Instanton counting on the ALE spaces and the root of unity limit of q-Nekrasov function

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August 10, 2011
Summer Institute 2011

based on [arXiv:1105.6091] and so on

- Progress on $\mathcal{N} = 2$ supersymmetric gauge theories

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'94 Seiberg-Witten

"*Solution*" of $\mathcal{N}=2$ theory

'02 Nekrasov

Instanton counting and gauge theory partition function

'09 Alday-Gaiotto-Tachikawa

Relation between 2d/4d theories, meaning of SW curve

- Mainly based on $\mathbb{R}^4$, S^4...

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- Extention to other spaces:

Instanton counting on the ALE space

  - [Nakajima] [Kronheimer-Nakajima] [Fucito-Morales-Poghossian]

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Instanton counting on the ALE space

  - [Nakajima] [Kronheimer-Nakajima] [Fucito-Morales-Poghossian]

- AGT relation
  $\longrightarrow$ 2d CFT: para-Liouville/Toda (super Liouville)
  - [Belavin-Feigin] [Nishioka-Tachikawa] [Bonelli-Maruyoshi-Tanzini] [Belavin-Belavin-Bershtein]

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ADHM construction

ADHM equation

For k-instanton configuration in $\mathrm{SU}(n)$ gauge theory,

$$[B_1, B_2] + IJ = 0, \quad [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = 0$$

- ADHM data: (B_1, B_2, I, J)

$$B_1, B_2 \in \mathrm{Hom}(V, V), \qquad I \in \mathrm{Hom}(W, V), \qquad J \in \mathrm{Hom}(V, W)$$

dim $V = k$: #instantons, dim $W = n$: gauge group rank

- Stringy configuration: kD0 & nD4 branes

B_1, B_2: kD0-kD0, I: nD4-kD0, J: kD0-nD4

ADHM construction

Instanton moduli space

$$\mathcal{M}_{n,k} = \{(B_1, B_2, I, J) | \text{ ADHM eq. } \} / \mathrm{U}(k)$$

- $\mathrm{U}(k)$ symmetry

$$(B_1, B_2, I, J) \longrightarrow (gB_1g^{-1}, gB_2g^{-1}, gI, Jg^{-1}), \qquad g \in \mathrm{U}(k)$$

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Localization formula

The integral over the moduli space concentrates on **fixed points** of the isometries (maximal torus action), up to gauge transformation.

- Isometries: $\mathrm{U}(1)^2 \times \mathrm{U}(1)^{n-1} \subset \mathrm{SO}(4) \times \mathrm{SU}(n)$

$$(B_1, B_2, I, J) \longrightarrow (e^{i\epsilon_1}B_1, e^{i\epsilon_2}B_2, Ie^{-ia_l}, e^{i(\epsilon_1+\epsilon_2)}e^{ia_l}J)$$

- Fixed points classified by n-tuple partition

$$\vec{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \cdots, \lambda^{(n)}), \qquad |\vec{\lambda}| = \sum_{l=1}^{n} |\lambda^{(l)}| = k$$

- Character of the tangent space at the fixed point

$$\chi_{\vec{\lambda}} = \sum_{l,m}^{n} \sum_{\lambda^{(l)}} \left(T_{a_{lm}} T_1^{-\lambda_i^{(m)}+j} T_2^{\check{\lambda}_j^{(l)}-i+1} + T_{a_{ml}} T_1^{\lambda_i^{(l)}-j+1} T_2^{-\check{\lambda}_j^{(m)}+i} \right)$$

Instanton partition function

$$Z = \sum_{\vec{\lambda}} Q^{|\vec{\lambda}|} \prod_{l,m}^{n} \prod_{\lambda^{(l)}} \frac{1}{a_{lm} - \epsilon_1(\lambda_i^{(m)} - j) + \epsilon_2(\check{\lambda}_j^{(l)} - i + 1)} \times \frac{1}{a_{lm} - \epsilon_1(\lambda_i^{(m)} - j + 1) + \epsilon_2(\check{\lambda}_j^{(l)} - i)}$$

- Asymptotics of the combinatorial partition function
  - limit shape, matrix model $\longrightarrow$ Seiberg-Witten curve

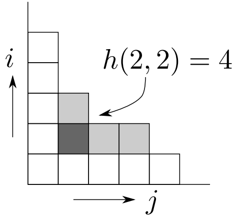
U(1) partition function

$$Z = \sum_\lambda Q^{|\lambda|} \prod_{(i,j)\in\lambda} \frac{1}{h(i,j)^2}, \qquad \lambda: \text{ partition}$$

