F - Theory and Grand Unificatio

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1. Motivation
2. F - Theory
3. Partially Twisted 7-brane Worldvolume Theory
4. Matter Curves
5. Yukawa Couplings

1. Motivation

Standard Model + SUSY

$\downarrow$

Gauge Coupling Unification

Grand Unification

$\downarrow$

GUT $\in$ String Landscape ?

This is a long-standing problem.
The construction of MSSM and GUTs has been tried in

- Heterotic Strings on Calabi-Yau 3-folds,
- Intersection Branes in Type II String Theory,

and so on.

Heterotic strings suffer from the moduli problem

Flux compactifications in type IIB resolve the moduli problem.

In fact, the Gukov-Vafa-Witten potential

4-form flux $G_4 = dC_3$

$$\int G_4 \wedge \underset{\sim}{\Omega}$$

(4,0)-form of CY_4

gives masses to the complex structure moduli of CY_4.

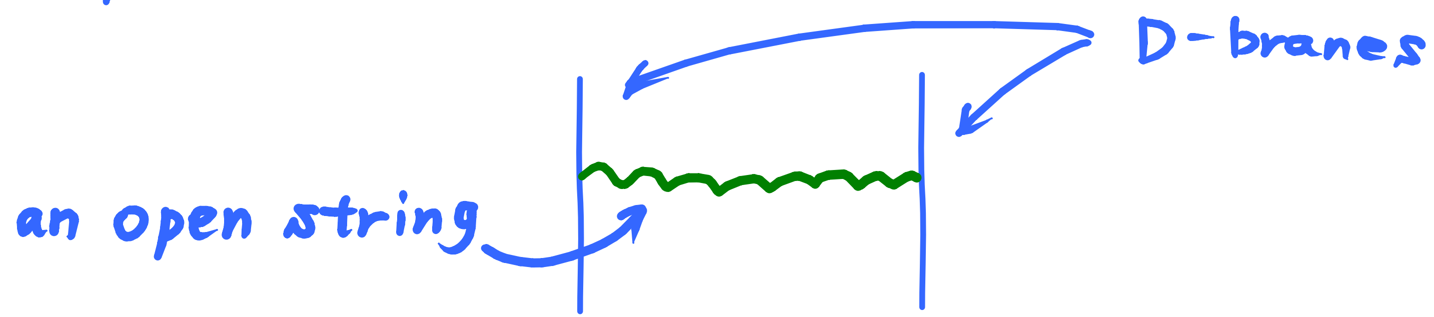
In type IIB, a grand unified model is the worldvolume theory on branes.

Yukawa couplings are

$$W \sim \underbrace{f_{abc}}_{\substack{\text{structure const.}\\ \text{of broken}\\ \text{gauge group } G_{int}}} \; \mathrm{tr}_{\underbrace{H}_{\substack{\text{unbroken}\\ \text{gauge group}\\ H}}} \left[\Phi^a \Sigma^b \Phi^c \right]$$

under $G \rightarrow H \; (\times G_{int})$

Only D-branes and/or orientifolds give only SU(n) or SO(n) with their representations of at most two indices



(perturbative) type IIB

→ No up-type Yukawa coupling

$$\varepsilon_{ijklm}\, 10_M^{ij} \cdot 10_M^{kl} \cdot 5_H^{m}$$

the structure constant of $G \supset E_6$

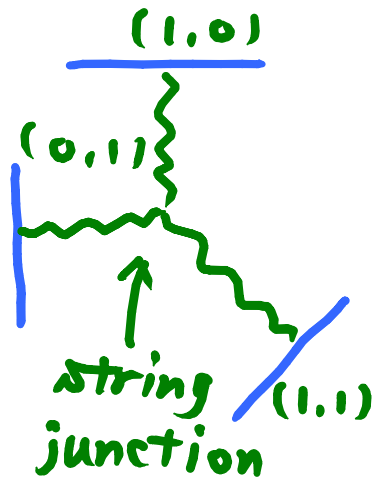
But, mutually non-local (p,q) 7-branes can be connected by string junctions so that they can yield

spin reps of $SO(n)$

and

$E_{6,7,8}$ gauge group

to give the up type Yukawa couplings of GUT models.



Upon compactification of Type IIB to 4 dimensions, on a 6-dim manifold B_6,

F-theory is the Type IIB theory with

$$\tau(u) = \underbrace{C_0}_{\text{RR 0-form}}(u) + i/\underbrace{g_B}_{\text{dilaton}}(u)$$

varying over B_6. $(u \in B_6)$

Mutually non-local (p, q) 7-branes, non-perturbative objects in Type IIB theory, are encoded geometrically in F-theory.

An F-theory GUT is an 8-dim. gauge theory on the worldvolume of the 7-branes.

Advantages

- possible mechanism to fix all complex structure moduli
- available Yukawa couplings at the classical level

Reference

- Donagi & Wijnholt, arXiv:0802.2969, ...
- Beasley, Heckman & Vafa, arXiv:0802.3391, ...

  My presentation is based mainly on this paper.
- Hayashi, Tatar, Toyoda, Watari, & Yamazaki, arXiv:0805.1057
- Hayashi, Kawano, Tatar, & Watari, arXiv:0901.4941
- Donagi & Wijnholt, arXiv:0904.1218
- Cecotti, Cheng, Heckman & Vafa, arXiv:0910.0477
- Cecotti, Cordova, Heckman & Vafa, arXiv:1010.5780

and many other important papers.

2. F-theory

F-theory may be obtained as a limit of M-theory.

As a warm-up, let us consider M-theory compactified on $T^2 \times B_6$.

Then, one finds

$$\boxed{\begin{tikzcd} \mathrm{M}/T^2 \times B_6 \arrow[d, phantom, "\wr"] & \\ \mathrm{IIA}/S^1 \times B_6 \arrow[r, "T\text{-dual}"] & \mathrm{IIB}/S^1 \times B_6 \arrow[d, "\text{decompactify}"] \\ & \mathrm{IIB}/B_6 \end{tikzcd}}$$

On the $M/T^2 \times B_6$ side,

the metric is given by

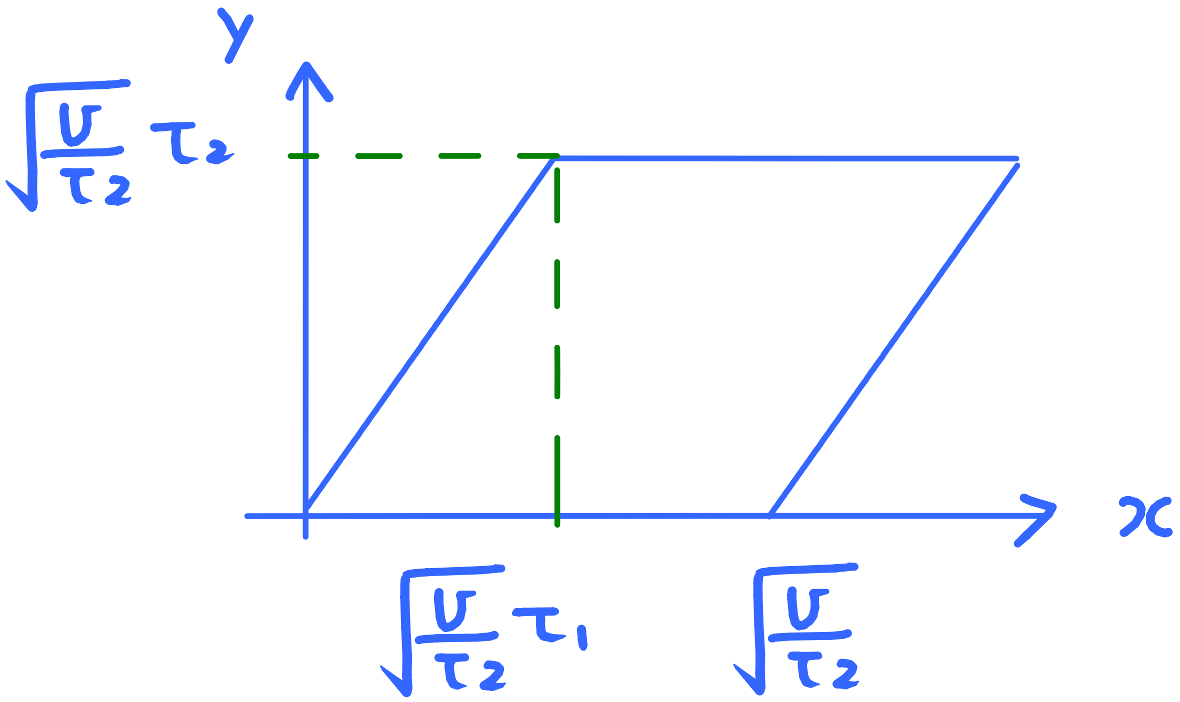
$$ds_M^2 = \underbrace{-(dx^0)^2 + (dx^1)^2 + (dx^2)^2}_{(2+1)\,\mathrm{dim\ Minkowski}}$$

$$+ ds_{B_6}^2$$

$$+ \underbrace{\frac{V}{\tau_2}\left[(dx + \tau_1 dy)^2 + \tau_2^2 dy^2\right]}_{T^2}$$

One can see that the T^2 has

$$\begin{cases} \text{area} & U \\ \text{complex structure} & \tau = \tau_1 + i\tau_2 \end{cases}$$



Via the above chain of dualities,
one obtains in terms of IIB strings

- the metric in the Einstein frame

$$ds_B^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + \frac{\ell_s^4}{v}(dy)^2 + ds_{B_6}^2$$

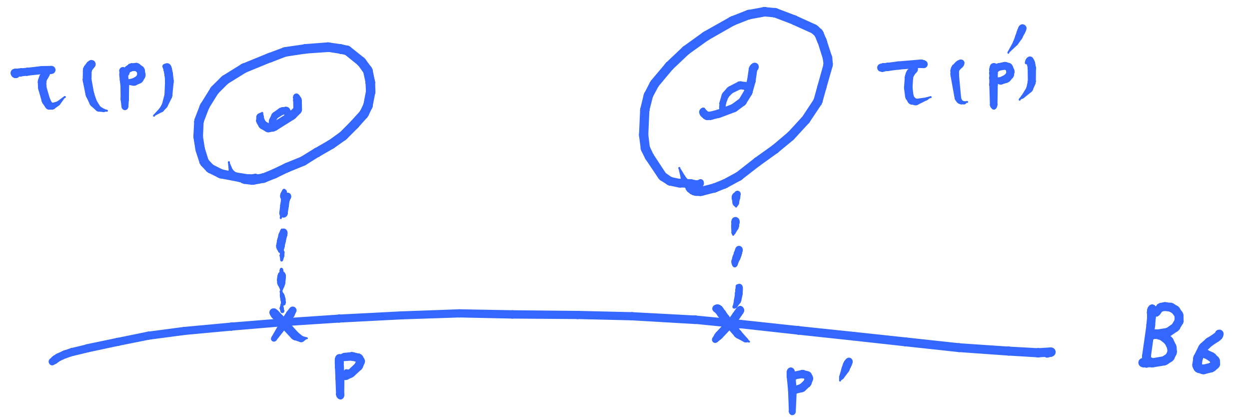
- the 0-form RR gauge field & the dilaton

$$C_0 + \frac{i}{g_B} = \tau = \tau_1 + i\tau_2$$

The $v \to 0$ limit yield 4-dim. IIB$/B_6$.

The generalization of this argument to a T^2-fibration over B_6 makes the complex structure τ dependent on the local complex coordinates (u, v, z) of B_6 as

$$\tau \to \tau(u, v, z),$$



and so the 0-form gauge field C_0 and the dilaton g_B vary over B_6 as

$$C_0 \longrightarrow C_0(u,v,z)$$
$$g_B \longrightarrow g_B(u,v,z)$$

Therefore, in the $v \to 0$ limit, one obtains 4-dim. IIB$/B_6$ with C_0 and g_B varying over B_6.

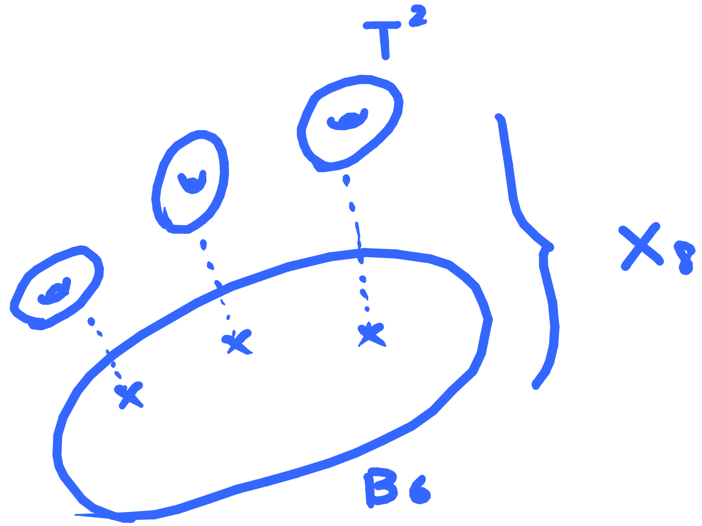
Then, F-theory is defined as the $v \to 0$ limit

$IIB / \underbrace{B_6}_{\text{Kähler mfd}}$ w/ $C_0(u,v,z)$, $g_B(u,v,z)$

$\cong$

$\underbrace{F / X_8}$

F-theory compactified on an elliptically fibered X_8



$\mathcal{N} = 1$ supersymmetry

in $(3+1)$ dimensions

$$\Downarrow$$

X_8 must be

a Calabi-Yau 4-fold

Let us explain another description of the elliptically fibered X_8.

A 2-$\dim_{\mathbb{R}}$ torus T^2 has another name — an elliptic curve and is described on $\mathbb{C}^2$ as a constraint

$$\boxed{y^2 = x^3 + fx + g}$$

with $(x, y) \in \mathbb{C}^2$.

