra from the current algebra

the $\mathfrak{sl}(N)$ current algebra $\Longrightarrow$ the W-algebra $W(\widehat{SU(N)}, [n_\ell])$

by the quantum Hamiltonian (Drinfeld-Sokolov) reduction with an enbedding $\rho : SU(2) \to SU(N)$, where $N = \underline{n_1} \oplus \underline{n_2} \oplus \cdots \oplus \underline{n_M}$ under the subgroup $\rho(SU(2))$.

$$W(\widehat{SU(N)}, [N]) \simeq W_N \text{ algebra}, \quad W(\widehat{SU(N)}, [1^N]) \simeq A^{(1)}_{N-1}$$

$W(\widehat{SU(N)}, [n_\ell]) \oplus \mathrm{Heis.} \simeq W(\widehat{U(N)}, [n_\ell])$ is generated by

$$\{U^{\ell}_{\tilde{\ell},(s)}(z)\}, \quad 1 \leq \ell, \tilde{\ell} \leq M,$$

$$s = \frac{|n_\ell - n_{\tilde{\ell}}|}{2} + 1, \frac{|n_\ell - n_{\tilde{\ell}}|}{2} + 2, \cdots, \frac{n_\ell + n_{\tilde{\ell}}}{2}$$

Highest weight representations

In the R-sector, the universal enveloping algebra is generated by $U^{\ell}_{\tilde{\ell},(s),n}$ $(n \in \mathbb{Z})$. Let $|u_{\ell,(s)}\rangle$ be a highest weight vector with eigenvalues $u_{\ell,(s)}$ of $U^{\ell}_{\ell,(s),0}$ (no summation on ℓ). We declare

$$U^{\ell}_{\tilde{\ell},(s),n}|u_{\ell,(s)}\rangle = 0$$

for $\ell > \tilde{\ell}, n \geq 0$ or $\ell \leq \tilde{\ell}, n > 0$.

The Verma module $\mathcal{V}$ depends on the composition (n_ℓ) of N.

Definition of the Whittaker vector

In general a coherent state is defined by

$$U^{\ell}_{\tilde{\ell},(s),n}|\Psi\rangle = u^{\ell}_{\tilde{\ell},(s),n}|\Psi\rangle, \qquad U^{\ell}_{\tilde{\ell},(s),n} \text{ : annihilation operators}$$

A choice $u^{\ell}_{\tilde{\ell},(s),n} \neq 0$ only for $U^{\ell+1}_{\ell,(s),0}$, $1 \leq \ell \leq M-1$ and $U^{1}_{M,(s),1}$ is consistent with the commutation relations.

$$Z^{\vec{n}}_{\text{inst}}(a_{s,\ell}, \epsilon_{1,2}; \vec{q}) = \sum_{\vec{Y}} \frac{1}{\prod_{i=1}^{\dim(\vec{Y})} w_i(\vec{Y})} \prod_{\ell=1}^{M} q_{\ell}^{k_{\ell}(\vec{Y})}.$$

where q_ℓ has mass dimension $n_\ell + n_{\ell+1} = 2s_{\max}$.
Mass dimension of 4D theory = Scaling dimension of 2D theory.
Hence we propose

$$U^{\ell+1}_{\ell,(s_{\max}),0}|\Psi\rangle = c_\ell\sqrt{q_\ell}|\Psi\rangle, \qquad U^{M+1}_{M,(s),0} \equiv U^{1}_{M,(s),1}$$

Examples of the Whittaker vector

1. $(3) \Longrightarrow W_3$ algebra,
   [Mironov-Morozov, 0908.2596, Taki, 0912.4789]
   $$W_1^{(3)}|\Psi\rangle \sim \sqrt{q_1}|\Psi\rangle, \quad [q_1] = 6$$
2. $(1,1,1) \Longrightarrow SU(3)$ current algebra,
   [Kozcaz-Pasquetti-Passerini-Wyllard, 1008.1412]
   $$J_0^{12}|\Psi\rangle \sim \sqrt{q_1}|\Psi\rangle, \quad J_0^{23}|\Psi\rangle \sim \sqrt{q_2}|\Psi\rangle, \quad J_1{}^{31}|\Psi\rangle \sim \sqrt{q_3}|\Psi\rangle$$
   $$[q_1] = [q_2] = [q_3] = 2$$
3. $(2,1) \Longrightarrow W_3^{(2)}$ algebra, [Wyllard, 1011.0289]
   $$G_0^{+}|\Psi\rangle \sim \sqrt{q_1}|\Psi\rangle, \quad G_1{}^{-}|\Psi\rangle \sim \sqrt{q_2}|\Psi\rangle, \qquad [q_1] = [q_2] = 3$$
   $$(1,2) \Longrightarrow G^{\pm} \leftrightarrow G^{\mp}$$