- Plancherel measure & hook length representation

$$\mu_n(\lambda) = \frac{(\dim\lambda)^2}{n!} = n! \prod_{(i,j)\in\lambda} \frac{1}{h(i,j)^2}$$

  - hook length: $h(i,j) = \lambda_i - j + \check{\lambda}_j - i + 1$



- cf. Schur polynomial

Contents

Orbifold and ALE space

- Orbifold: $\mathbb{R}^4 \simeq \mathbb{C}^2 \longrightarrow \mathbb{C}^2/\Gamma$
  - Γ: a finite subgroup of $\mathrm{SU}(2) \longrightarrow$ ADE classification
- Resolving the singularity $\longrightarrow$ ALE spaces
  - [Eguchi-Hanson] [Gibbons-Hawking] [Kronheimer]
  - [Eguchi-san's talk] [Nishinaka-san's talk]
- eg. $\underline{A_{r-1}}$

$$\Gamma = \mathbb{Z}_r : (z_1, z_2) \longrightarrow (\omega_r z_1, \bar{\omega}_r z_2), \quad e^{ia_l} \longrightarrow \omega_r^{p_l} e^{ia_l}$$

The primitive r-th root of unity: $\omega_r = e^{\frac{2\pi}{r} i}$

  - cf. surface operators [Kanno-san's talk]

All the observables are Γ-invariant

- ADHM data:

$$(B_1, B_2, I, J) \longrightarrow (\omega_r B_1, \bar{\omega}_r B_2, I\bar{\omega}_r^{p_l}, \omega_r^{p_l} J)$$

- Torus action:

$$(T_1, T_2, T_{a_l}) \longrightarrow (\omega_r T_1, \bar{\omega}_r T_2, \omega_r^{p_l} T_{a_l})$$

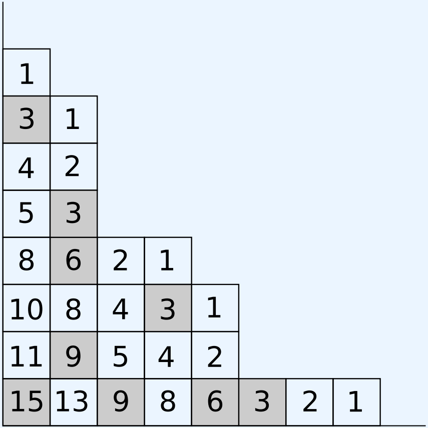
- Character:

$$\begin{aligned} & T_{a_{lm}} T_1^{-\lambda_i^{(m)}+j} T_2^{\check{\lambda}_j^{(l)}-i+1} \\ \longrightarrow \quad & \omega_r^{-(p_{ml}+\lambda_i^{(m)}+j+\check{\lambda}_j^{(l)}-i+1)} T_{a_{lm}} T_1^{-\lambda_i^{(m)}+j} T_2^{\check{\lambda}_j^{(l)}-i+1} \end{aligned}$$

Γ-invariant sector

$$p_{ml} + \lambda_i^{(m)} + j + \check{\lambda}_j^{(l)} - i + 1 \equiv 0 \pmod{r}$$

Γ-invariant sector for $\mathrm{U}(1)$ theory



Numbers in boxes stand for their hook lengths $\lambda_i - j + \check{\lambda}_j - i + 1$.
Shaded boxes are invariant under the action of $\Gamma = \mathbb{Z}_3$.

- How to extract the Γ-invariant sector?

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q-deformation

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q-deformation

- Remark: 5d theory $\mathbb{R}^4 \times S^1 \longrightarrow q = e^R$
  - Equivaliant K-theory

Orbifold projection

- q-deformation parameter: $q = e^{-g}$

$$\frac{1}{1-q^x} = \frac{1}{1-e^{-gx}} = \frac{1}{1-(1-gx+\cdots)} \simeq \frac{1}{gx} \to \mathcal{O}(g^{-1})$$