In fact, since

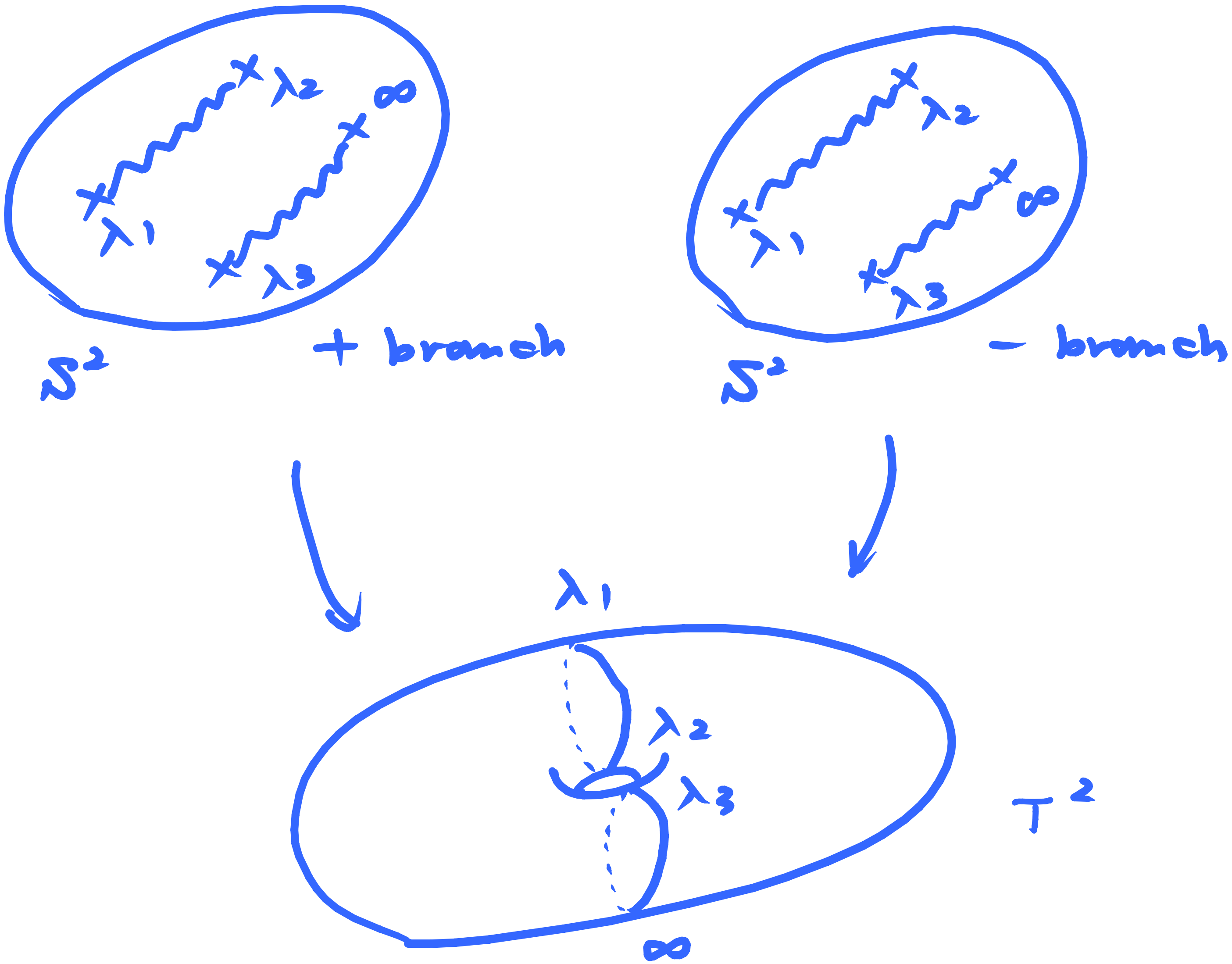
$$x^3 + fx + g = (x-\lambda_1)(x-\lambda_2)(x-\lambda_3),$$

the constraint means that y is a double-valued function of x;

$$y = \pm\sqrt{(x-\lambda_1)(x-\lambda_2)(x-\lambda_3)}$$

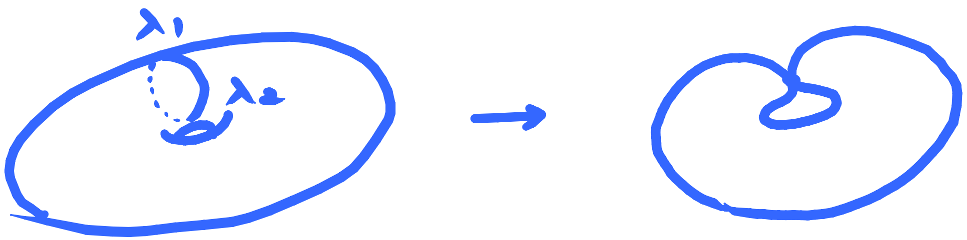
Therefore, introduce two copies of 2-dim. sphere as the domain to make $y = f(x)$ a single-valued function

and further introduce two cuts on each of the spheres (branches) to paste them along the cuts. They end up with a torus, as one expects.



λ_2
∞
λ_1
λ_3
S^2
+ branch
λ_2
∞
λ_1
λ_3
S^2
− branch
λ_1
λ_2
λ_3
T^2
∞

As can be seen from the figure, when the roots $\lambda_{1,2,3}$ degenerate, say, $\lambda_1 = \lambda_2$, one of the one-cycles of T^2 collapses.



It occurs if and only if the discriminant

$$\boxed{\begin{aligned} \Delta &= 27 g^2 + 4 f^3 \\ &\propto (\lambda_1 - \lambda_2)^2 (\lambda_2 - \lambda_3)^2 (\lambda_3 - \lambda_1)^2 \end{aligned}}$$

vanishes.

It is known that the complex structure τ of the torus can be given in terms of f & g via $SL(2,\mathbb{Z})$ modular invariant j-function $j(\tau)$ as

$$j(\tau) = \frac{4\cdot(24f)^3}{\Delta}$$

In this talk, we don't need the defition of the j-function $j(\tau)$, but except for the large τ_2 limit

$$j(\tau) \underset{\tau_2\to\infty}{\sim} e^{-2\pi i\tau} + 744 + O(e^{2\pi i\tau})$$

Using this formalation of T^2, one can easily move to the T^2-fibration over B_6.

For the local cpx. coordinates (u, v, z) of B_6,

$$y^2 = x^3 + f(u,v,z)\,x + g(u,v,z)$$

locally describes X_8, and one can see that the discriminant Δ also varies over B_6.

For example, at a point $u = u_i$ of B_6, suppose that

$$\Delta(u) \sim 4(u-u_i)^N, \quad 24f(u) \sim 1.$$

then, one finds that a one-cycle of T^2 collapses at $u = u_i$, and

$$j(\tau) = \frac{4(24f)^3}{\Delta} \underset{u \sim u_i}{\sim} \frac{1}{(u-u_i)^N} \to \infty.$$

Since $j(\tau) \to \infty$ in the $\tau_2 \to \infty$ limit,

$$j(\tau) \underset{\tau_2 \to \infty}{\sim} e^{-2\pi i \tau} = \frac{1}{(u-u_i)^N}$$

gives

$$\tau \underset{u \sim u_i}{\sim} \frac{N}{2\pi i} \log(u-u_i) .$$

Recalling that

$$\tau = C_0 + \frac{i}{g_B},$$

one can see that

$$g_B = \frac{2\pi}{N \log\left(\frac{1}{|u-u_i|}\right)} \xrightarrow[u \to u_i]{} 0 .$$

Thus, around $u = u_i$, the perturbative IIB strings gives a good description.

Furthermore, going around $u = u_i$,
one gets a monodoromy

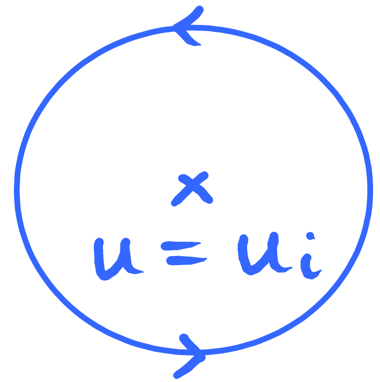
$$\tau \to \tau + N,$$

in other words,

$$C_0 \to C_0 + N$$

$$\Updownarrow$$

$$\oint_{u=u_i} dC_0 = N$$



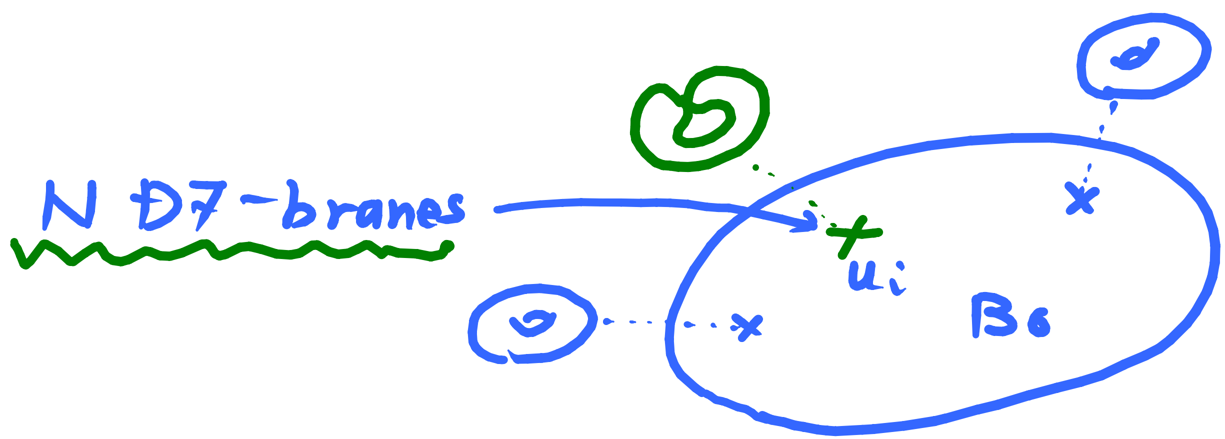
$$\tau \to \tau + N$$

Since D7-brane charge is given by

$$N_{D7} = \oint_{u=u_i} dC_0 ,$$

the monodoromy means that

N D7-branes are situated at $u = u_i$,
where the fiber torus degenerates.



Also, the constraint

$$\Delta(u, v, z) = 0$$

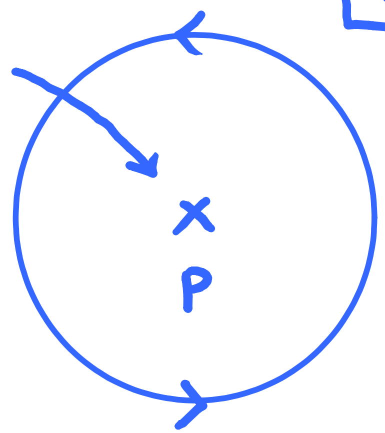
is complex co-dimension 1, and the solutions to the constraint span 8-dimensional space in 10-dim space-time. This is consistent with the D7-brane interpretation of the monodoromy.

$x^0 \sim x^9$	$0\ 1\ 2\ 3$	$4\ 5\ 6\ 7\ 8\ 9$
	$\longleftarrow \mathbb{R}^{1,3} \longrightarrow$	$\longleftarrow B_6 \longrightarrow$
		$\cup$
D7-branes	$\circ\ \circ\ \circ\ \circ$	$\underbrace{S_4}_{\text{complex 2-dim space}}$

More generally, going around a point p with $\Delta(p) = 0$, τ has a monodromy

$$\tau \to \frac{a\tau + b}{c\tau + d},$$

$\Delta(p)=0$



B_6

with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \underbrace{SL(2,\mathbb{Z})}_{\text{S-duality of Type IIB strings}}$

For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1-pq & p^2 \\ -q^2 & 1+pq \end{pmatrix}$,

$(p, q \in \mathbb{Z})$

one can see that

a $[p, q]$ 7-brane is located at $p \in B_6$.

(a $[1, 0]$ 7-brane = a D7-brane)

Let us consider an example. Suppose that

$$y^2 = x^3 + f x + g$$

where

$$\begin{cases} f = f_0 \quad (\text{const.}), \\ g = g_0 + \dfrac{a}{54 g_0} z^{N+1} \quad (a, g_0 : \text{const.}), \end{cases}$$

with

$$27 g_0^2 + 4 f_0^3 = 0 .$$

The discriminant Δ is given by

$$\Delta = 27 g^2 + 4 f^3$$

$$\underset{z \sim 0}{\sim} a z^{N+1} + O(z^{N+2}) .$$

It means that the fiber degenerates at $z = 0$.

After the shift $x \to x - \sqrt{-\frac{f}{3}}$, the equation $y^2 = x^3 + fx + g$ yields

$$y^2 \cong x^3 - \sqrt{-3f_0}\, x^2 + \frac{a}{27 g_0} z^{N+1} + O(z^{N+2}),$$

and one can find a singularity at

$$(x, y, z) = (0, 0, 0)$$

for $N \geqq 1$.

In fact,

$$F(x,y,z) = 0$$

with

$$F(x,y,z) = -y^2 + x^3 - \sqrt{-3g_0}\, x + \frac{a}{27g_0} z^{N+1} + O(z^{N+2})$$

has a singularity at $(x,y,z) = (0,0,0)$, because $(x,y,z) = (0,0,0)$ is a solution to

$$F = \frac{\partial}{\partial x} F = \frac{\partial}{\partial y} F = \frac{\partial}{\partial z} F = 0 .$$

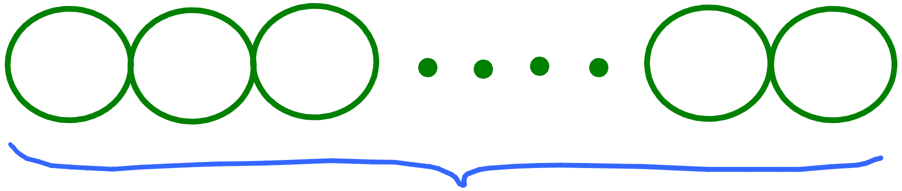
It is a singularity not only of the fiber, but also of the total space.

Near the singularity $(x,y,z)=(0,0,0)$, after appropriate rescaling and shifting of (x,y,z), one has

$$x^2+y^2+z^{N+1}=0.$$

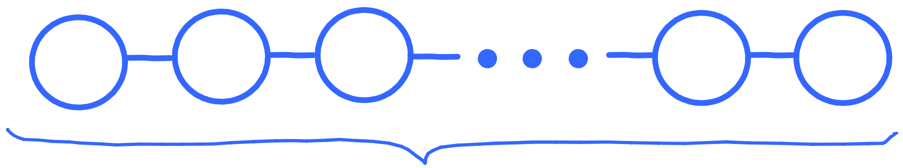
It is known to be an A_N type singularity.

At an A_N type singularity, N copies of 2-dim. spheres (2-cycles) collapse into the origin, and intersect each other like this



N 2-cycles

which corresponds to A_N type Dynkin diagram.



N nodes

In order to see this, let us consider an A_1 singularity

$$F = x^2 + y^2 + z^2$$

in terms of

$$\begin{cases} x = x_1 + i\,y_1 \\ y = x_2 + i\,y_2 \\ z = x_3 + i\,y_3 \end{cases} \quad \left(\begin{array}{l} x_{1,2,3} \in \mathbb{R} \\ y_{1,2,3} \in \mathbb{R} \end{array} \right).$$

To see a non-collapsing sphere, let us deform $F = 0$ into $F = \epsilon^2$;

$$\boxed{x^2 + y^2 + z^2 = \epsilon^2 \qquad (\epsilon > 0)}$$

$$\Updownarrow$$

$$\begin{cases} x_1^2 + x_2^2 + x_3^2 = \epsilon^2 + y_1^2 + y_2^2 + y_3^2, \\ x_1 y_1 + x_2 y_2 + x_3 y_3 = 0. \end{cases}$$

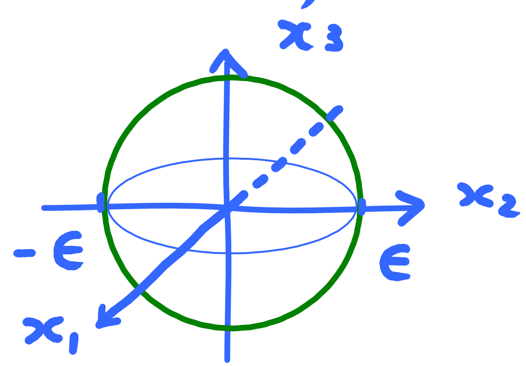
Equivalently, one may write

$$\begin{cases} |\vec{x}|^2 = \epsilon^2 + |\vec{y}|^2, \\ \vec{x}\cdot\vec{y} = 0, \end{cases}$$

where

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

At $y = 0$, one can see a 2-dim sphere S^2 of radius ϵ. So, in the limit $\epsilon \to 0$, the sphere collapses

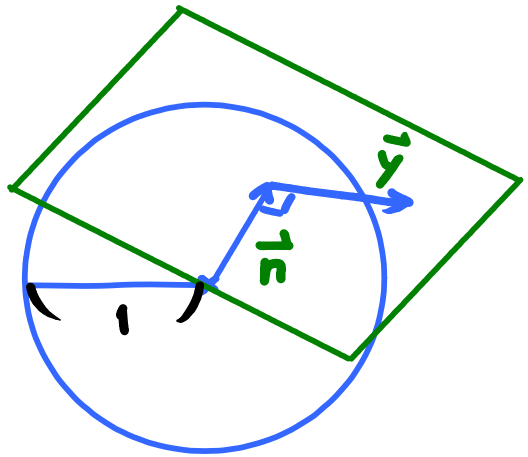


Further, with $\vec{n} = \sqrt{\dfrac{1}{\epsilon^2 + |\vec{y}|^2}}\, \vec{x}$,

one has

$$\boxed{|\vec{n}| = 1, \quad \vec{n} \cdot \vec{y} = 0.}$$

This is a TS^2 bundle over the base S^2.
Therefore, it contains a 2-dim. sphere
give by $y = 0$.



Let's move on to an A_2 singularity

$$x^2 + y^2 + z^3 = 0$$

deformed into

$$\boxed{x^2 + y^2 + (z-2\epsilon)\, z\, (z+2\epsilon) = 0}$$

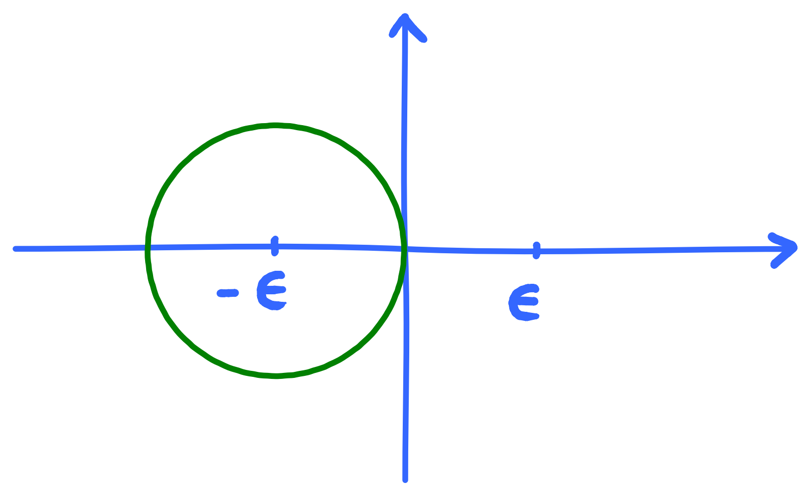
with $\epsilon > 0$.

Let us look at the part $z \neq 2\epsilon$, and rescale $(x, y) \mapsto \sqrt{z-2\epsilon}\,(x, y)$ with a shift $z = z' - \epsilon$.

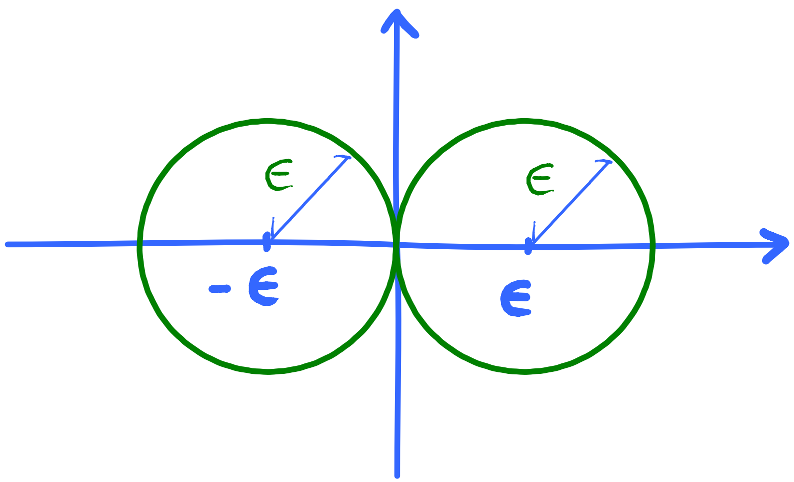
One finds an A_1-singularity

$$x^2 + y^2 + (z')^2 - \epsilon^2 = 0$$

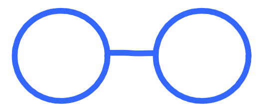
at $(x, y, z) = (0, 0, -\epsilon)$. $(z' = 0)$



Let us see the other part $z \neq -2\epsilon$, so that one can rescale $(x, y) \mapsto \sqrt{z + 2\epsilon}\,(x, y)$ and shift $z = z' + \epsilon$. One finds another A_1-singularity at $(x, y, z) = (0, 0, +\epsilon)$.



$\Longleftrightarrow$



A_2 Dynkin diagram

More generally, at a point, say $(x, y, z) = (0, 0, 0)$, where a fiber degenerates;

$$\Delta = 0,$$

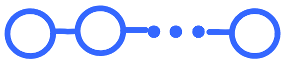
the total space, locally described by

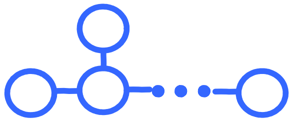
$$y^2 = x^3 + f x + g,$$

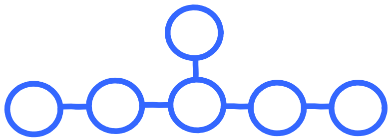
can develop one of the singularities

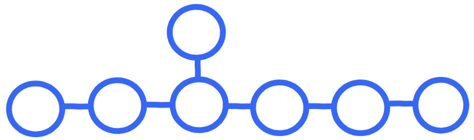
$$\boxed{\begin{aligned} A_N &: \quad y^2 = x^2 + z^{N+1} \\ D_N &: \quad y^2 = x^2 z + z^{N-1} \\ E_6 &: \quad y^2 = x^3 + z^4 \\ E_7 &: \quad y^2 = x^3 + x z^3 \\ E_8 &: \quad y^2 = x^3 + z^5 \end{aligned}}$$

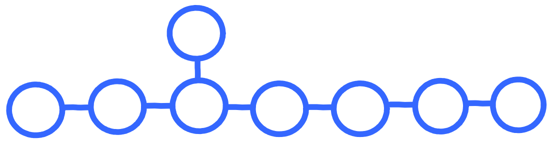
Each of these singularities has the counterpart of the Dynkin diagram of a Lie algebra like A_N type singularities.

A_N : 

D_N : 

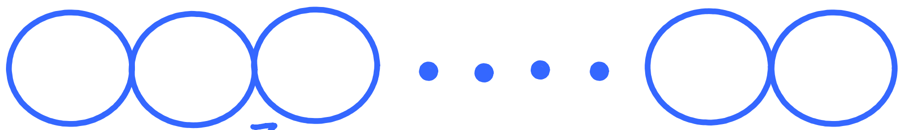
E_6 : 

E_7 : 

E_8 : 

It turns out that the Lie algebra corresponds to the one of the gauge group of the worldvolume theory on the 7-branes.

To see this, at a singularity,



i-th 2-cycle $\leftrightarrow$ ω_i : two-form

$$C_3 = \sum_i \underbrace{A^{(i)}}_{} \wedge \omega_i$$

; 3-form gauge field in M-theory

one-form gauge fields taking value in the Cartan subalgebra of the gauge group.

On an M2-brane wrapped the i-th 2-cycle, the Chern-Simons coupling

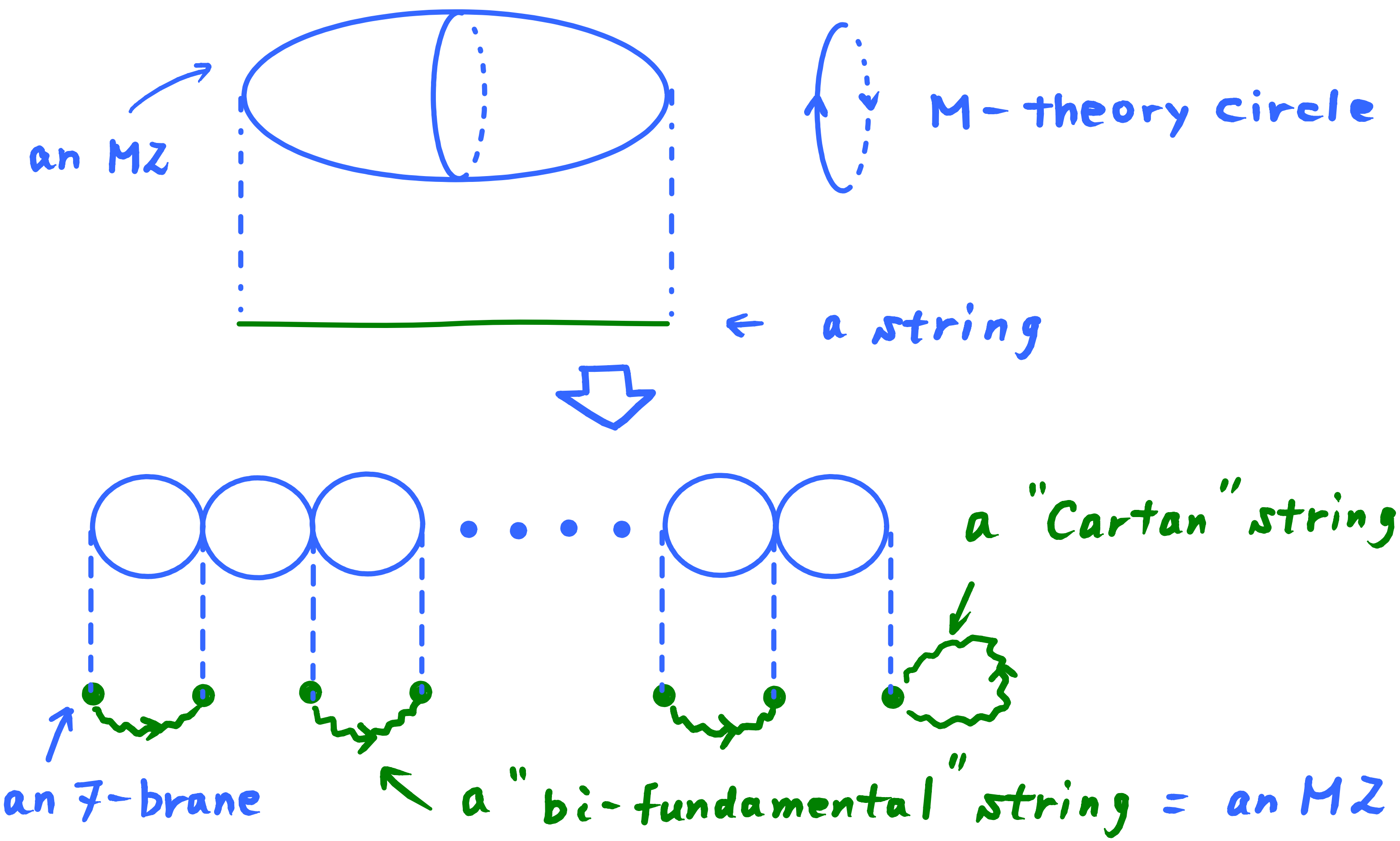
$$\int_{\Sigma_i} C_3$$

on the worldvolume of the M2 yields

$$\int_{\Sigma_i} \left(A^{(i)} \wedge \omega_i \right) = \int A_\mu^{(i)} dx^\mu ,$$

meaning that the M2 gives a particle charged under $A^{(i)}$. It is a gauge boson like the W boson, corresponding to a root of the Lie algebra.

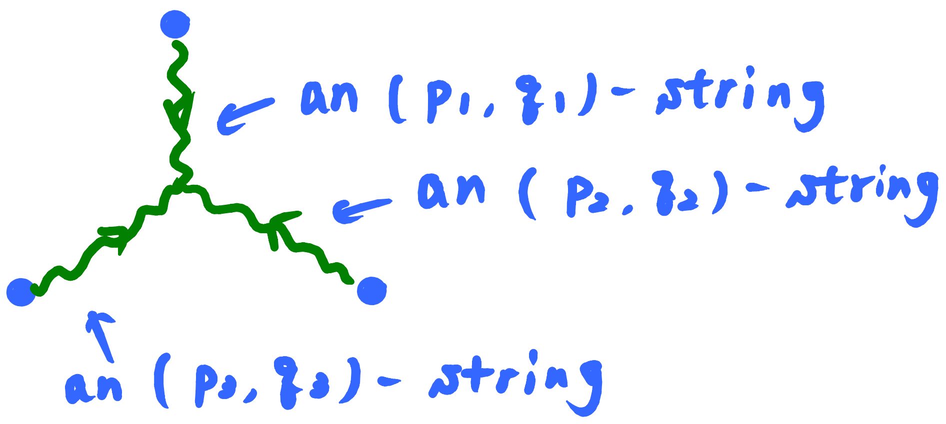
In terms of strings, since



A (p,q)-string can has its end on a $[p,q]$ 7-brane.