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- The root of unity limit: $q = \omega_r e^{-g}$

$$\begin{aligned} \frac{1}{1-(\omega_r e^{-g})^x} &= \frac{1}{1-\omega_r^x(1-gx+\cdots)} \\ &\simeq \begin{cases} 1/(1-\omega_r^x) \to \mathcal{O}(g^0) & x \not\equiv 0 \pmod{r} \\ 1/(gx) \to \mathcal{O}(g^{-1}) & x \equiv 0 \pmod{r} \end{cases} \end{aligned}$$

the root of unity limit $\Longleftrightarrow$ orbifold projection

q-Nekrasov function

$$Z = \prod_{(l,i)\neq(m,j)}^{\infty} \frac{(Q_{lm} q^{\lambda_i^{(l)} - \lambda_j^{(m)}} t^{j-i}; q)_\infty}{(Q_{lm} q^{\lambda_i^{(l)} - \lambda_j^{(m)}} t^{j-i+1}; q)_\infty} \frac{(Q_{lm} t^{j-i+1}; q)_\infty}{(Q_{lm} t^{j-i}; q)_\infty}$$

$$q = e^{R\epsilon_2}, \qquad t = e^{-R\epsilon_1} = q^\beta, \qquad Q_{lm} = e^{Ra_{lm}} = q^{b_{lm}}$$

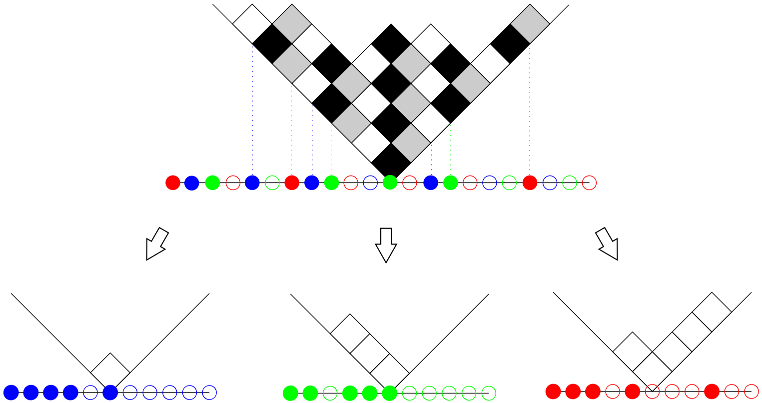
- $\mathbb{C}^2/\mathbb{Z}_r$

$$q \longrightarrow \omega_r q, \qquad t \longrightarrow \omega_r t$$

- $\mathbb{C}/\mathbb{Z}_r \times \mathbb{C}/\mathbb{Z}_s \qquad (u = r/s \in \mathbb{N})$

$$q \longrightarrow \omega_r q, \qquad t \longrightarrow \omega_s t = \omega_r^u t$$

Partition rearranged

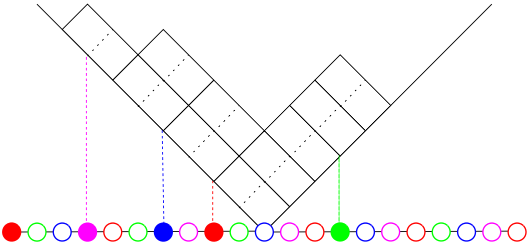


n-tuple partition $\longrightarrow$ nr-tuple partition ($\longrightarrow$ r-tuple partition)

- [Dijkgraaf-Sułkowski]

Partition rearranged

- $\mathbb{C}/\mathbb{Z}_4 \times \mathbb{C}/\mathbb{Z}_2 \longrightarrow (r,s) = (4,2),\ u = r/s = 2$



Fractional exclusive statistics

Matrix model

- Partition function: $(\tilde{q}, \tilde{t}) \equiv (\omega_r q, \omega_r t)$

$$Z \longrightarrow \prod_{(v,i)\neq(w,j)} \frac{(\tilde{q}^{x_i^{(v)}-x_j^{(w)}};\tilde{q})_\infty}{(\tilde{t}\tilde{q}^{x_i^{(v)}-x_j^{(w)}};\tilde{q})_\infty}$$

  - [Klemm-Sułkowski]

Matrix model

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  - [Klemm-Sułkowski]

q-Vandermonde determinant

$$\Delta_{q,t}^2(x) = \prod_{i\neq j} \frac{(x_i/x_j;q)_\infty}{(tx_i/x_j;q)_\infty}$$