(a $(1,0)$-string = a fundamental string)

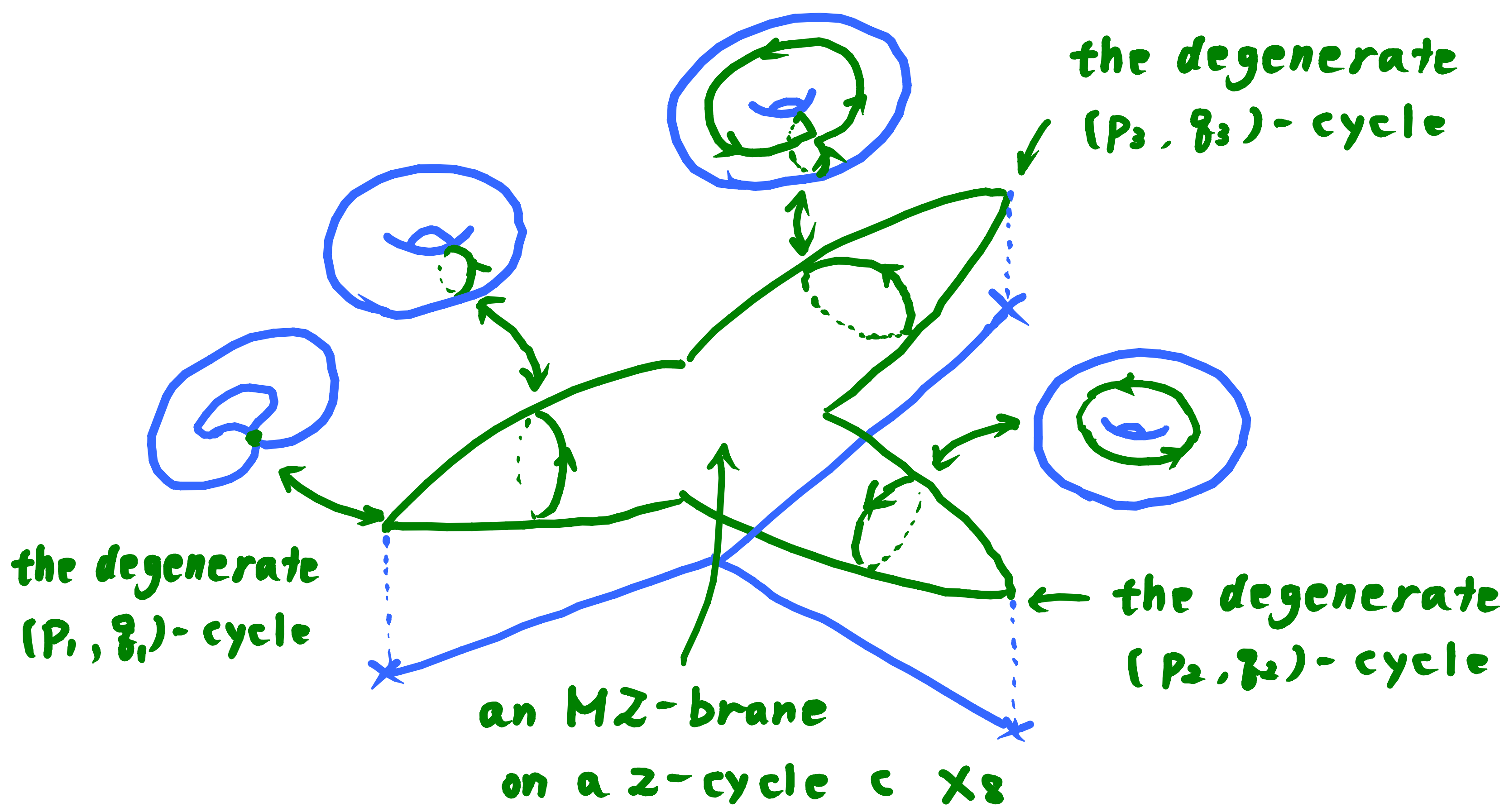
(p,q)-strings can form a string junction;



with

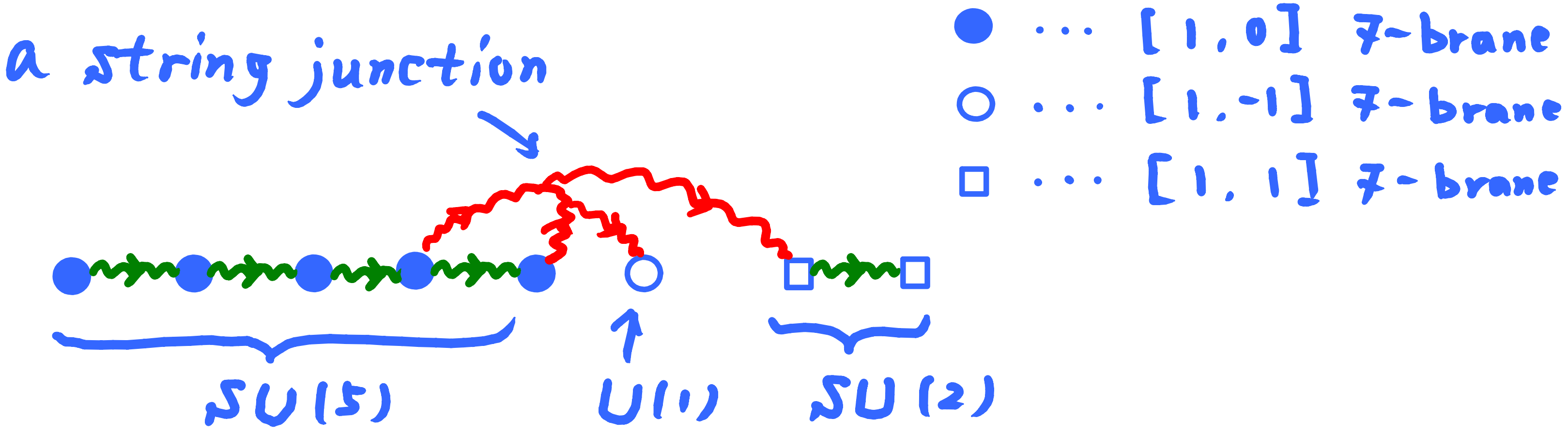
$$\begin{cases} p_1+p_2+p_3=0, \\ q_1+q_2+q_3=0. \end{cases}$$

From the M-theory point of view,



String junctions (= M2 branes) can provide gauge bosons corresponding to the roots of the gauge group on the 7-branes.

For example, for an E_6 gauge group,



The singularities can be deformed, like the A_N type singularities as

$$A_N : \quad y^2 = x^2 + z^{N+1} + \alpha_2 z^{N-1} + \cdots + \alpha_N z + \alpha_{N+1}$$

$$D_N : \quad y^2 = x^2 z + z^{N-1} + \delta_2 z^{N-2} + \cdots + \delta_{2N-2} - 2\gamma_N x$$

$$E_6 : \quad y^2 = x^3 + z^4 + \varepsilon_2 x z^2 + \varepsilon_5 x z + \varepsilon_6 z^2 + \varepsilon_8 x + \varepsilon_9 z + \varepsilon_{12}$$

$E_{7,8}$: $\cdots$ (omitted)

to obtain non-collapsing 2-cycles.

Since a deformation makes a singularity milder, it should correspond to breaking of the gauge group on the worldvolume theory of 7-branes.

In fact, the deformation parameters can be given in terms of the VEV of an adjoint Higgs of the gauge group. (Katz & Morrison '91)

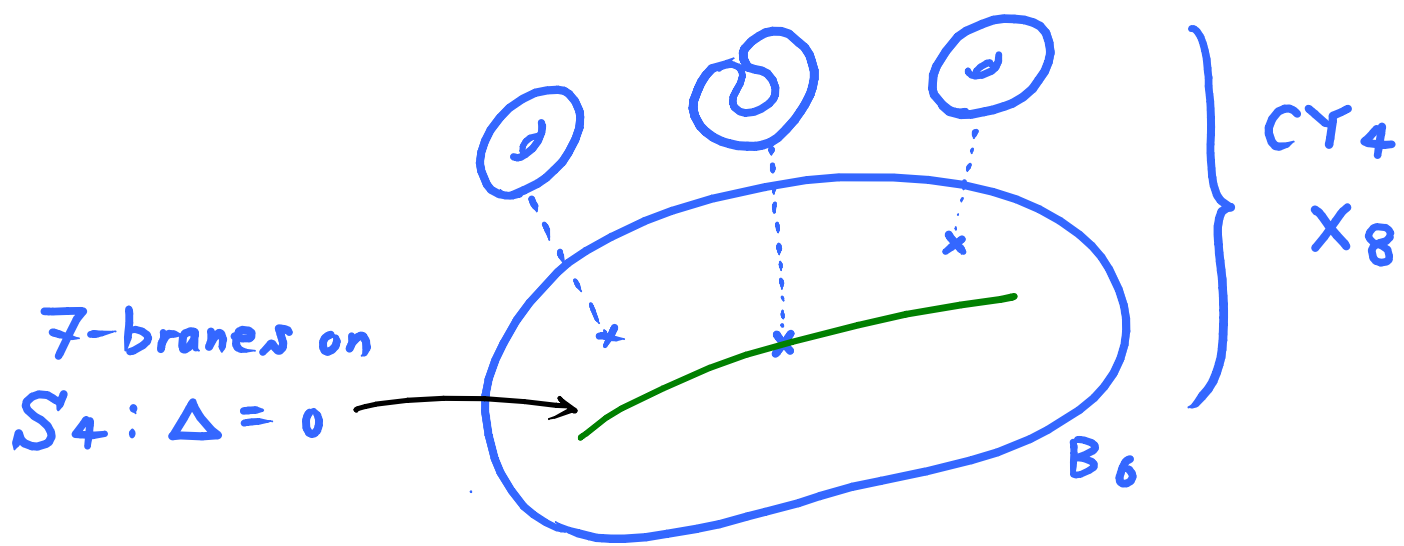
Let's wrap up what we have seen so far.

F-theory compactified on an elliptically fibered CY_4, X_8, over the base B_6 is described by

$$y^2 = x^3 + f(u,v,z)x + g(u,v,z).$$

At a point of B_6 where $\Delta = 0$, the fiber collapses, and one finds 7-branes there.

A solution to $\Delta = 0$ spans complex 2-dim. Kähler submanifold $S_4 \subset B_6$, and so the 7-branes extend along $(3+1)$-dim. Minkowski space and wrap on the cpx. 2-dim S_4.



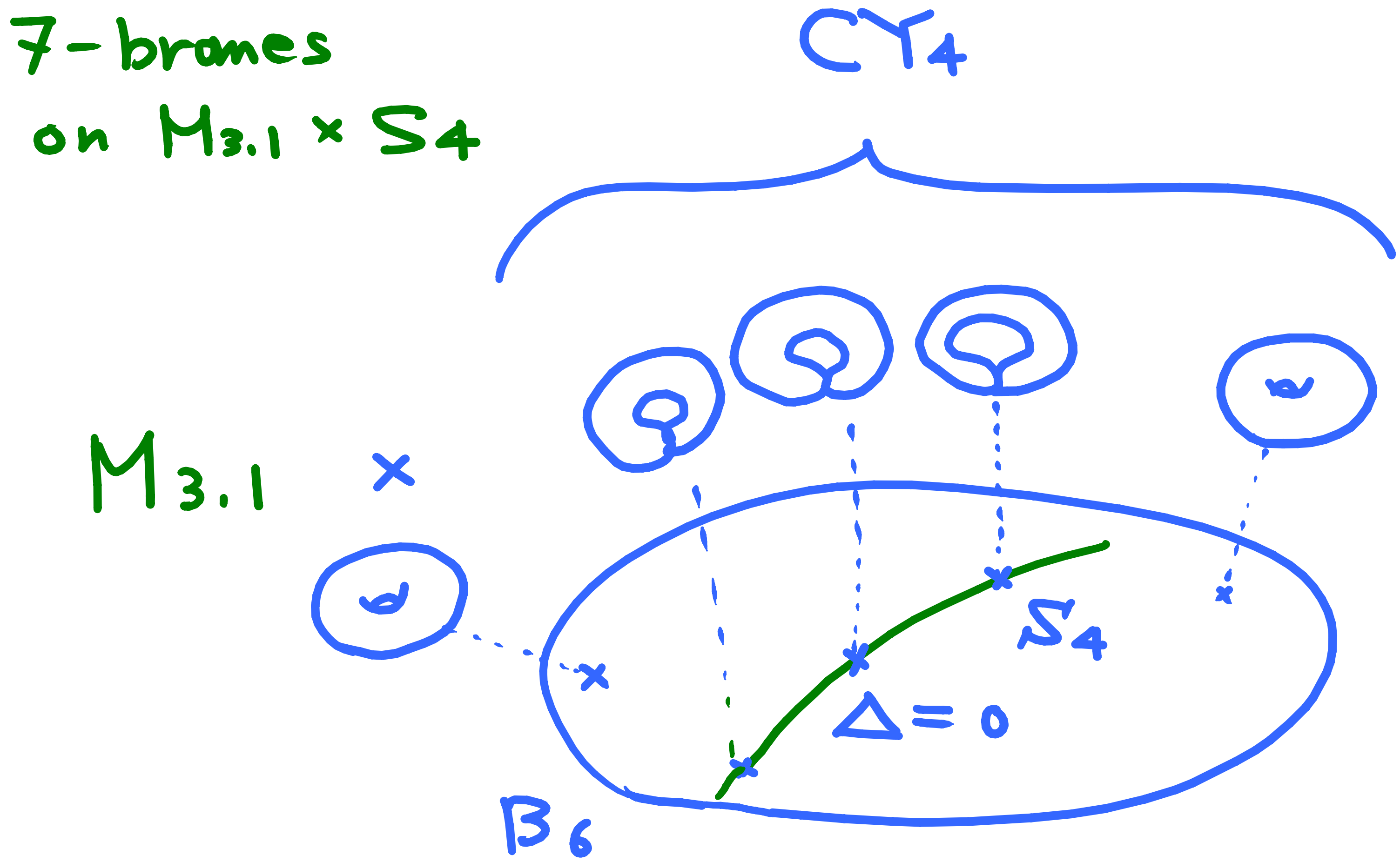
At a point satisfying $\Delta = 0$, the total space given by $y^2 = x^3 + f x + g$ can develop one of the ADE type singularities.

It corresponds to the gauge group in the worldvolume theory of the 7-branes at the singularity.

The ADE singularities can be deformed by the complex structure moduli, and the cpx. str. moduli can be given in terms of a field in the adjoint rep. of the corresponding gauge group.

The cpx. str. moduli deforms the singularity i.e., the configuration of 7-branes. It suggests that the cpx. str. moduli is a Higgs field on the worldvolume theory.

A singularity		7-branes
position	$\longleftrightarrow$	position
type	$\longleftrightarrow$	gauge group
deformation (complex structure moduli)	$\longleftrightarrow$	VEV of an adjoint Higgs


7-branes
on $M_{3,1} \times S_4$
CY_4
$M_{3,1}$ ×
S_4
$\Delta = 0$
B_6

3. Partially Twisted 7-Brane Worldvolume Theory

What is the worldvolume theory on the 7-branes at a singularity?

We have seen that the gauge group is determined by the type of the singularity.

The worldvolume theory on Dp-branes is 10-dim. $\mathcal{N}=1$ Supersymmetric Yang-Mills theory dimensionally reduced to $(p+1)$ dimensions.

Following this, let us dimensionally reduce it by 2 dimensions to give $(7+1)$-dim. SYM.

It has the global symm. $SO(1,7) \times U(1)_J$, where $SO(1,7)$ is $(7+1)$-dim. Lorentz group and $U(1)_J$ is R-symmetry.

Since 7-branes are wrapped on 2-dim $\mathcal{S}_4$, one needs the Kaluza-Klein reduction of the (7+1)-dim. SYM onto $\mathcal{S}$.

Since $\mathcal{S}_4$ is a generic Kähler mfd, the holonomy group of $\mathcal{S}$ is $U(2) \subset SO(4)$.

The K-K. reduction reduces the Lorentz group $SO(1,7)$ as

$$SO(1,7) \rightarrow SO(1,3) \times SO(4)$$

and embeds the holonomy group $U(2)$ into this $SO(4)$.

Since the supercharge Q of 10-dim. $\mathcal{N}=1$ SYM transforms as the 16-dim. rep. under the Lorentz group $SO(1,9)$, and under the global symm. $SO(1,7) \times U(1)_J$, as $(8_-, +\frac{1}{2}) \oplus (8_+, -\frac{1}{2})$,

one can see that under the global symm.

$$\begin{array}{ccccc} SO(1,3) & \times & SO(4) & \times & U(1)_J \\ \wr\wr & & \wr\wr & & \\ \text{"}SU(2)_L\text{"} \times \text{"}SU(2)_R\text{"} & & SU(2)_L \times SU(2)_R & & \end{array}$$

the supercharge Q transforms as

$(8_-, \frac{1}{2})$

$\rightarrow (2,1;1,2;\frac{1}{2}) \oplus (1,2;2,1;\frac{1}{2})$

$(8_+, -\frac{1}{2})$

$\rightarrow (2,1;2,1;-\frac{1}{2}) \oplus (1,2;1,2;-\frac{1}{2})$.

Since the holonomy group $U(2) \simeq SU(2) \times U(1)$ is identified with the subgroup of $SO(4)$ as

$$U(2) \simeq SU(2) \times U(1) \longleftrightarrow SU(2)_L \times \underset{SU(2)_R}{\underset{\cap}{U(1)_R}}$$

$$(SO(4) \simeq SU(2)_L \times SU(2)_R)$$

all the components of the supercharge Q are non-trivial representations of the holonomy group $U(2)$, and no unbroken supersymmetries remain upon the usual K.-K. reduction.

However, it contradicts with the fact that the compactification onto CY_4 must yield $\mathcal{N}=1$ supersymmetry in $(3+1)$ dimensions.

Instead of the usual K.-K. reduction, one can identify the holonomy group

$SU(2) \times U(1)$ as $SU(2)_L \times U(1)_{top}$

$$\begin{array}{ccccc} SU(2)_L & \times & U(1)_{top} & & \\ \uparrow & & \uparrow & \text{a linear combination} & \\ SU(2)_L & \times & U(1)_R & \times & U(1)_J \\ & & \cap & & \\ \underbrace{\quad\quad\quad}_{} & & SU(2)_R & & \end{array}$$

$$\underbrace{SU(2)_L \times SU(2)_R}_{SO(4)}$$

to give singlet supercharges under the holonomy group $U(2)$.

Surprisingly enough, the liner combination of $U(1)_R$ and $U(1)_J$ is uniquely determined up to the conventional choices and yields $\mathcal{N}=1$ supersymmetry in $(3+1)$ dimensions. This procedure is called partially twisting.

Similarly to topological twisting, this twisting also changes spins of the field contents but only along the "internal" space $\mathcal{S}$.

10 dim.

v_M : gauge field ($M = 0, 1, \cdots, 9$)

λ : gaugino (Majorana-Weyl spinor
with 16 real components)

$\downarrow$

8 dim.

v_I : gauge field ($I = 0, 1, \cdots, 7$)

$\Phi = \frac{1}{\sqrt{2}} (v_8 - i v_9)$: a complex scalar

λ : gaugino (Weyl spinor with
8 complex components)

8 dim. $\qquad\qquad$ (3+1) dim.

$v_I \longrightarrow v_{\mu=0,1,2,3}$: gauge field

$$\longrightarrow \left(\begin{array}{l} \bar{A}_1 = \frac{1}{\sqrt{2}}(v_4 - i v_5) \\ \bar{A}_2 = \frac{1}{\sqrt{2}}(v_6 - i v_7) \end{array} \right.$$

$$\Rightarrow \bar{A}_m \, dz^m$$

$$\Phi \longrightarrow \frac{1}{2} \bar{\varphi}_{\bar{m}\bar{n}} \, d\bar{z}^{\bar{m}} \wedge d\bar{z}^{\bar{n}}$$

$$= \bar{\varphi}_{12} \, d\bar{z}^1 \wedge d\bar{z}^2$$

where

(z^1, z^2) : local complex coordinates of S_4

8 dim $\qquad\qquad\qquad$ (3+1) dim.

$$\lambda \;\begin{cases} \longrightarrow & \bar{\lambda}^{\dot{\alpha}} \;:\; \text{gaugino} \\ \longrightarrow & \psi_{\bar{m}\alpha}\, d\bar{z}^{\bar{m}} \\ \longrightarrow & \frac{1}{2}\, \bar{\chi}^{\dot{\alpha}}_{\bar{m}\bar{n}}\, d\bar{z}^{\bar{m}} \wedge d\bar{z}^{\bar{n}} \end{cases}$$

One thus obtains

- gauge vector multiplet

  (v_μ, λ),

- matter chiral multiplet

  $(A_{\bar m}, \psi_{\bar m \alpha})$,

  $(\varphi_{mn}, \chi_{mn\alpha})$,

all of which transform as adj. rep. under the gauge transformation.

Introducing auxiluary fields

$$D\,, \quad F_{\bar{m}}\, d\bar{z}^{\bar{m}}\,, \quad H_{mn}\, dz^m \wedge dz^n\,,$$

one can form superfields as

$$V(x,\theta,\bar{\theta}) = -\theta\sigma^\mu\bar{\theta}\, v_\mu(x) - i\,\bar{\theta}^2\,\theta\cdot\lambda(x) + i\,\theta^2\,\bar{\theta}\cdot\bar{\lambda}(x) + \frac{1}{2}\theta^2\bar{\theta}^2 D(x)\,,$$

$$A_{\bar{m}}(y,\theta) = A_{\bar{m}}(y) + \sqrt{2}\,\theta\cdot\psi_{\bar{m}}(y) + \theta^2 F_{\bar{m}}(y)\,,$$

$$\Phi_{mn}(y,\theta) = \varphi_{mn}(y) + \sqrt{2}\,\theta\cdot\chi_{mn}(y) + \theta^2 H_{mn}(y)\,,$$

$$(\, y^\mu \equiv x^\mu - i\,\bar{\theta}\bar{\sigma}^\mu\theta\,)$$

In order to obtain the worldvolume action, one compactifies $(7+1)$-dim. SYM onto $M_{3,1} \times \Sigma$ by the K.-K. reduction and further twists the fields of the theory.

Infact. the action is given by

$$S = \int d^4x \int d^2z \, d^2\hat{z} \sqrt{g_\Sigma} \, \mathrm{tr} \, L$$

$$L = \int d^4\theta \, K + \int d^2\theta \, W + \int d^2\bar{\theta} \, \bar{W}$$

$$K = g^{m\bar{n}} g^{k\bar{l}} \bar{\Phi}_{\bar{n}\bar{l}} e^{2gV} \Phi_{mk} e^{-2gV}$$

$$+ g^{m\bar{n}} \left(\bar{A}_m - \frac{i}{g} \partial_m e^{2gV} \cdot e^{-2gV} \right) e^{2gV}$$

$$\times \left(A_{\bar{n}} - \frac{i}{g} \bar{\partial}_{\bar{n}} e^{-2gV} \cdot e^{2gV} \right) e^{-2gV}$$

$$+ \frac{1}{2g^2} g^{m\bar{n}} \left(\partial_m e^{2gV} \right) \left(\bar{\partial}_{\bar{n}} e^{-2gV} \right)$$

$$W = - g^{m\bar{n}} g^{k\bar{l}} F_{\bar{n}\bar{l}} \Phi_{mk}$$

where

$$F_{\bar{m}\bar{n}} = \partial_{\bar{m}} A_{\bar{n}} - \partial_{\bar{n}} A_{\bar{m}} + i g [A_{\bar{m}}, A_{\bar{n}}]$$

Let us now have a closer look at the relation of a singularity with the gauge symmetry breaking.

At an A_n type singularity, say $(x,y,z) \sim (0,0,0)$,

$$y^2 = x^3 + fx + g$$

degenerates into

$$y^2 = x^2 + z^{n+1},$$

after shifting and rescaling, as seen before.

It can be deformed as

$$y^2 = x^2 + \det(z - \phi)$$

by an adjoint Higgs ϕ.

Let us now demonstrate that ϕ is not a scalar, but

$$\phi \in K_s \text{ (cannonical bundle on } S_4\text{)}.$$

To this end, let us begin with the C.-Y. condition of 4-dim X_8, which requires (4,0)-form Ω_X of X_8 to be a trivial line bdl. Since, for local coordinates (z^1, z^2) of S_4,

$$\Omega_X \sim \frac{dx \wedge dz}{y} \wedge dz^1 \wedge dz^2,$$

and

$$dz^1 \wedge dz^2 \in K_s^{-1}$$

the C.-Y. condition requres that

$$\frac{dx \wedge dz}{y} \in K_s .$$

It means that x, y, z take values in some powers of K_s ; i.e.,

$$(x, y, z) \in (K_s^a, K_s^b, K_s^c),$$

and the constraint

$$F = x^2 + y^2 + z^{n+1}$$

also takes value in K_s^f

$$F \in K_s^f$$

Therefore, one gets two conditions

$$2a = 2b = (n+1)c = f$$

and

$$\frac{dx \wedge dz}{y} \in K_S^{a-b+c} \Rightarrow a-b+c = 1$$

to give

$$(x, y, z) \in \left(K_S^{\frac{n+1}{2}}, K_S^{\frac{n+1}{2}}, K_S\right).$$

Recalling the deformed constraint

$$x^2 + y^2 + \det(z - \phi) = 0 .$$

One can conclude that

$$\phi \in K_S .$$

This is an important prediction on the field content of the world volume theory.
It says that there must exist an adj. Higgs taking value in K_S on S_4, i.e., $(2,0)$-form field on S_4.
The "twisted" theory is consistent with this, because we certainly get, in the theory,

$$\varphi_{mn}\, dz^m \wedge dz^n \in K_S .$$

In order to obtain chiral matters of GUT groups, there are two options one can choose; one is the use of instanton solutions on S, the other is to introduce another set of 7-branes on 2-dim$_{\mathbb{C}}$ S' intersecting with the 7-branes on S to give rise to new d.o.f. at the intersection $S \cap S'$.

We begin with the former choice.

In the worldvolume theory, one has the D-term condition

$$\boxed{g^{m\bar{n}} F_{m\bar{n}} = \frac{ig}{2} g^{m\bar{n}} g^{k\bar{l}} [\varphi_{mk}, \bar{\varphi}_{\bar{n}\bar{l}}],}$$

and the F-term conditions

$$\boxed{\begin{aligned} F_{mn} = F_{\bar{m}\bar{n}} &= 0, \\ g^{m\bar{n}} D_{\bar{n}} \varphi_{mk} &= 0. \end{aligned}}$$

A solution to them gives a supersymmetric background.