- cf. Macdonald polynomial

- Jack limit: $t = q^\beta$

$$\frac{(e^z;q)_\infty}{(te^z;q)_\infty} = \prod_{n=0}^{\beta-1}(1-e^z q^n) = \prod_{n=0}^{\beta-1}\left(1-e^{z+nR\epsilon_2}\right)$$

$$\xrightarrow{z\gg1} (1-e^z)^\beta$$

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$$\xrightarrow{z \gg 1} (1-e^z)^\beta$$

$$\Delta_{q,t}^2(x) \longrightarrow \prod_{i\neq j}\left(1-\frac{x_i}{x_j}\right)^\beta \sim \prod_{i<j}(x_i-x_j)^{2\beta}$$

β-ensemble

- Uglov limit: $q \to \omega_r q$, $t \to \omega_r q^\beta$ [Uglov '98]

$$
\begin{aligned}
\Delta_{q,t}^2(x) \quad &\to \quad \prod_{i \neq j} \left(1 - \frac{x_i}{x_j}\right)^{\frac{\beta-1}{r}+1} \left(1 + \cdots + \frac{x_i^{r-1}}{x_j^{r-1}}\right)^{\frac{\beta-1}{r}} \\
&\sim \quad \prod_{i<j} (x_i - x_j)^{2\frac{\beta-1}{r}+2} (x_i^{r-1} + x_i^{r-2} x_j + \cdots + x_j^{r-1})^{2\frac{\beta-1}{r}} \\
&= \quad \prod_{i<j} (x_i - x_j)^2 (x_i^r - x_j^r)^{2(\beta-1)/r}
\end{aligned}
$$

- Double root of unity limit:

$$q \longrightarrow \omega_r q, \qquad t \longrightarrow \omega_s q^\beta, \qquad u := r/s \in \mathbb{N}, \quad 1 \le u \le r$$

$$\begin{aligned} \Delta^2_{q,t}(x) \quad \longrightarrow \quad & \prod_{i<j} \left[(x_i - x_j)^{2(\beta-u)/r+2} \prod_{n=1}^{u-1} (x_i - \omega_r^n x_j)^2 \right. \\ & \left. \times \left(x_i^{r-1} + \cdots + x_j^{r-1} \right)^{2(\beta-u)/r} \right] \\ = \quad & \prod_{i<j} \left[(x_i - x_j)^2 (x_i^r - x_j^r)^{2(\beta-u)/r} \prod_{n=1}^{u-1} (x_i - \omega_r^n x_j)^2 \right] \end{aligned}$$

- eg. $\underline{u = r}$: $(q, t) \to (\omega_r q, t)$

$$\Delta^2_{q,t}(x) \longrightarrow \prod_{i<j} (x_i^r - x_j^r)^{2\beta/r}$$

  - orbifold: $\mathbb{C} \times \mathbb{C}/\mathbb{Z}_r$ [Kanno-san's talk]

Matrix measure

$$\Delta^2(x) = \prod_{v=0}^{r-1}\prod_{i<j}\Bigg[\left(\frac{2}{R}\sinh\frac{rR}{2}\left(x_i^{(v)}-x_j^{(v)}\right)\right)^{2(\beta-1)/r} \times\left(\frac{2}{R}\sinh\frac{R}{2}\left(x_i^{(v)}-x_j^{(v)}\right)\right)^{2}\Bigg]$$
$$\times \prod_{v<w}^{r-1}\prod_{i,j}\Bigg[\left(\frac{2}{R}\sinh\frac{rR}{2}\left(x_i^{(v)}-x_j^{(w)}\right)\right)^{2(\beta-1)/r} \times\left(\frac{2}{R}\sinh\frac{R}{2}\left(x_i^{(v)}-x_j^{(w)}+\frac{2\pi i}{r}(v-w)\right)\right)^{2}\Bigg]$$

- The case of $\beta = 1 \longrightarrow$ almost free
- cf. Chern-Simons matrix model on the lens space $S^3/\mathbb{Z}_r$
  [Aganagic-Klemm-Mariño-Vafa] [Halmagyi-Yasnov]
  $\longrightarrow$ ABJM matrix model [Moriyama-san's talk]