A simple solution is given by an intersecting brane background ;

$$[\varphi_{mn}, \bar{\varphi}_{\bar{k}\bar{l}}] = 0 ,$$

with

$$D^n \varphi_{nm} = 0 .$$

Then, the gauge field $A_{\bar{m}}$ can be trivial

$$A_{\bar{m}} = 0$$

or anti-self dual (ASD) instanton solutions satisfying

$$F_{mn} = F_{\bar{m}\bar{n}} = 0 , \quad g^{m\bar{n}} F_{m\bar{n}} = 0 .$$

In the ASD instanton background $A_{\bar{m}}$, suppose that the solution $A_{\bar{m}}$ takes value in $G_{inst} \subset G$, where G is the gauge group of the "twisted" theory. The gauge group G is broken by this into $H \subset G$;

$$G \longrightarrow H$$

The adj. rep. adj(G) of G is decomposed under $H \otimes G_{inst}$ into

$$\mathrm{adj.}(G) = \bigoplus_i (R_i, V_i).$$

The Lagrangian of the fermionic fields is give by

$$L = i D_\mu \chi^a_{mn} \sigma^\mu \bar{\chi}^a_{\bar{k}\bar{l}} g^{m\bar{k}} g^{n\bar{l}} + i D_\mu \lambda^a \sigma^\mu \lambda^a$$
$$- i D_\mu \bar{\psi}^a_m \bar{\sigma}^\mu \psi^a_{\bar{n}} g^{m\bar{n}}$$

along $\mathbb{R}^{1,3}$

$$- \sqrt{2}\, g^{m\bar{n}} \bar{\psi}^a_m \bar{D}_{\bar{n}} \bar{\lambda}^a - \sqrt{2}\, g^{m\bar{n}} D_m \lambda^a \cdot \psi^a_{\bar{n}}$$
$$+ g^{m\bar{k}} g^{n\bar{l}} \chi^a_{mn} \bar{D}_{\bar{k}} \psi^a_{\bar{l}} + g^{m\bar{k}} g^{n\bar{l}} \bar{\chi}^a_{\bar{k}\bar{l}} D_m \bar{\psi}^a_n$$

along $S4$

mass terms from the (3+1)-dim. Minkowski point of view

In terms of the form notation on S_4,

$$\psi = \psi^a_{\bar{n}} T^a d\bar{z}^{\bar{n}},$$

$$\bar{\chi} = \bar{\chi}^a_{\bar{m}\bar{n}} T^a d\bar{z}^{\bar{m}} \wedge d\bar{z}^{\bar{n}},$$

$$\lambda = \lambda^a T^a,$$

and the covariant derivative for a p-form

$$\bar{D}\phi = \bar{\partial}\phi + ig\left(A \wedge \phi - (-)^p \phi \wedge A\right),$$

with

$$A = A^a_{\bar{m}} T^a d\bar{z}^{\bar{m}},$$

the massless modes of the fermions are given as the solutions to

$$\bar{D}\bar{\lambda} = 0 , \quad \bar{D}^{\dagger}\bar{\lambda} = 0 ,$$

$$\bar{D}\psi = 0 , \quad \bar{D}^{\dagger}\psi = 0 ,$$

$$\bar{D}\bar{\chi} = 0 , \quad \bar{D}^{\dagger}\bar{\chi} = 0 ,$$

in other words, as harmonic forms of $\bar{D}$ and $\bar{D}^{\dagger}$!

$$H \otimes G_{int.} \subset G$$

The zero modes for (R_i, V_i) are given by

$$\bar{\lambda} \in H^0_{\bar{\partial}}(\Sigma, V_i),$$

$$\psi \in H^1_{\bar{\partial}}(\Sigma, V_i),$$

$$\bar{\chi} \in H^2_{\bar{\partial}}(\Sigma, V_i),$$

and their complex conjugates by

$$\lambda \in H^0_{\partial}(\Sigma, V_i^*),$$

$$\bar{\psi} \in H^1_{\partial}(\Sigma, V_i^*),$$

$$\chi \in H^2_{\partial}(\Sigma, V_i^*).$$

Since $H^p_\partial(\Sigma, V_i^*)$ is dual as a vector space to $H^p_{\bar\partial}(\Sigma, V_i)$, i.e.,

$$H^p_\partial(\Sigma, V_i^*) \simeq H^p_{\bar\partial}(\Sigma, V_i)^\vee,$$

dual

the complex conjugates may be rewritten as

$$\lambda \in H^0_{\bar\partial}(\Sigma, V_i)^\vee,$$

$$\bar\psi \in H^1_{\bar\partial}(\Sigma, V_i)^\vee,$$

$$\chi \in H^2_{\bar\partial}(\Sigma, V_i)^\vee.$$

Collecting only the left-handed spinors among them, one obtains

$$\lambda \in H^0_{\bar\partial}(S, V_i)^\vee ,$$
$$\psi \in H^1_{\bar\partial}(S, V_i) ,$$
$$\chi \in H^2_{\bar\partial}(S, V_i)^\vee .$$

Since the adj. rep. adj.(G) of G is real, the decomposition under $H \times G_{inst}$ gives the V_i^* rep. as well as the V_i rep..

Therefore, $(R_i, V_i) \oplus (R_i^*, V_i^*)$ yields the zero modes;

$$
\begin{aligned}
\lambda &\in H^0_{\bar\partial}(S, V_i)^\vee, \\
\psi &\in H^1_{\bar\partial}(S, V_i), \\
\chi &\in H^2_{\bar\partial}(S, V_i)^\vee. \\
\tilde\lambda &\in H^0_{\bar\partial}(S, V_i^*)^\vee, \\
\tilde\psi &\in H^1_{\bar\partial}(S, V_i^*), \\
\tilde\chi &\in H^2_{\bar\partial}(S, V_i^*)^\vee.
\end{aligned}
$$

the V_i^* rep.

the V_i rep

The V_i reps.

$$H^0_{\bar\partial}(S, V_i^*)^\vee \oplus H^1_{\bar\partial}(S, V_i) \oplus H^2_{\bar\partial}(S, V_i^*)^\vee$$

and the V_i^* reps.

$$H^0_{\bar\partial}(S, V_i)^\vee \oplus H^1_{\bar\partial}(S, V_i^*) \oplus H^2_{\bar\partial}(S, V_i)^\vee$$

gives the **net number of generations**

$$\begin{aligned} \tau_{V_i} &\equiv \#(V_i \text{ reps.}) - \#(V_i^* \text{ reps.}) \\ &= h^0(S, V_i^*) + h^1(S, V_i) + h^2(S, V_i^*) \\ &\quad - h^0(S, V_i) - h^1(S, V_i^*) - h^2(S, V_i), \end{aligned}$$

where

$$h^p(S, V_i) = \dim H^p_{\bar\partial}(S, V_i) = \dim H^p_{\bar\partial}(S, V_i)^\vee .$$

It may be rewritten as

$$\begin{aligned} \tau_{V_i} &= [h^0(S, V_i^*) - h^1(S, V_i^*) + h^2(S, V_i^*)] \\ &\quad - [h^0(S, V_i) - h^1(S, V_i) + h^2(S, V_i)], \\ &= \chi(S, V_i^*) - \chi(S, V_i). \end{aligned}$$

The combination

$$\chi(S, V_i) = h^0(S, V_i) - h^1(S, V_i) + h^2(S, V_i)$$

is the Euler character of the vector bundle V_i over S $(= S_4)$.

The Euler character $\chi(S, V)$ can be calculated by the index formula

$$\chi(S, V) = \int_S ch(V)\, Td(S),$$

where $ch(V)$ is the Chern character, and $Td(S)$ is the Todd class.

They may be given in terms of the Chern classes;

$$\begin{aligned} C(V) &\equiv \det\left(1 + \frac{i}{2\pi} F\right) \\ &= 1 + C_1(V) + C_2(V) + \cdots \end{aligned}$$

with F a connection of the vector bundle V.

In fact,

$$
\begin{aligned}
Ch(V) &= \operatorname{rank}(V) + c_1(V) \\
&\quad + \frac{1}{2}\left[c_1(V)^2 - c_2(V)\right] + \cdots, \\
Td(S) &= 1 + \frac{1}{2} c_1(TS) \\
&\quad + \frac{1}{12}\left[c_1(TS)^2 + c_2(TS)\right] + \cdots,
\end{aligned}
$$

and

$$
\begin{aligned}
\chi(S,V) = \int_S \Big[& \frac{\operatorname{rank}(V)}{12}\left[c_1(TS)^2 + c_2(TS)\right] \\
& + \frac{1}{2} c_1(V)\, c_1(TS) \\
& + \frac{1}{2}\left[c_1(V)^2 - 2c_2(V)\right] \Big].
\end{aligned}
$$

Using the fact

$$\begin{cases} c_1(V^*) = -\,c_1(V), \\ c_2(V^*) = c_2(V), \end{cases}$$

one finds the net number of generations

$$\begin{aligned} \tau_{V_i} &= \chi(S, V_i^*) - \chi(S, V_i) \\ &= -\int_S c_1(V)\, c_1(TS). \end{aligned}$$

Beasley, Heckman, and Vafa have argued that the limit where S_4 is contracted into a point inside B_6 corresponds to the decoupling limit of gravity.

The contractivity requires a complex surface S_4 to be the Hirzebruch surface $\mathbb{F}$ or the del Pezzo surface dP_n.

For $\mathbb{F}_n$ and $d\mathbb{P}_n$,

$$H^2_{\bar\partial}(\mathcal{S}, V) = 0,$$

and further with a non-trivial irreducible representation V,

$$H^0_{\bar\partial}(\mathcal{S}, V) = 0.$$

Then, the generation number itself is given by the Euler character;

$$n_{V_i} \equiv h^1_{\bar\partial}(\mathcal{S}, V_i) = -\chi(\mathcal{S}, V_i),$$

$$n_{V_i^*} \equiv h^1_{\bar\partial}(\mathcal{S}, V_i^*) = -\chi(\mathcal{S}, V_i^*).$$

Returning to a generic complex surface S_4, one can obtain Yukawa couplings among the zero modes by substituting their solutions into the superpotential.

$$W = -g^{m\bar{n}} g^{k\bar{l}} \,\mathrm{tr}\,[\, \underbrace{F_{\bar{n}\bar{l}}}_{\partial_{\bar{n}} A_{\bar{l}} - \partial_{\bar{l}} A_{\bar{n}} + ig[A_{\bar{n}}, A_{\bar{l}}]} \Phi_{mk} \,]$$

$$= \cdots - ig\, g^{m\bar{n}} g^{k\bar{l}} \,\mathrm{tr}\,[\, A_{\bar{n}} [\, A_{\bar{l}} ,\ \Phi_{mk} \,]\,] .$$

Thus, the Yukawa coupling constants are given in terms of

the structure constants of G

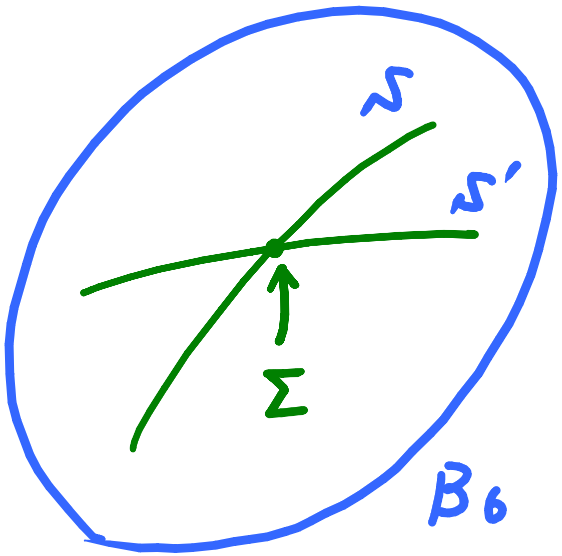
and

the overlaps of the zero mode solutions.

For $\mathbb{F}_n$ or dP_n, however, since $H^2_{\bar{\partial}}(S, V) = 0$, there are no zero mode solutions of Φ_{mn}, and thus, no Yukawa couplings are available.

4. Matter Curves

Another choice to obtain chiral matters is to use intersecting 7-branes.



Generically, $\Sigma = S \cap S'$ is one complex dimensional, and on S, it is described locally by

$$\alpha(z_1, z_2) = 0$$

with the local coordinates (z_1, z_2) of S.

For example, suppose that A_n type singularity

$$x^2 + y^2 + z^{n+1}(z - \alpha(z_1, z_2))^{m+1} = 0$$

is supported on S.

Since, on Σ inside S,

$$\alpha(z_1, z_2) = 0,$$

one can see that the A_n singularity enhances to A_{n+m+1} type singularity as

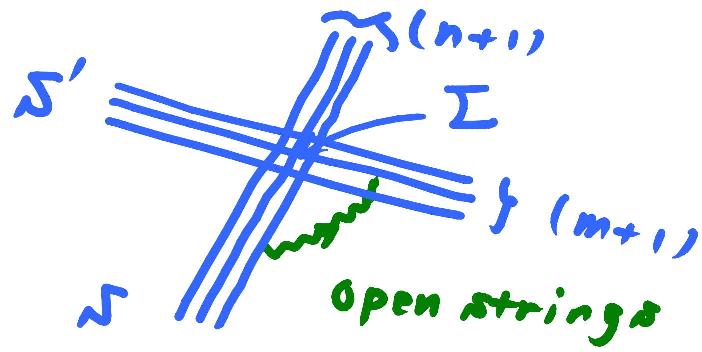
$$x^2 + y^2 + z^{n+m+2} = 0$$

on Σ.

In the 8-dim. worldvolume theory, on $\Sigma \subset S$, the gauge group $SU(n+1)$ is enhanced as

$$U(n+1) \longrightarrow U(n+m+2).$$

It is a familiar in the intersecting D-brane senario, and the enhancement occurs due to open string "bi-fundamentals".



Since $\alpha(z_1, z_2)$ corresponds to the deformation parameter (complex structure moduli), the Higgs field φ_{mn} in the worldvolume theory is given by

$$\langle \varphi_{12} \rangle = \begin{pmatrix} \underbrace{\begin{matrix} 0 & & \\ & \ddots & \\ & & 0 \end{matrix}}_{n+1} & \\ & \underbrace{\begin{matrix} \alpha(z_1,z_2) & & \\ & \ddots & \\ & & \alpha(z_1,z_2) \end{matrix}}_{m+1} \end{pmatrix},$$

breaking the gauge group

$$U(n+m+2) \longrightarrow \underset{\substack{\uparrow \\ \text{on } S}}{U(n+1)} \times \underset{\substack{\uparrow \\ \text{on } S'}}{U(m+1)}$$

In the background $\langle \varphi_{mn} \rangle$, one can find new massless d.o.f. localized near Σ. In order to see the d.o.f., let's look at the e.o.m. of the fermions in the worldvolume theory ;

- $-i\bar{\sigma}^{\mu} D_{\mu} \chi_{mn} + i\sqrt{2}\, g [\varphi_{mn}, \bar{\lambda}] + D_m \bar{\psi}_n - D_n \bar{\psi}_m = 0,$

- $-i\bar{\sigma}^{\mu} D_{\mu} \lambda - \sqrt{2}\, g^{m\bar{n}} D_{\bar{n}} \bar{\psi}_m - i\sqrt{2}\, g\, g^{m\bar{n}} g^{k\bar{\ell}} [\varphi_{mk}, \bar{\varphi}_{\bar{n}\bar{\ell}}] = 0,$

- $-i\bar{\sigma}^{\mu} D_{\mu} \psi_{\bar{n}} + \sqrt{2}\, D_{\bar{n}} \bar{\lambda} - 2ig\, g^{k\bar{\ell}} [\bar{\varphi}_{\bar{n}\bar{\ell}}, \bar{\psi}_k] - 2 g^{k\bar{\ell}} D_k \bar{\chi}_{\bar{\ell}\bar{n}} = 0.$

The background

$$A_{\bar{m}} = \bar{A}_{m} = 0\,, \quad \varphi_{12} = \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & \alpha & \\ & & & & \ddots \\ & & & & & \alpha \end{pmatrix}$$

satisfies the F-term and D-term conditions, if

$$\frac{\partial}{\partial \bar{z}_1} \alpha(z_1, z_2) = \frac{\partial}{\partial \bar{z}_2} \alpha(z_1, z_2) = 0\,.$$

Substituting the background into the e.o.m. of fermions, one can see that the zero modes are the solutions to

- $D_1 \bar{\psi}_2 - D_2 \bar{\psi}_1 = 0$,
- $D_{\bar{1}} \bar{\psi}_1 + D_{\bar{2}} \bar{\psi}_2 = -2ig[\varphi_{12}, \bar{\chi}_{\bar{1}\bar{2}}]$,
- $D_2 \bar{\chi}_{\bar{1}\bar{2}} = ig[\bar{\varphi}_{\bar{1}\bar{2}}, \bar{\psi}_2]$,
- $D_1 \bar{\chi}_{\bar{1}\bar{2}} = ig[\bar{\varphi}_{\bar{1}\bar{2}}, \bar{\psi}_1]$,

where we assume that $g_{m\bar{n}} = \delta_{m\bar{n}}$, and $\bar{\lambda} = 0$.

More specifically, let us take

$$\alpha(z_1, z_2) = M^2 z_1$$

with M a mass parameter.

Let us recall that at $z_1 \neq 0$, the gauge group $U(n+m+2)$ is broken to $U(n+1) \times U(m+1)$.

Since the fermions transform as

$$\mathrm{adj}(U(n+m+2)) = \mathrm{adj}(U(n+1)) \oplus \mathrm{adj}(U(m+1)) \oplus (\underset{U(n+1)}{\square}, \underset{U(m+1)}{\bar{\square}}) \oplus (\bar{\square}, \square),$$

let us have a look at the $(\bar{\square}, \square)$ components of them in their e.o.m.;

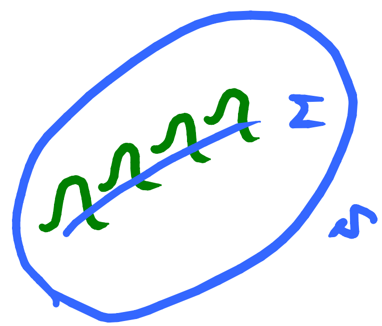
- $\partial_1 \bar{\Psi}_2 - \partial_2 \bar{\Psi}_1 = 0,$
- $\partial_{\bar{1}} \bar{\Psi}_1 + \partial_{\bar{2}} \bar{\Psi}_2 = -2igM^2 z_1 \bar{\chi}_{\bar{1}\bar{2}},$
- $\partial_m \bar{\chi}_{\bar{1}\bar{2}} = igM^2 \bar{z}_{\bar{1}} \bar{\Psi}_m.$

One can verify that

$$\bar{\chi}_{\bar{1}\bar{2}} = \underbrace{\bar{f}}_{\text{any anti-hol. function}}(\bar{z}_{\bar{2}})\, e^{-\sqrt{2}\, g M^2 |z_1|^2} \cdot \underbrace{u}_{\substack{(3+1)\text{-dim.}\\ \text{spinor}}}(x^\mu)$$

$$\bar{\psi}_1 = i\sqrt{2}\, \bar{\chi}_{\bar{1}\bar{2}}$$

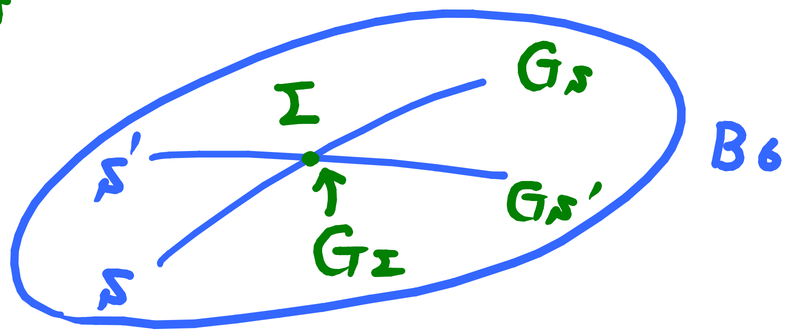
$$\bar{\psi}_2 = 0$$



yield a solution to the equations.
The solution is indeed localized on $\Sigma \subset S$. The intersection Σ is called a matter curve because of extra matters.

More generically, the gauge groups G_S and $G_{S'}$ are supported, respectively, on S and S', and on $\Sigma \subset S \cap S'$, since the singularity is enhanced, $G_S \times G_{S'}$ is also enhanced to the corresponding gauge group G_Σ on Σ;

$$G_\Sigma \supset G_S \times G_{S'}$$



The adj. rep. adj (G_I) is decomposed under $G_S \times G_{S'}$ into their irreducible reps. as

$$\mathrm{adj}(G_I) = \mathrm{adj}(G_S) \oplus \mathrm{adj}(G_{S'}) \oplus_i (R_i, R_i').$$

adj (G_S) and adj $(G_{S'})$ are the d.o.f. of the worldvolume theory, respectively, on S and S'. (R_i, R_i') is the d.o.f. localized on $\Sigma \subset S \cap S'$. B.H.V. call them "bi-fundamental" matters.

The effective Lagrangian describing the localized bi-fundamental matters on Σ should be obtained by substituting the zero mode solutions with the background $\langle \varphi_{mn} \rangle$ into the worldvolume theory.

(C.C.H.V. '10)

Instead of it, B.H.V. discussed an alternative procedure to give the Lagrangian of the bi-fundamental matters on Σ.

Such a Lagrangian should be a supersymmetric field theory in $\mathbb{R}^{1,3} \times$ 2-dim Σ.

(3+1)-dim. Minkowski space

without any gauge fields.

It uniquely specifies a candidate theory.

6-dim. supersymmetric field theories necessarily include gauge fields for $\mathcal{N} \geq 2$. It leads us to consider hypermultiplets in 6-dim. $\mathcal{N}=1$ SUSY theory.

The supercharges $Q^{i=1,2}$ of 6-dim. $\mathcal{N}=1$ SUSY theories are $SU(2)$ Majorana-Weyl spinors ;

$$\left\{\begin{aligned} &\Gamma^7 Q^i = + Q^i \\ &(Q^i)^\dagger \Gamma^0 = (Q^j)^T \varepsilon_{ji} \underbrace{C_6}_{\text{charge conjugation}} \end{aligned}\right.$$

$SU(2)$ inv. tensor

and transform under the $SU(2)_R$ transformation in the fundamental representation. $(i = 1, 2)$

The bosonic part of the super Poincaré algebra is

$$SO(1,5) \times SU(2)_R$$

with translation.

Since Σ is assumed to be a Kähler manifold, the holonomy group is $U(1)$.

Upon the Kaluza-Klein reduction of the 6-dim. theory onto Σ,

$$SO(1,5) \times SU(2)_R \to SO(1,3) \times U(1)_J \times SU(2)_R$$

and the holonomy group $U(1)$ is usually identified with $U(1)_J$.

Since the supercharge Q^i is in the $(4_+, 2)$ rep. of $SO(1,5) \times SU(2)_R$, and is decomposed under $SO(1,3) \times U(1)_J \times SU(2)_R$ into

$$\underbrace{(\overset{\text{"}SU(2)_L\text{"}}{2}, \overset{\text{"}SU(2)_R\text{"}}{1}}_{SO(1,3)} ; +\tfrac{1}{2}, 2) \oplus (1, 2 ; -\tfrac{1}{2}, 2),$$

the holonomy group $U(1)$ of Σ breaks all the supersymmetries.

Therefore, let us consider the partial twist of the theory to obtain supersymmetry. Once again, up to the conventional choices, the twisting is uniquely determined.

The resulting theory has 4-dim. $\mathcal{N}=1$ supersymmetries.

To this end, one may identify the holonomy group $U(1)$ as a linear combination of $U(1)_J$ and $U(1)_R \subset SU(2)_R$ to yield singlet supercharges under it.

$$\begin{array}{c} SU(2)_R \\ \cup \\ \underbrace{U(1)_J \times U(1)_R} \\ \downarrow \text{ a linear combination} \\ U(1) \end{array}$$

$U(1)$ identified with the holonomy.

A 6-dim. hypermultiplet consists of

ϕ_i ; scalar field and a doublet of $SU(2)_R$

and

Ψ ; Weyl spinor ($\Gamma^7 \Psi = \Psi$) and a singlet.

Upon the K.-K. reduction onto Σ, renaming

$$H = \phi_2 , \quad \tilde{H} = \phi_1^*$$

and assuming that

$$\phi_2 \in (R_i , R_i^{*\prime}) \text{ of } G_S \times G_S{}',$$

(ϕ_i , Ψ) may be rewritten in terms of 4-dim.
$\mathcal{N}=1$ superfields.

$$H(y,\theta) = H(y) + \sqrt{2}\,\theta\cdot\chi(y) + \theta^2 F(y) \quad \in (R_i, R_i'^*)$$

$$\tilde{H}(y,\theta) = \tilde{H}(y) + \sqrt{2}\,\theta\cdot\tilde{\chi}(y) + \theta^2 \tilde{F}(y) \quad \in (R_i^*, R_i')$$

$(y^\mu \equiv x^\mu - i\bar{\theta}\sigma^\mu\theta)$.

The partial "twisting" requires $(\overset{(\sim)}{H}, \overset{(\sim)}{\chi}, \overset{(\sim)}{F})$ to be chiral spinors on 2-dim. Σ.

Thus, they take values in $\sqrt{K_\Sigma}$.

The Lagrangian is given by

$$L_I = \int d^4\theta \, K(H, \tilde{H}, H^\dagger, \tilde{H}^\dagger)$$

$$+ \int d^2\theta \, W(H, \tilde{H}) + \int d^2\bar{\theta} \, \bar{W}(H^\dagger, \tilde{H}^\dagger)$$

with

$$K = \mathrm{tr}\left[H^\dagger e^{2gV} H e^{-2gV'} + \tilde{H}^\dagger e^{2gV'} \tilde{H} e^{-2gV} \right]$$

$$W = \sqrt{2}\, \tilde{H} \bar{D}_{\bar{z}} H$$

where

$$\bar{D}_{\bar{z}} H = \bar{\partial}_{\bar{z}} H + ig \left(\frac{\partial \bar{z}^{\bar{m}}}{\partial \bar{z}} \right) A_{\bar{m}} H - igH \left(\frac{\partial \bar{z}^{\bar{m}}}{\partial \bar{z}} \right) A'_{\bar{m}}$$

$$\left(z \equiv \frac{1}{\sqrt{2}} (x^4 + i x^5) \right)$$

Note that there are no couplings with Φ_{mn} in the Lagrangian L_Σ, because the pull-back of the 2-form Φ_{mn} onto Σ is trivially zero;

$$\Phi_{z\bar{z}} = \left(\frac{\partial Z^m}{\partial z}\right)\left(\frac{\partial Z^n}{\partial \bar{z}}\right)\Phi_{mn} = 0.$$

In the presence of the matter curve Σ, the D- and the F-term conditions are modified to

- $g^{m\bar{n}} F_{m\bar{n}} - i g\, g^{m\bar{n}} g^{k\bar{l}} [\Phi_{mk}, \bar{\Phi}_{\bar{n}\bar{l}}]$
  $= i g \,(H H^\dagger - \tilde{H}^\dagger \tilde{H})\, \delta_\Sigma$,

  delta function supported on Σ

- $g^{m\bar{n}} \bar{D}_{\bar{n}} \Phi_{mk} = -\frac{i}{\sqrt{2}}\, g\, g_{k\bar{z}} \left(\frac{\partial \bar{z}^{\bar{l}}}{\partial \bar{z}} \right) H \tilde{H}\, \delta_\Sigma$,
- $F_{\bar{m}\bar{n}} = 0$,
- $\bar{D}_{\bar{z}} H = 0$, $\quad \bar{D}_{\bar{z}} \tilde{H} = 0$.

The matter curve gives source terms (surface operators) to the bulk theory on S.

To give rise to chiral matters from matter curves, one can use the anti-self dual instanton background $A_{\bar{m}}$, which takes value in $G_{inst} \subset G_S$ and breaks G_S as

$$G_S \longrightarrow H_S.$$

Suppose that the rep. R_i of G_S is decomposed under $H_S \times G_{inst}$ into

$$R_i = \bigoplus_j (R_{ij}, V_{ij}).$$

Then, for the hypermultiplet $(H, \tilde{H})$ on Σ, since

$$\begin{aligned} H &\in (R_i, R_i'^{*}) \\ \tilde{H} &\in (R_i^{*}, R_i') \end{aligned} \quad \text{of } G_N \times G_N',$$

they are decomposed under $H_N \times G_{inst} \times G_N'$ into

$$\begin{aligned} H &\in \bigoplus_j (R_{ij}, V_{ij}; R_i'^{*}), \\ \tilde{H} &\in \bigoplus_j (R_{ij}^{*}, V_{ij}^{*}; R_i'). \end{aligned}$$

Looking at the fermionic part

$$-\frac{1}{\sqrt{2}}\,\mathrm{tr}\left[\tilde{X}\,\bar{D}_{\bar{z}}\,X\right]$$

of the F-term on Σ

$$\sqrt{2}\int d^2\theta\ \mathrm{tr}\left[\tilde{H}\,\bar{D}_{\bar{z}}\,H\right],$$

one can see that the masses of H, $\tilde{H}$ are given by the eigenvalue of $\bar{D}_{\bar{z}}$.