Relation to 2d CFT

- Instanton counting on the ALE space $\mathbb{C}^2/\mathbb{Z}_r$
  $\longrightarrow$ 2d CFT: para-Liouville/Toda theories
  [Belavin-Feigin] [Nishioka-Tachikawa] [Bonelli-Maruyoshi-Tanzini]
  [Belavin-Belavin-Bershtein] cf. [Estienne-Bernevig]
  - eg. $\mathrm{SU}(2)$ on $\mathbb{C}^2/\mathbb{Z}_2$: superconformal

$$\frac{\mathrm{SU}(r+1)_n}{\mathrm{U}(1)} = \frac{\mathrm{SU}(n)_1^{\otimes r+1}}{\mathrm{SU}(n)_{r+1}} \longrightarrow \frac{\mathrm{SU}(n)_l \otimes \mathrm{SU}(n)_1^{\otimes r}}{\mathrm{SU}(n)_{r+l}}$$

$$c = r(n-1)\left(1 + \frac{n(n+1)}{r^2} Q^2\right), \quad Q^2 = \begin{cases} (b+1/b)^2 & \text{(Liouville)} \\ -(p-q)^2/pq & \text{(minimal)} \end{cases}$$

- The root of unity limit of q-Virasoro/$\mathcal{W}$ algebra?
  - [Bouwknegt-Pilch]

Contents

Summary and Outlook

- Combinatorial instanton partition function for the ALE space is given by the root of unity limit of q-Nekrasov function.
- A new kind of matrix model describing β-ensemble
  - Large-N limit analysis
- Corresponding reduction from Macdonald polynomial and q-Virasoro/$\mathcal{W}$ algebra

Summary and Outlook

- Vortices on orbifolds $\longrightarrow$ in preparation [Kimura-Nitta]
- Relation to the wheel condition: [Feigin-Jimbo-Miwa-Mukhin]

$$t^{k+1}q^{r-1} = 1$$

- $S^4/\mathbb{Z}_r$ [Pestun]
- Squashed lens space $S^3/\mathbb{Z}_r$ [Hama-Hosomichi-Lee]
- Superconformal index $S^3/\mathbb{Z}_r \times S^1$ and 2d qYM [Gadde-Rastelli-Razamat-Yan]

Appendix: Instanton construction for $\mathbb{C}/\mathbb{Z}_r \times \mathbb{C}/\mathbb{Z}_s$

- Orbifolding: $\mathbb{C}/\mathbb{Z}_r \times \mathbb{C}/\mathbb{Z}_s$ with $u := r/s \in \mathbb{N}$

$$(z_1, z_2) \sim (\omega_r z_1, \bar{\omega}_s z_2)$$

- Decomposition of vector spaces

$$W = \bigoplus_{i=0}^{r-1} W_i, \qquad V = \bigoplus_{i=0}^{r-1} V_i$$

- ADHM data

$$B_{1,i} \in \mathrm{Hom}(V_i, V_{i+1}), \quad B_{2,i} \in \mathrm{Hom}(V_i, V_{i-u}),$$

$$I_i \in \mathrm{Hom}(W_i, V_i), \quad J_i \in \mathrm{Hom}(V_i, W_{i-u+1})$$

Appendix: Instanton construction for $\mathbb{C}/\mathbb{Z}_r \times \mathbb{C}/\mathbb{Z}_s$

- ADHM equation

$$B_{1,i-u}B_{2,i} - B_{2,i+1}B_{1,i} + I_{i-u+1}J_i = 0,$$

$$B_{1,i-1}B_{1,i-1}^\dagger - B_{1,i}^\dagger B_{1,i} + B_{2,i+u}B_{2,i+u}^\dagger - B_{2,i}^\dagger B_{2,i} + I_i I_i^\dagger - J_i^\dagger J_i = 0$$

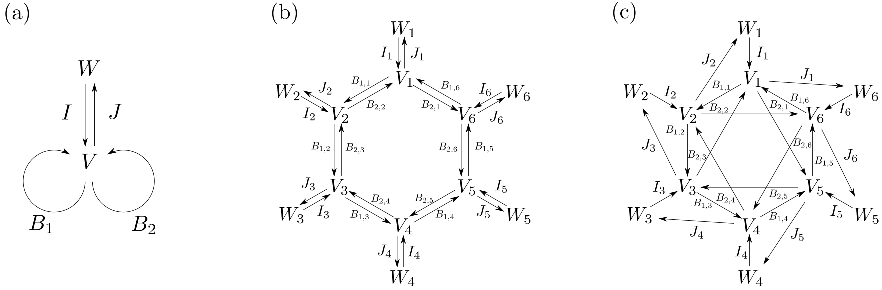

Figure: Quiver diagrams: (a) $\mathbb{C}^2$, (b) $\mathbb{C}^2/\mathbb{Z}_6$ and (c) $\mathbb{C}/\mathbb{Z}_6 \times \mathbb{C}/\mathbb{Z}_3$.