Therefore, the zero mode solutions of H and $\widetilde{H}$ are given by

$$\chi \in H^0_{\bar\partial}(\Sigma, K_\Sigma^{\frac{1}{2}} \otimes V_{ij}),$$

$$\tilde\chi \in H^0_{\bar\partial}(\Sigma, K_\Sigma^{\frac{1}{2}} \otimes V_{ij}^*)$$

$$\cong \quad \text{(Serre duality)}$$

$$H^1_{\bar\partial}(\Sigma, K_\Sigma^{\frac{1}{2}} \otimes V_{ij})^\vee .$$

Thus,

$$n_H = \#(\text{the zero modes of } H)$$
$$= \dim H^0_{\bar\partial}(\Sigma, K_\Sigma^{\frac{1}{2}} \otimes V_{ij})$$
$$= h^0(\Sigma, K_\Sigma^{\frac{1}{2}} \otimes V_{ij}),$$

$$n_{\tilde H} = \#(\text{the zero modes of } \tilde H)$$
$$= h^1(\Sigma, K_\Sigma^{\frac{1}{2}} \otimes V_{ij}),$$

give the net number of generations in terms of the Euler character χ as

$$n_H - n_{\tilde H} = h^0(\Sigma, K_\Sigma^{\frac{1}{2}} \otimes V_{ij}) - h^1(\Sigma, K_\Sigma^{\frac{1}{2}} \otimes V_{ij})$$
$$= \chi(\Sigma, K_\Sigma^{\frac{1}{2}} \otimes V_{ij}).$$

The Euler character may be calculated by

$$\chi(\Sigma, K_\Sigma^{\frac{1}{2}} \otimes V_{ij}) = \int_\Sigma \underbrace{ch(K_\Sigma^{\frac{1}{2}} \otimes V_{ij})}_{\mathrm{rank}(K_\Sigma^{\frac{1}{2}} \otimes V_{ij}) + c_1(K_\Sigma^{\frac{1}{2}} \otimes V_{ij})} \underbrace{Td(\Sigma)}_{(1 + \frac{1}{2} c_1(T\Sigma))}$$

$$= (1-g)\,\mathrm{rank}(K_\Sigma^{\frac{1}{2}} \otimes V_{ij}) + \int_\Sigma c_1(K_\Sigma^{\frac{1}{2}} \otimes V_{ij})$$

where $\int_\Sigma c_1(T\Sigma) = 2 - 2\underbrace{g}_{\text{The genus of } \Sigma}$

In the matter curve theory on Σ, one can obtain Yukawa couplings from the F-term on Σ,

$$\sqrt{2}\int d^2\theta \, \mathrm{tr}\left[\tilde{H}\,\bar{D}_{\bar{z}}\,H\right]$$

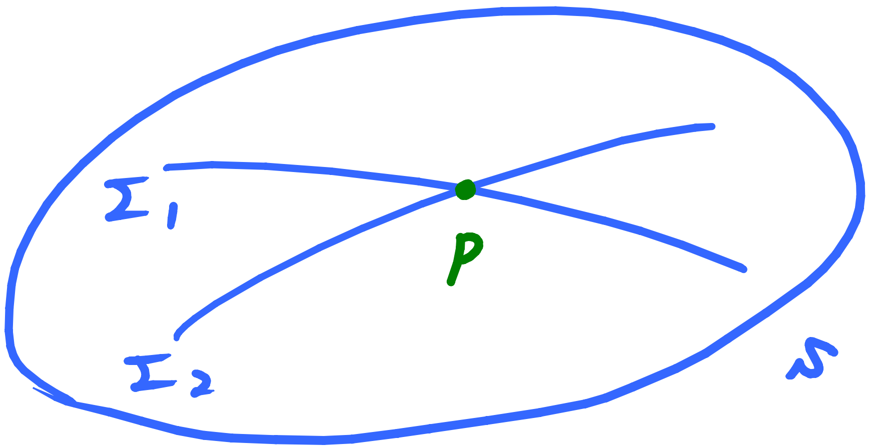
$$= i\sqrt{2}\,g\int d^2\theta \, \mathrm{tr}\left[\tilde{H}A_{\bar{z}}H + \tilde{H}HA'_{\bar{z}} + \cdots\right]$$

$$= -\frac{i}{\sqrt{2}}\,g\,\mathrm{tr}\Big[\tilde{\chi}A_{\bar{z}}\chi + \tilde{\chi}\cdot\chi A'_{\bar{z}} + H\tilde{\chi}\cdot\psi_{\bar{z}}$$

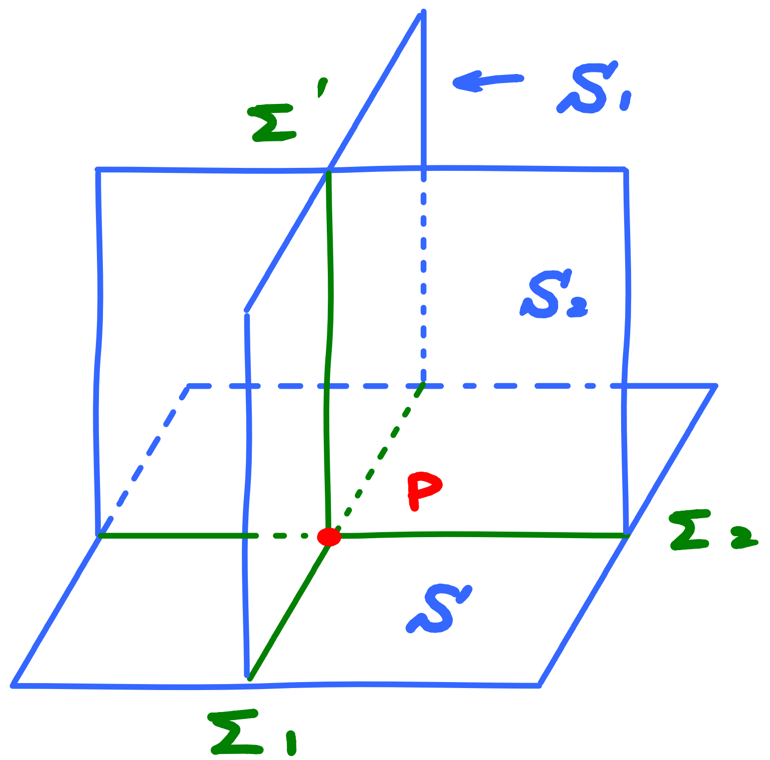
$$+ \tilde{H}\psi_{\bar{z}}\cdot\chi - \tilde{H}\chi\cdot\psi'_{\bar{z}} - H\psi'_{\bar{z}}\tilde{\chi}\Big].$$

5. Yukawa Couplings

Finally, let us touch on the third source to obtain the Yukawa couplings by considering the collision of two matter curves Σ_1 and Σ_2 at a point p on S.



From the viewpoint of B_6, it could be seen, for example, as



One of the simplest examples of such matter curve collisions is given by the singularity

$$x^2 + y^2 + z^n \underbrace{(z-u)}_{z_1} \underbrace{(z-v)}_{z_2} = 0$$

with the discriminant

$$\Delta \simeq z^n (z-u)(z-v).$$

One can see the locations of the 7-branes;

n 7-branes on S : $z = 0$,

1 7-brane on S_1 : $z_1 = u = 0$,

1 7-brane on S_2 : $z_2 = v = 0$.

and the matter curves

$$\Sigma_1 = S \cap S_1 \quad : \quad z = u = 0,$$

$$\Sigma_2 = S \cap S_2 \quad : \quad z = v = 0,$$

and

$$\Sigma' = S_1 \cap S_2 \quad : \quad u = v.$$

The deformation of the singularity is given by the Higgs

$$\langle \varphi_{12} \rangle = \begin{pmatrix} \left.\begin{matrix} 0 & & \\ & \ddots & \\ & & 0 \end{matrix}\right\} n & & \\ & u & \\ & & v \end{pmatrix}.$$

At the point $p = \Sigma_1 \cap \Sigma_2$; $z = u = v = 0$,

the gauge group is enhanced to $U(n+2)$.

On Σ_1, $U(n+2) \longrightarrow U(n+1) \times U(1)$,

On Σ_2, $U(n+2) \longrightarrow U'(n+1) \times U'(1)$,

and

On S,

$$\begin{array}{l} U(n+1) \times U(1) \searrow \\ \qquad\qquad\qquad\qquad U(n) \times U(1) \times U'(1). \\ U'(n+1) \times U'(1) \nearrow \end{array}$$

Since

$$\mathrm{adj}.(U(n+2))$$

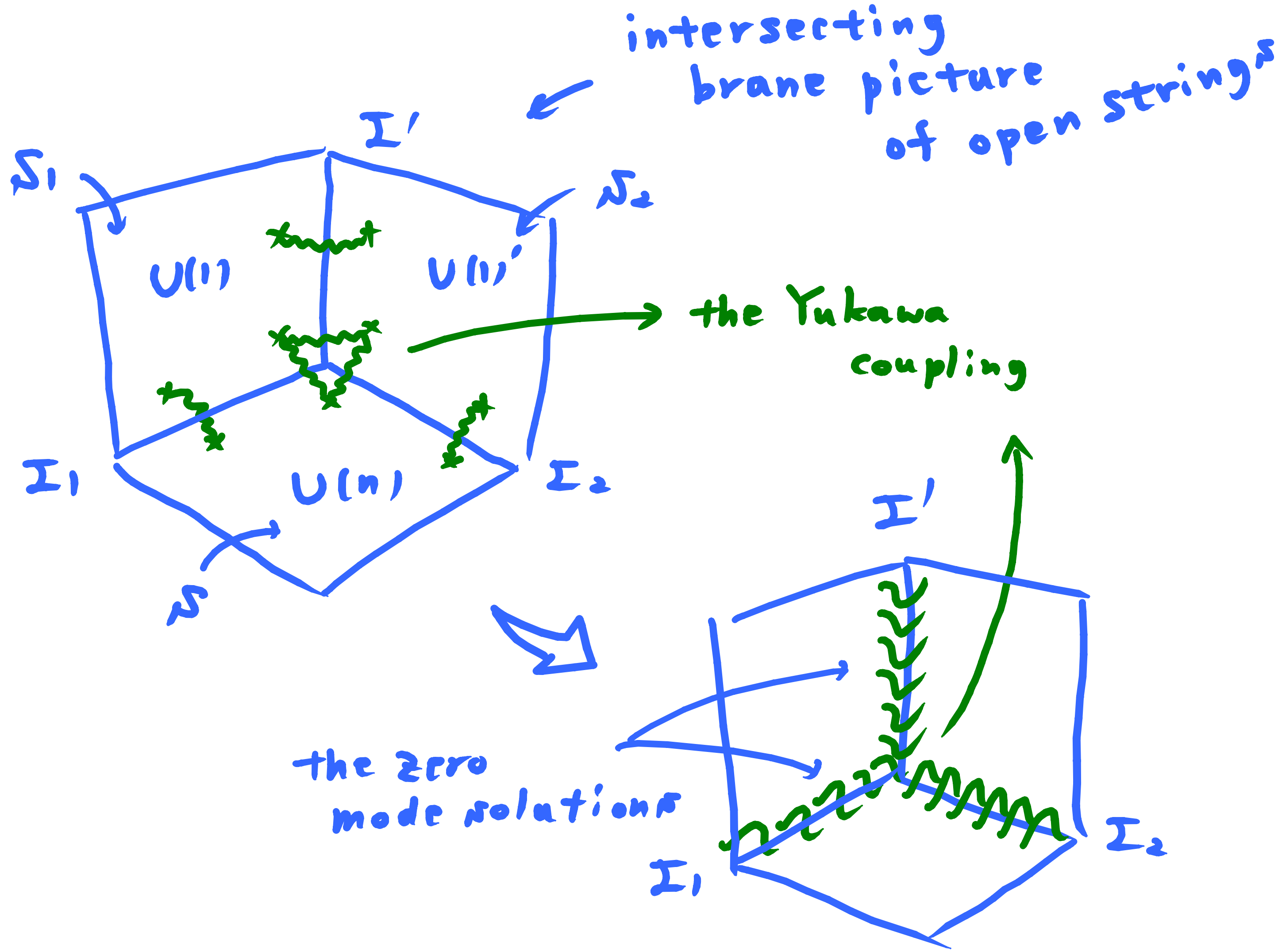
$$= \mathrm{adj}(U(n)) \oplus \mathrm{adj}(U(1)) \oplus \mathrm{adj}(U(1)')$$

$$\oplus\, (n, 1, 0) \oplus (\bar{n}, -1, 0) \qquad \text{on } \Sigma_1$$

$$\oplus\, (n, 0, 1) \oplus (\bar{n}, 0, -1) \qquad \text{on } \Sigma_2$$

$$\oplus\, (1, n, -n) \oplus (1, -n, n) \qquad \text{on } \Sigma'$$

bi-fundamentals (localized matters)


intersecting brane picture of open strings
S_1
I'
S_2
$U(1)$
$U(1)'$
the Yukawa coupling
I_1
$U(n)$
I_2
S
I'
the zero mode solutions
I_1
I_2

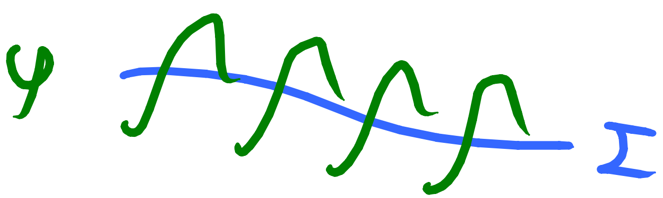
The zero modes on a matter curve Σ take the form like

$$\bar{\chi}_{\bar{1}\bar{2}} = \underbrace{\varphi(u,v)\,\psi(x^\mu)}$$

$$\psi_{\bar{1}} = i\sqrt{2}\,\varphi(u,v)\,\psi(x^\mu)$$

the same 4-dim. spinor

where $\varphi(u,v)$ has a Gaussian profile along Σ,



Substituting the zero modes into the 8-dim. worldvolume action, one would find that the F-term

$$\sim \int d^2\theta \, g^{m\bar{n}} g^{k\bar{l}} \operatorname{tr}[F_{\bar{n}\bar{l}} \Phi_{mk}]$$

$$= \int d^2\theta \, i g \, g^{m\bar{n}} g^{k\bar{l}} \operatorname{tr}[A_{\bar{n}}[A_{\bar{l}}, \Phi_{mk}]]$$

$$+ \cdots,$$

yields the Yukawa couplings among the 4-dim. fields $\psi(x^\mu)$.

More generically, the gauge groups

$$G_p \text{ at } p \;\supset\; G_{\Sigma_i} \times U(1) \text{ on } \Sigma_i$$

$$\supset\; G_S \times U(1) \times U(1) \text{ on } S$$

give hypermultiplets $(H_i, \tilde{H}_i)$ on each Σ_i

and the Yukawa couplings at p.

$\propto$ the structure constants of G_p.

More generic configurations of 7-branes could also lead to the Yukawa couplings.